

## CONVERGENCE ANALYSIS OF A PERTURBED ITERATIVE SCHEME (PIS) FOR SOLUTION OF NONLINEAR SYSTEMS

**S.K. DEY**

Visiting Professor  
von Karman Institute for Fluid Dynamics  
B-1640 Rhode-ST-Genese  
Belgium

Present Address:  
Department of Mathematics  
Eastern Illinois University  
Charleston, Illinois 61920  
U.S.A.

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ABSTRACT. In the references [1, 2, 3] a perturbed iterative scheme (PIS) has been studied both theoretically and computationally to solve nonlinear equations. In this article a more general analysis of its convergence properties has been done.

KEY WORDS AND PHRASES. *Perturbed Functional Iterations, Nonlinear Equations, Numerical Solution.*

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### 1. INTRODUCTION.

A perturbed iterative scheme (PIS) has been developed in [1, 3] to solve nonlinear equations. It is a functional iterative scheme obtained by adding a predetermined perturbation parameter to nonlinear Gauss-Seidel iterations. PIS has a simple algorithm. Other similar iterations [6, 7, 8] had more restricted applications and complicated algorithms. Theorems on convergence properties of

PIS, derived in [1] for one nonlinear system, were modified in [3] for coupled nonlinear systems. In this article more generalized convergence properties of PIS have been analyzed by applying decaying matrices [5]. A practical demonstration of this concept is also given. Theorems on convergence of PIS derived in [1, 2, 3] may be interpreted as particular cases of those derived in this paper.

## 2. ALGORITHM OF PIS [1].

Let us consider a nonlinear system:

$$F_i(x_1, x_2, \dots, x_n) = 0, \quad i = 1, 2, \dots, n \quad (2.1)$$

This equation may be expressed as  $F(x) = 0$  where  $x = (x_1 \ x_2 \ \dots \ x_n)^T$ ,

$F : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$  ( $\mathbb{R}^n$  is the real  $n$ -dimensional space). We assume that (2.1)

admits a root  $x = x^* \in D$ . Our objective is to develop a perturbed iterative scheme (PIS) to solve (2.1) and compute  $x^*$ . If (2.1) is written as:

$$x_i = G_i(x_1, x_2, \dots, x_n), \quad i = 1, 2, \dots, n \quad (2.2)$$

nonlinear Gauss-Seidel iteration at some  $k$ th step is:

$$x_i^k = G_i^k \quad (2.3)$$

where  $x_i^k$  = value of  $x_i$  at the  $k$ th (not exponent) iteration and

$$G_i^k = G_i(x_1^k, x_2^k, \dots, x_{i-1}^k, x_i^{k-1}, \dots, x_n^{k-1}). \quad (2.4)$$

Let  $G : D \times D \subset \mathbb{R}^n \times \mathbb{R}^n \rightarrow D$ . Then (2.3) may be expressed as

$$x^k = G(x^k, x^{k-1}) \quad (2.5)$$

$$\text{and, } x^* = G(x^*, x^*) \quad (2.6)$$

DEFINITION 1: In such a case,  $x^*$  is called a fixed image of  $G$  on  $D \times D$ .

Now (2.3) may be perturbed as follows:

$$x_i^k = w_i^k + G_i^k \quad (2.7)$$

To compute the perturbation parameter  $w_i^k$ , we assume that  $\forall i, k$ : (i)  $w_i^k$ 's are small, such that terms of the order  $(w_i^k)^2$  may be neglected, (ii)  $(\partial G_i / \partial x_i)^k \neq 1$ , and (iii)  $(\partial^2 G_i / \partial x_i^2)^k$  is bounded.

If (2.7) converges after  $(k-1)$  iterations,  $x_i^{k-1} = x_i^k = x_i^*$ . Then,

$w_i^k + G_i^k = G_i(x_1^k \dots x_{i-1}^k, w_i^k + G_i^k, x_{i+1}^{k-1} \dots x_n^{k-1})$ . Expanding the right side by

Taylor's series and using the above assumptions we get

$$w_i^k = (\bar{G}_i^k - G_i^k) / \{1 - \partial_i G_i^k\} \tag{2.8}$$

where  $\bar{G}_i^k = G_i(x_1^k \dots x_{i-1}^k, G_i^k, x_{i+1}^{k-1} \dots x_n^{k-1})$  and

$$\partial_i G_i^k = (\partial G_i / \partial x_i) x_1^k \dots x_{i-1}^k, G_i^k, x_{i+1}^{k-1} \dots x_n^{k-1}.$$

Thus in (2.7),  $w_i^k$ 's are computed in terms of quantities known a priori. The convergence criterion is:

$$\max_i |w_i^k| < \epsilon \tag{2.9}$$

where  $\epsilon$  is positive and arbitrarily small.

