Internat. J. Math. & Math. Sci.
Vol. 4 No. 1 (1981) 137-146

OSCILLATION IN SECOND ORDER FUNCTIONAL EQUATIONS WITH DEVIATING ARGUMENTS

BHAGAT SINGH

Department of Mathematics
University of Wisconsin Center
Manitowoc, Wisconsin 54220 U.S.A.

(Received October 16, 1979)

ABSTRACT. For the pair of functional equations

(A) $$(r(t)y'(t))' + p(t)h(h(g(t))) = f(t)$$

and

(B) $$(r(t)y'(t))' - p(t)h(y(g(t))) = 0$$

sufficient conditions have been found to cause all solutions of equation (A) to be oscillatory. These conditions depend upon a positive solution of equation (B).

KEY WORDS AND PHRASES: Oscillatory, Nonoscillatory, Sublinear, Superlinear

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES:

1. INTRODUCTION.

Our main goal, in this work, is to seek the oscillatory behavior of the equation

$$(r(t)y'(t))' + p(t)h(y(g(t))) = f(t) , \tag{1.1}$$

via the nonoscillation of the equation

$$(r(t)y'(t))' - p(t)h(y(g(t))) = 0. \tag{1.2}$$

Oscillation properties of equation (1.1) were studied by Kartsatos [3] and Kusano and Onose [4] by first "homogenizing" it and then using the techniques known for homogeneous equations. In fact a function $\lambda(t)$ was sought to satisfy

$$(r(t)(y(t) - \lambda(t))' = f(t). \tag{1.3}$$

A similar approach was later used by this author [9] in finding conditions for the oscillation of the equation

$$(r(t)y'(t))^{(n-1)} + (-1)^{n+1}p(t)y(g(t)) = f(t). \tag{1.4}$$

Recently Rankin [8] presented a new approach to study the oscillatory behavior of the ordinary differential equation

$$y''(t) + p(t)y(t) = f(t), \tag{1.5}$$

by using the transformation

$$y(t) = \phi(t)z(t) , \tag{1.6}$$

where $\phi(t)$ is a positive solution of the equation

$$y''(t) + p(t)y(t) = 0. \tag{1.7}$$

Transformations usually do not carry over to functional equations (1.1) and (1.2). The failure in study of equation (1.1) leads us to this work in which we present a different approach to study the oscillation of equation (1.1) which may be sublinear, superlinear, retarded or advanced.

Since our results do not depend on the integral size of $p(t)$, they are different from those of Kartsatos [3], Kusano and Onose [4] and this author [9]. Our results are also different than those of Rankin [8]. In fact the following example shows that Rankin's results are not true for the pair of retarded equations

$$y''(t) + \frac{3}{16t^2(t-\pi)^{3/4}} y(t-\pi) = -100t \sin 10t + 20 \cos 10t + \frac{3}{8}\frac{(t-\pi)^{1/4}}{t^2} + \frac{3(t-\pi)^{1/4}}{16t^2} \sin 10t, \quad (1.8)$$

and

$$y''(t) + \frac{3}{16t^2(t-\pi)^{3/4}} y(t-\pi) = 0. \quad (1.9)$$

Equation (1.9) has the nonoscillatory solution $\phi(t) = t^{3/4}$ which satisfies the conclusion of Rankin's main theorem ([8, Theorem 2]) namely

$$\liminf_{t\to\infty} \int_T^t \frac{1}{\phi^2(x)} \int_T^x \phi(s) f(s) ds dx = -\infty, \quad (1.10)$$

$$\limsup_{t\to\infty} \int_T^t \frac{1}{\phi^2(x)} \int_T^x \phi(s) f(s) ds dx = \infty, \quad (1.11)$$

and $\int_T^\infty \frac{1}{\phi^2(x)} dx < \infty,$

for any large $T > 0$; where

$$f(t) = -100t \sin (10t) + 20 \cos(10t) + \frac{3(t-\pi)^{1/4}}{8t^2} + \frac{3(t-\pi)^{1/4}}{16t^2} \sin(10t).$$

But equation (1.8) has the nonoscillatory solution

$$y(t) = 2t + t \sin(10t).$$

2. DEFINITIONS AND ASSUMPTIONS

Throughout this study we assume the following:

(i) $g(t)$, $r(t)$, $p(t)$, $h(t)$ and $f(t)$ are $C[R,R]$ where R denotes the real line;

(ii) $r(t)>0$, $r'(t)\leq 0$ and $p(t)>0$ for $t>t_0>0$ where we shall assume t_0 to be fixed arbitrarily. t_0 will be referred to in this study without any further mention;

(iii) $g(t)\to\infty$ as $t\to\infty$;

(iv) sign $h(t)$ = sign t.