### 3. ANALYSIS OF CONVERGENCE.

By convergence we mean  $\lim_{k \rightarrow \infty} x^k = x^*$ . Now (2.7) may be expressed as:

$$x^k = w^k + G(x^k, x^{k-1}) \tag{3.1}$$

where  $w^k = (w_1^k \dots w_n^k)^T \in R^n$ . Let  $G$  be continuous on  $D \times D$ .

**THEOREM 1:** A necessary condition for convergence is that, for some norm

$$\lim_{k \rightarrow \infty} \|w^k\| = 0 \tag{3.2}$$

**PROOF:** Assuming that (2.7) converges to  $x^*$ ,  $\lim_{k \rightarrow \infty} x^k = x^*$  and

$\lim_{k \rightarrow \infty} G(x^k, x^{k-1}) = G(x^*, x^*) = x^*$ . Then, (3.1) gives  $\lim_{k \rightarrow \infty} w^k = \emptyset$ , which implies

(3.2).

To prove that (3.2) may also be a sufficient condition for convergence, the following concept is used:

**DEFINITION 2:** A sequence of commuting square matrices  $A_1, A_2, \dots, A_n$  is called a sequence of decaying matrices or simply D-matrices if

$$\lim_{k \rightarrow \infty} A_1 A_2 \dots A_k = \emptyset \tag{3.3}$$

Each  $A_k$  is called a D-matrix. Obviously, the elements of  $A_k$  are variable and they change as  $k$  changes. Several properties of D-matrices are given in [5].

The one we need is:

LEMMA 1: A sufficient condition such that  $A_k$  is a D-matrix is that for some particular norm and  $\forall k > K$ ,

$$\|A_k\| \leq \alpha < 1 \quad (3.4)$$

PROOF: For any norm,  $\|A_1 A_2 \dots A_k\| \leq \|A_1\| \|A_2\| \dots \|A_k\|$ . Also,

$\|A_k\| = 0$  iff  $A = \emptyset$ . Proof is now rather trivial.

Let  $H : D \times D \subset \mathbb{R}^n \times \mathbb{R}^n \rightarrow D \forall (x^k, y^k), (\alpha, \beta) \in D \times D$ . Let us express

$$H(x^k, y^k) - H(\alpha, \beta) = A_k(x^k - \alpha) + B_k(y^k - \beta) \quad (3.5)$$

where  $A_k, B_k$  are square matrices ( $n \times n$ ) and,  $a_{i,j} =$  an element of  $A_k = a_{i,j}^k(x^k, y^k, \alpha, \beta)$ ;  $b_{i,j}^k =$  an element of  $B_k = b_{i,j}^k(x^k, y^k, \alpha, \beta)$ . As  $k$  changes,  $x^k, y^k$  and hence  $a_{i,j}^k, b_{i,j}^k$  change. Let  $|x| = (|x_1| |x_2| \dots |x_n|)^T$ .

DEFINITION 2: If  $\forall (x^k, y^k), (\alpha, \beta) \in D \times D, A_k, B_k$  are continuous,

$$C_k = |(I - A_k)^{-1}| \quad (3.6)$$

is bounded, and

$$E_k = |(I - A_k)^{-1} B_k| \quad (3.7)$$

is continuous and form a sequence of D-matrices, the mapping  $H : D \times D \subset \mathbb{R}^n \times \mathbb{R}^n \rightarrow D$  is called a D-mapping on  $D \times D$ .

THEOREM 2: If in (3.1)  $G$  is a D-mapping and (3.2) is true, PIS (3.1) will converge to  $x^*$ , a fixed image of  $G$  on  $D \times D$ . Furthermore, if  $\rho(E_k)$  (the spectral radius of  $E_k$ )  $< 1 \forall k > K$ ,  $x^*$  is the unique fixed image of  $G$  on  $D \times D$ .