The term "solution" refers to nontrivial continuously extendable solutions of equations under consideration over the interval $[t_0,\infty)$. We call a function $Q(t)\ \varepsilon\ C\,[[t_0,\infty),\,R]$ oscillatory if $Q(t)$ has arbitrarily large zeros on $[t_0,\infty)$; otherwise $Q(t)$ is called nonoscillatory. Equations (1.1) and (1.2) are called sublinear or superlinear

$$0 < \frac{h(t)}{t^\alpha} \le k$$

if $0 < \alpha \le 1$ or $\alpha < 1$ respectively where k is constant and α is the ratio of odd integers.

3. MAIN RESULTS

THEOREM 1: In addition to (i)-(iv) suppose there exists a function $\phi(t)$ which is continuous for $t\ge t_0$ and satisfies $(r(t)\phi'(t))' \ge 0$ ($\not\equiv 0$ in any interval),

$$\liminf_{t\to\infty} \int^t \frac{1}{\phi^2(s)} \int^s \phi(x)f(x)\,dx\,ds = -\infty, \tag{3.1}$$

$$\limsup_{t\to\infty} \int^t \frac{1}{\phi^2(s)} \int^s \phi(x)f(x)\,dx\,ds = \infty \tag{3.2}$$

and

$$\int^\infty \frac{1}{\phi^2(t)}\,dt < \infty . \tag{3.3}$$

Then all solutions of equation (1.1) are oscillatory.

PROOF: Suppose to the contrary that equation (1.1) has a nonoscillatory solution $y(t)$. Without any loss of generality suppose $T>t_0$ is large enough so that for $t>T$, $y(g(t))>0$ and $y(t)>0$. Rewriting equation (1.1) after multiplication with $\phi(t)$ we have

$$(r(t)\phi(t)y'(t))' - (r(t)\phi'(t))y'(t) + p(t)\phi(t)h(y(g(t))) = \phi(t)f(t). \tag{3.4}$$

Integrating (3.4) for $t \geq T$ we have

$$r(t)\phi(t)y'(t) - r(T)\phi(T)y'(T) - r(t)\phi'(t)y(t)$$

$$+ r(T)\phi'(T)y(T) + \int_T^t (r(s)\phi'(s))'y(s)ds$$

$$+ \int_T^t p(s)\phi(s)h(y(g(s)))ds = \int_T^t \phi(s)f(s)ds. \tag{3.5}$$

Set

$$K = r(T)\phi'(T)y(T) - r(T)\phi(T)y'(T). \tag{3.6}$$

Dividing (3.5) by $\phi^2(t)$ and rearranging terms we have

$$\frac{r(t)y'(t)}{\phi(t)} + \frac{K}{\phi^2(t)} - \frac{r(t)\phi'(t)y(t)}{\phi^2(t)} + \frac{1}{\phi^2(t)} \int_T^t (r(s)\phi'(s))'yds$$

$$+ \frac{1}{\phi^2(t)} \int_T^t p(s)\phi(s)h(y(g(s)))ds = \frac{1}{\phi^2(t)} \int_T^t \phi(s)f(s)ds. \tag{3.7}$$

Integrating (3.7) between T and t we have

$$\frac{r(t)y(t)}{\phi(t)} - \frac{r(T)y(T)}{\phi(T)} + \int_T^t \frac{r(s)\phi'(s)y(s)}{\phi^2(s)} ds - \int_T^t \frac{r'(s)y(s)}{\phi(s)} ds$$

$$+ \int_T^t K/\phi^2(s)ds - \int_T^t \frac{r(s)\phi'(s)y(s)}{\phi^2(s)} ds$$

$$+ \int_T^t \frac{1}{\phi^2(x)} \int_T^x [(r(s)\phi'(s))'y(s) + p(s)\phi(s)h(y(g(s)))]dsdx$$

$$= \int_T^t \frac{1}{\phi^2(x)} \int_T^x \phi(s)f(s)dsdx \tag{3.8}$$

which leads to

$$\frac{r(t)y(t)}{\phi(t)} - \frac{r(T)y(T)}{\phi(T)} - \int_T^t \frac{r'(s)y(s)}{\phi(s)} ds$$

$$+ \quad K \int_T^t 1/\phi^2(s)\,ds$$

$$+ \quad \int_T^t \frac{1}{\phi^2(x)} \int_T^x [(r(s)\phi'(s))'y(s) + p(s)\phi(s)h(y(g(s)))]\,ds\,dx$$

$$= \quad \int_T^t \frac{1}{\phi^2(x)} \int_T^x \phi(s)f(s)\,ds\,dx. \tag{3.9}$$

Since third, fourth and fifth terms on the left hand side of (3.9) are either nonnegative or finite, we immediately reach a contradiction in view of (3.1) and (3.2). The proof is complete.