PROOF: Subtracting (2.6) from (3.1) and using (3.5) we get  $x^k - x^* = w^k + A_k(x^k - x^*) + B_k(x^{k-1} - x^*)$ .

$$\text{Then, } |x^k - x^*| \leq C_k |w^k| + E_k |x^{k-1} - x^*| \quad (3.8)$$

where  $C_k$  and  $E_k$  are defined in (3.6) and (3.7) respectively. Now, using (3.8) recurrently,

$$\begin{aligned}
 |x^k - x^*| &\leq \sum_{j=1}^k E_k E_{k-1} \dots E_{j+1} C_j |w^j| + E_k E_{k-1} \dots E_1 |x^0 - x^1| \\
 &= (E_k E_{k-1} \dots E_{k_0+1}) \sum_{j=1}^{k_0} (E_{k_0} E_{k_0-1} \dots E_{j+1}) C_j |w^j| \\
 &+ \sum_{j=k_0+1}^k (E_k E_{k-1} \dots E_{j+1}) C_j |w^j| + E_k E_{k-1} \dots E_1 |x^0 - x^1|. \quad (3.9)
 \end{aligned}$$

(3.2) implies that for some  $k \geq k_0+1$ ,  $|w^k| < \epsilon$ . Also, since  $E_k$  is a D-matrix,  $\lim_{k \rightarrow \infty} E_1 E_2 \dots E_k = \emptyset$  and  $\lim_{k \rightarrow \infty} E_{k_0+1} E_{k_0+2} \dots E_k = \emptyset$ . Hence  $C_j$  being bounded  $\forall j$ , from (3.9) we get  $\lim_{k \rightarrow \infty} |x^k - x^*| = \emptyset$  which establishes convergence.

In order to prove uniqueness, we assume that  $x = y^*$  is a second root. Then,  $x^* - y^* = G(x^*, x^*) - G(y^*, y^*) = A_*(x^* - y^*) + B_*(x^* - y^*)$ . Thus,  $|x^* - y^*| = |(I - A_*)^{-1} B_*| |x^* - y^*| = E_* |x^* - y^*|$ . Since  $\rho(E_k) < 1, \forall k > K$  and  $E_k \rightarrow E_*$  as  $k \rightarrow \infty, \rho(E_*) < 1$ . This gives  $(I - E_*)^{-1} = \sum_{j=0}^{\infty} E_*^j > \emptyset$ . Hence  $(I - E_*) |x^* - y^*| = \emptyset$ , implies  $x^* = y^*$ .

It may be observed that  $A_k$  and  $B_k$  are respectively lower and upper triangular matrices with variable elements. If  $A_k + B_k$  is a tridiagonal matrix it is rather easy to check when  $G$  is a D-mapping [9]. However, in general  $(I - A_k)^{-1}$  is not quite simple to find. In such cases, theorem 2 cannot be used and the following approach may be taken.

**THEOREM 3:** Let  $\forall k$ ,

$$G(x^k, x^{k-1}) - G(x^*, x^*) = A_k(x^k - x^*) + B_k(x^{k-1} - x^*) \quad (3.10)$$

where  $(I - A_k)^{-1}$  is bounded. Let for some particular norm and  $\forall k > K$ ,

$\|A_k\| + \|B_k\| \leq \alpha < 1$ . Furthermore if (3.2) is true, PIS given by (3.1) will converge to  $x^*$  and  $x^*$  is the unique fixed image of  $G$  on  $D \times D$ .

**PROOF:** Let  $M_k = (I - A_k)^{-1} B_k$ . Since  $\|A_k\| + \|B_k\| \leq \alpha < 1$ ,  $\|A_k\| \leq \alpha < 1$  and  $(\alpha - \|A_k\|)(1 - \|A_k\|)^{-1} < \alpha$ . Also,

$$\begin{aligned} \|M_k\| &= \|(I - A_k)^{-1} B_k\| \leq \|B_k\| / (1 - \|A_k\|) \leq (\alpha - \|A_k\|) / (1 - \|A_k\|) \\ &< \alpha < 1. \text{ Hence } M_k \text{ is a D-matrix. Subtracting (2.6) from (3.1) and using (3.10)} \\ \text{we get: } x^k - x^* &= w^k + A_k(x^k - x^*) + B_k(x^{k-1} - x^*). \text{ Thus, } \|x^k - x^*\| \\ &\leq \|(I - A_k)^{-1}\| \|w^k\| + \|M_k\| \|x^{k-1} - x^*\|. \text{ If } \theta = \max_k \|(I - A_k)^{-1}\|, \\ \|x^k - x^*\| &\leq \theta \|w^k\| + \alpha \|x^{k-1} - x^*\|. \end{aligned} \quad (3.11)$$

(3.2) suggests that for some  $k \geq k_0 + 1$ ,  $\|w^k\| < \epsilon$ . Hence applying (3.11)

recurrently we get:

$$\|x^k - x^*\| \leq \theta \cdot \alpha^{k-k_0} \sum_{j=1}^{k_0} \alpha^{k_0-j} \|w^j\| + \epsilon \cdot \theta \cdot (1 - \alpha)^{-1} + \alpha^k \|x^0 - x^*\|.$$

Since  $0 < \alpha < 1$ , as  $k \rightarrow \infty$ ,  $\|x^k - x^*\| \rightarrow 0$ .