COROLLARY 1. Suppose (i)-(iv) hold. Further suppose that equation a positive solution $\phi(t)$ satisfying (3.1), (3.2) and (3.3). Then all solutions of equation (1.1) are oscillatory.

PROOF. Since $(r(t)\phi'(t))' \geq 0$, conclusion follows from Theorem 1.

EXAMPLE 1. Consider the equations

$$y''(t) + e^{\pi}y(t-\pi) = 4e^{2t}\cos t + 3e^{2t}\sin t - e^{2t-\pi}\sin t\,, \tag{3.10}$$

and

$$y''(t) - e^{\pi}y(t-\pi) = 0\,, \tag{3.11}$$

for $t > \pi$. Equation (3.11) has $y(t) = e^t$ as a solution which satisfies (3.1), (3.2) and (3.3). Thus all solutions of equation (3.10) are oscillatory. In fact $y(t) = e^{2t}\sin t$ is one such solution.

REMARK. In Rankin's work $\phi''(t) < 0$ where as here $\phi''(t) > 0$ when $r(t) \equiv 1$.

THEOREM 2. Suppose $r(t) \equiv 1$ and (i)-(iv) hold. Further suppose that equation (1.2) has a positive solution $\phi(t)$ such that $\phi'(t) \geq 0$ ($\not\equiv 0$ in any subinterval) for $t > t_0$. Let (3.1) and (3.2) of Theorem (1) hold. Then all solutions of equation (1.1) are oscillatory.

PROOF: Since

$$\phi''(t) \geq 0, \quad \phi'(t) \geq 0 \quad \text{and} \quad \phi(t) \geq 0 , \tag{3.12}$$

for $t > t_0$, there exist positive numbers c_1 and c_2 such that $\phi(t) \geq c_1 t + c_2$, and consequently $\phi(t)$ satisfies (3.3). The proof is complete. We now have the following corollary.

COROLLARY 2: Suppose equation (1.2) has a positive nonoscillatory solution $z(t)$ such that $z'(t) > 0$. Further suppose that equation (1.1) has a nonoscillatory solution. Then either

$$\liminf_{t\to\infty} \int^t \frac{1}{z^2(s)} \int^s z(x) f(x)\,dx\,ds > -\infty \tag{3.13}$$

or

$$\limsup_{t\to\infty} \int^\infty \frac{1}{z^2(s)} \int^s z(x) f(x)\,dx\,ds < \infty . \tag{3.14}$$

EXAMPLE 2. The equation

$$y''(t) + \frac{2}{t^2} y(t) = -\sin t + \frac{4}{t^2} + \frac{2\sin t.}{t^2} \tag{3.15}$$

has the nonoscillatory solution $y(t) = 2 + \sin t$. Now consider

$$y''(t) - \frac{2}{t^2}\, y(t) = 0 , \tag{3.16}$$

which has $z(t) = t^2$ as a nonoscillatory solutions satisfying the conditions and conclusion of Corollary 2.

4. ASYMPTOTIC NONOSCILLATION

Example 2 shows that when (3.1) and (3.2) are relaxed then equation (1.1) may have nonoscillatory solutions. In this section we give conditions when nonoscillatory solutions of (1.1) approach limits.

THEOREM 3: Suppose (i)-(iv) hold. Let $\phi(t)$ be a positive solution of equation (1.2) such that $\phi'(t) \geq 0$ ($\not\equiv 0$ in any subinterval of t for $t > t_0$),

$$\liminf_{t\to\infty} \int^{t} \frac{1}{\phi^2(x)} \int^{x} \phi(s)f(s)\,ds\,dx < 0 , \tag{4.1}$$

and

$$\limsup_{t\to\infty} \int^{t} \frac{1}{\phi^2(x)} \int^{x} \phi(s)f(s)\,ds\,dx > 0. \tag{4.2}$$

Let $y(t)$ be a bounded solution of equation (1.1). If $y(t)$ is nonoscillatory then $y(t)$ tends to a finite limit.