To prove uniqueness, let  $x = y^*$  be the other root. Then,  $x^* - y^*$   
 $= G(x^*, x^*) - G(y^*, y^*) = A_*(x^* - y^*) + B_*(x^* - y^*)$  or  $(1 - \|M_k\|) \|x^* - y^*\| \leq 0$ .  
 Since  $\|M_k\| < \alpha < 1 \forall k > K$  and as  $k \rightarrow \infty$ ,  $M_k \rightarrow M_*$ , we have  $\|x^* - y^*\| \leq 0$  giving  
 $x^* = y^*$ .

It may be easy to prove:

**THEOREM 4:** A sufficient condition such that nonlinear Gauss-Seidel iterations  $x^k = G(x^k, x^{k-1})$  will converge to  $x = x^*$  is that  $E_k = |(I - A_k)^{-1} B_k|$  is a D-matrix where  $A_k$  and  $B_k$  are defined in (3.5).

**PROOF:** In (3.9) we set  $w^k = 0$  and the proof follows.

#### 4. AN APPLICATION.

Solve:  $0.25x_1^2 - x_1 + 0.75x_2 = 0$ ,  $0.5x_1x_2 - x_2 + 0.005 = 0$ . In  
 $[-0.5, 0.5] \times [-0.5, 0.5]$ , this system has a root given by  $x_1 = 3.7606 \times 10^{-3}$  and  
 $x_2 = 5.00942 \times 10^{-3}$ . We express the system as  $x_1 = G_1(x_1, x_2)$ ,  $x_2 = G_2(x_1, x_2)$ ,  
 where  $G_1(x_1, x_2) = 0.25x_1^2 + 0.75x_2$ ;  $G_2(x_1, x_2) = 0.5x_1x_2 + 0.005$ . Denoting the  
 roots by  $x_1^*$ ,  $x_2^*$ :  $G_1(x_1^{k-1}, x_2^{k-1}) - G_1(x_1^*, x_2^*) = 0.25(x_1^{k-1} + x_1^*)(x_1^{k-1} - x_1^*)$   
 $+ 0.75(x_2^{k-1} - x_2^*)$ , and  $G_2(x_1^k, x_2^{k-1}) - G_2(x_1^*, x_2^*) = 0.5x_2^{k-1}(x_1^k - x_1^*)$   
 $+ 0.5x_1^*(x_2^{k-1} - x_2^*)$ . Thus,  $G(x^k, x^{k-1}) - G(x^*, x^*) = A_k(x^k - x^*) + B_k(x^{k-1} - x^*)$

where  $A_k = \begin{bmatrix} 0 & 0 \\ 0.5x_2^{k-1} & 0 \end{bmatrix}, B_k = \begin{bmatrix} 0.25(x_1^{k-1} + x_1^*) & 0.75 \\ 0 & 0.5x_1^* \end{bmatrix}$

Then,  $E_k = |(I - A_k)^{-1}B_k| = \begin{bmatrix} 0.25|x_1^{k-1} + x_1^*| & 0.75 \\ 0.125|x_2^{k-1}(x_1^{k-1} + x_1^*)| & |0.375x_2^{k-1} + 0.5x_1^*| \end{bmatrix}$

Obviously, if  $\forall k (x_1^k, x_2^k) \in D \times D, D = [-0.5, 0.5], \|E_k\|_\infty \leq \alpha < 1$  (since  $x_1^* = 0$ ). Thus  $E_k$  is continuous and a D-matrix. Also, since  $C_k = |(I - A_k)^{-1}|$  is bounded, G is a D-mapping. Hence, iterations will converge iff (3.2) is true.

The algorithm, now, requires the following to be computed sequentially at any kth iteration: (i)  $G_1^k = 0.25(x_1^{k-1})^2 + 0.75x_2^{k-1}$ ; (ii)  $[\partial G_1 / \partial x_1]_{G_1^k, x_2^{k-1}} = 0.5G_1^k$ ; (iii)  $\bar{G}_1^k = 0.25(G_1^k)^2 + 0.75x_2^{k-1}$ ; (iv)  $w_1^k = (\bar{G}_1^k - G_1^k) / \{1 - [\partial G_1 / \partial x_1]_{G_1^k, x_2^{k-1}}\}$ ; (v)  $x_1^k = w_1^k + G_1^k$ ; (vi)  $G_2^k = 0.5x_1^k x_2^{k-1} + 0.005$ ; (vii)  $[\partial G_2 / \partial x_2]_{x_1^k, G_2^k} = 0.5x_1^k$ ; (viii)  $\bar{G}_2^k = 0.5x_1^k G_2^k + 0.005$ ; (ix)  $w_2^k = (\bar{G}_2^k - G_2^k) / \{1 - [\partial G_2 / \partial x_2]_{x_1^k, G_2^k}\}$ ; (x)  $x_2^k = w_2^k + G_2^k$ . Convergence follows iff  $\max_{i=1,2} |w_i^k| < \epsilon$ . If  $\epsilon = 10^{-5}$  and

$(x_1^0, x_2^0) \in D \times D$ , PIS converges in 5 iterations. This simple example explains the convergence principle as developed in theorem 2. It shows that in order to prove that  $E_k$  is a D-matrix, some prior knowledge of the roots is required.

5. DISCUSSIONS.

PIS has some limitations [1]. In order to prove that G is a D-mapping some apriori knowledge of  $x^*$  is required (as is evident in section 4). PIS is not quite effective to solve equations with multiple roots. For example the system:  $x = \sin(x) \cos(y) + \sin(z), y = \sin(y) \cos(z) + \sin(x). z = \sin(z) \cos(x) + \sin(y)$  has three roots: (0, 0, 0), (1.249, 1.249, 1.249) and (-1.249, -1.249, -1.249). With  $(x^0, y^0, z^0) = (1, 1, 1), (9999, 9999, 9999), (0.1, -0.1, 0.005), (-9.0, -9.0, 0.00003), (76, 900, 8000), (-3, -3, 3)$  it converged to the second root, with

$(x^0, y^0, z^0) = (5555, 3333, 1111), (56, 92, -9), (-1, -1, 13), (49, 78, 100)$  it converged to the third root but the first root was not found. The reason for this is still under investigation. If, in some cases,  $w_i^k$ 's are identically equal to zero, PIS becomes less effective [1]. PIS may be interpreted as a combination of Gauss-Seidel and Lieberstein's methods [1]. It was compared with other similar methods in [1, 3]. Although Newton's method has a quadratic rate of convergence it requires in general initial estimates to be close to the root and evaluations of Jacobians at each iteration. PIS requires none. Thus for large systems whereas Newton's method is not practical to use, PIS can be easily used [1, 3]. In [1] it has been proved that PIS has a quadratic rate of convergence. Also, since PIS-algorithm is relatively simple, computer programming is rather easy. Since no matrices are stored, requirement for computer memory storage is also small.

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#### REFERENCES

1. Dey, S. K. Perturbed Iterative Solution of Nonlinear Equations with Applications to Fluid Dynamics, Journal of Computational and Applied Mathematics, Vol. 3, No. 1, (1977) 15-30.
2. Dey, S. K. Numerical Solution of Nonlinear Implicit Finite Difference Equations of Flow Problems by Perturbed Functional Iterations. Lecture series 1978-9. Computational Fluid Dynamics, Vol. II, Von Karman Institute for Fluid Dynamics, Rhode-St-Genese, Belgium, (1978).
3. Dey, S. K. Perturbed Iterative Solution of Coupled Nonlinear Systems with Applications to Fluid Dynamics, Journal of Computational and Applied Mathematics, Vol. 4, No. 2, (1978) 117-128.
4. Ortega, J. M. and Rheinboldt, W. C. Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, (1970), New York.
5. Dey, S. K. Numerical Instabilities of Nonlinear Partial Differential Equations, CFT 7800/SKD/IE, Von Karman Institute for Fluid Dynamics, Rhode-St-Genese, Belgium, (1978).

6. Deist, F. and Sefor, L. Solution of Systems of Nonlinear Equations by Parameter Variation, Computer Journal, Vol. 10, (1976) 78-82.
7. Freudenstein, F. and Roth, B. Numerical Solution of Systems of Nonlinear Equations, Journal of ACM, Vol. 10, (1963) 550-556.
8. Ortega, J. M. and Rockoff, M. Nonlinear Difference Equations and Gauss-Seidel Type Iterative Methods, SIAM Journal of Numerical Analysis, Vol. 3, (1966) 497-513.
9. Dey, S. K. Applications of General D-mappings to Certain Flow Problems, Eastern Illinois University, (to be published).

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