PROOF. Without any loss of generality, let $T \geq t_0$ be large enough so that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq T$. Suppose to the contrary that

$$\liminf_{t\to\infty} y(t) < \limsup_{t\to\infty} y(t). \tag{4.3}$$

Then there exists a sequence $\{T_n\}_{n=1}^{\infty}$ such that $T_n \to \infty$ as $n \to \infty$ and $y'(T_n) = o$. Let k be a large positive integer such that

$$\frac{y(T_k)r(T_k)}{\phi(T_k)} < \text{Min}\left[-\liminf_{t\to\infty} \int_{T_k}^{t} \frac{1}{\phi^2(x)} \int_{T_k}^{x} \phi(s)f(s)\,ds\,dx , \quad \limsup_{t\to\infty} \int_{T_k}^{t} \frac{1}{\phi^2(x)} \int_{T_k}^{x} \phi(s)f(s)\,ds\,dx \right]. \tag{4.4}$$

Following the proof of Theorem 1, we obtain from (3.9)

$$\frac{r(t)y(t)}{\phi(t)} - \int_{T_k}^{t} \frac{r'(s)y(s)\,ds}{\phi(s)} + r(T_k)\phi'(T_k)y(T_k) \int_{T_k}^{t} \frac{1}{\phi^2(x)}\,dx$$

$$+ \int_{T_k}^{t} \frac{1}{\phi^2(x)} \int_{T_k}^{x} p(s)\,(y(s)h(\phi(g(s))) + \phi(s)h(y(g(s))))\,ds\,dx$$

$$= \int_{T_k}^{t} \frac{1}{\phi^2(x)} \int_{T_k}^{x} \phi(s)f(s)\,ds\,dx + \frac{y(T_k)r(T_k)}{\phi(T_k)} . \tag{4.5}$$

In view of (4.1), (4.2) and (4.4), we reach a contradiction in (4.5). The proof is complete.

REMARK. Example 2 shows that conditions (4.1) and (4.2) cannot be weakened.

COROLLARY 3. Suppose conditions of Theorem 3 hold. Let $y(t)$ be any solution of equation (1.1) such that $\frac{y(t)}{\phi(t)} \to 0$ as $t \to \infty$. If $y(t)$ is nonoscillatory then $y(t)$ tends to a finite or infinite limit as $t \to \infty$.

REMARK. Recently Graef and Spikes [1], Hammett [2], Kusano and Onose [5,6], Philos and Starkos [7], this author [10,11] have studied asymptotic nonoscillation with regard to equation (1.1). However all these results make use of an integral condition on $p(t)$. Theorem 3 and Corollary 3 present a different approach.

REFERENCES

1. J. Graef and P. Spikes, Asymptotic behavior of solutions of a second order nonlinear differential equation, J. Differential Equations, 17 (1975), 461-476.

2. M. Hammett, Nonoscillation properties of a nonlinear differential equation, Proc. Amer. Math. Soc., 30 (1971), 92-96.

3. A.G. Kartsatos, On the maintenance of oscillations of nth order equations under the effect of a small forcing term, J. Differential Equations, 10 (1971), 355-563.

4. T. Kusano and H. Onose, Oscillations of functional differential equations with retarded arguments, 15 (1974), 269-277.

5. T. Kusano and H. Onose, Asymptotic behavior of nonoscillatory solutions of functional differential equations of arbitrary order, J. London Math. Soc., 14 (1976), 106-112.

6. T. Kusano and H. Onose, Nonoscillation theorems for differential equations with deviating argument, Pacific J. Math., 63 (1976), 185-192.

7. Ch. G. Philos and V.A. Staikos, Asymptotic properties of nonoscillatory solutions of differential equations with deviating argument, Pacific J. Math., 70 (1977), 221-242.

8. S. Rankin, Oscillation results for a nonhomogeneous equations, Pacific J. Math., 80 (1979), 237-244.

9. B. Singh, Impact of delays on oscillation in general functional equations, _Hiroshima Math. J._, 5 (1975), 351-361.

10. B. Singh, Forced nonoscillations in fourth order functional equations, _Funk. Ekva._, 19 (1976), 227-237.

11. B. Singh, Nonoscillations of forced fourth order retarded equations, _SIAM J. Appl. Math._, 28 (1975), 265-269.

12. B. Singh and T. Kusano, On asymptotic limits of nonoscillations in functional equations with retarded arguments, Hiroshima Math. J., (to appear).

Mathematical Problems in Engineering

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	March 1, 2009
First Round of Reviews	June 1, 2009
Publication Date	September 1, 2009

Guest Editors

Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru

Hindawi Publishing Corporation
http://www.hindawi.com