BILATERAL GENERATING FUNCTIONS FOR A NEW CLASS OF GENERALIZED LEGENDRE POLYNOMIALS

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ABSTRACT.   Recently Chatterjea (1) has proved a theorem to deduce a bilateral

generating function for the Ultraspherical polynomials.   In the present paper an

attempt has been made to give a general version of Chatterjea's theorem.   Finally,

the theorem has been specialized to obtain a bilateral generating function for a

class of polynomials $\{P_n (x; \alpha,\beta)\}$ introduced by Bhattacharjya (2).

1.   INTRODUCTION.

Using the following differential formula for the Ultraspherical polynomials

P_n^λ (x) due to Tricomi,

$$P_n^\lambda \left[x(x^2-1)^{-1/2} \right] = \frac{(-1)^n}{n!} (x^2-1)^{\lambda+\frac{n}{2}} D^n (x^2-1)^{-\lambda}, \qquad (1.1)$$

Chatterjea (1) has recently obtained a bilateral generating function for the

Ultraspherical polynomials in the form of following theorem.

THEOREM   1.   If

$$F(x,t) = \sum_{m=o}^{\infty} a^m t^m P_m^\lambda (x),$$

then

$$\rho^{-2\lambda} F\left(\frac{x-t}{\rho}, \frac{ty}{\rho} \right) = \sum_{r=o}^{\infty} t^r b_r(y) P_r^\lambda (x), \qquad (1.2)$$

where

$$b_r(y) = \sum_{m=o}^{\infty} \binom{r}{m} a_m y^m, \text{ and } \rho = (1-2xt+t^2)^{1/2}.$$

A closer look at the above relation (1.2) suggests the following interesting

general version of Chatterjea's theorem:

2.   Let F o G be used to denote the composition F o G(x) = F(G(x)). In

terms of this notation, we state

    THEOREM 2.   Suppose that there exist functions f, g, h and X and a

sequence of constants $\{c_n\}$ such that the sequence of functions $\{Q_n\}$ is generated

by the formula

$$c_n \ f \ g^n \ Q_n \circ X = D^n \ h, \qquad n = 0, 1, 2, \ldots, \qquad (2.1)$$

where $D \equiv d/dx$. Define the generating function

$$F(x, t) = \sum_{n=o}^{\infty} a_n \ t^n \ Q_n(x). \qquad (2.2)$$

Then

$$fF(X, gtz) \big|_{x+t} = f \sum_{n=o}^{\infty} c_n \ (gt)^n \ Q_n \circ X \ b_n(z),$$

where

$$b_n(z) = \sum_{k=o}^{\infty} \frac{a_k}{c_k \ (n-k)!} \ z^k.$$

PROOF.   By Taylor's theorem,

$$fF(X, gtz) \big|_{x+t} = e^{tD} \ fF(X, gtz). \qquad (2.3)$$

To evaluate the right hand side of (2.3), we shall use as our starting point the

relations (2.1) and (2.2), and the series expansion for e^{tD}. Thus

$$e^{tD} \ fF(X, gtz) = e^{tD} \ f \sum_{n=o}^{\infty} a_n \ (gtz)^n \ Q_n \circ X$$

$$= e^{tD} \sum_{n=o}^{\infty} \frac{a_n}{c_n} \ (tz)^n \ D^n \ h$$

$$= \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} (a_n/c_n) \ t^{n+m} \ z^n \ D^{n+m} \ h/m!$$

$$= \sum_{n=o}^{\infty} \sum_{m=o}^{\infty} (a_n/c_n) \ (gt)^{n+m} \ c_{n+m} \ f \ Q_{n+m} \circ X/m!$$

$$= f \sum_{n=o}^{\infty} c_n \ (gt)^n \ Q_n \circ X \ b_n(z),$$

where

$$b_n(z) = \sum_{k=o}^{\infty} \frac{a_k}{c_k(n-k)!} \ z^k.$$

It is worthwhile to remark here that if we choose $Q_n(x) = P_n^{\lambda}(x)$,

$f(x) = (x^2-1)^{-\lambda}$, $g(x) = (x^2-1)^{-1/2}$, $X(x) = x(x^2-1)^{-1/2}$, $h(x) = (x^2-1)^{-\lambda}$ and

$c_n = n! \ /(-1)^n$ then Theorem 2 would correspond to Chatterjea's theorem.

APPLICATIONS: Earlier, Bhattacharjya (2) introduced a new class of

generalized Legendre polynomials $\{P_n(x; \ \alpha, \ \beta)\}$ which are orthogonal with the

weight function $\dfrac{|x|^{\beta}}{\left(1-x^2\right)^{(\beta-\alpha)/2}}$. The Rodrigue's formulae for these polynomials are (2 , (6.6) and (6.8)):

$$P_{2m}(x^{-1/2}; \alpha, \beta) = \frac{x^{m+(\alpha+1)/2}(1-x)^{(\beta-\alpha)/2}}{(-2m-(\alpha-1)/2)_m},$$

$$\cdot D^m[(1-x)^{m-(\beta-\alpha)/2} x^{-m-(\alpha+1)/2}], \quad (2.4)$$

and

$$P_{2m+1}(x^{-1/2}; \alpha, \beta) = \frac{x^{m+1+\alpha/2}(1-x)^{(\beta-\alpha)/2}}{(-2m-(\alpha+1)/2)_m}$$

$$\cdot D^m[(1-x)^{m-(\beta-\alpha)/2} x^{-m-(\alpha+3)/2}] \quad (2.5)$$

Here we note that the sequences $\{P_{2n}(x^{-1/2}; \alpha-2n, \beta)\}$ and $\{P_{2n+1}(x^{-1/2}; \alpha-2n, \beta)\}$ are amenable to a method of Theorem 2 for finding bilateral generating functions.

Let $Q_n(x) = P_{2n}(x; \alpha-2n, \beta) \equiv P_{2n}(x)$. For simplicity of notation, set $y = -(\alpha+1)/2$ and $\delta = (\alpha-\beta)/2$. Then (2.1) holds with $f(x) = x^y(1-x)^{\delta}$, $g(x) = (1-x)^{-1}$, $X(x) = x^{-1/2}$ and $c_n = \phi(n) = (-n-(\alpha-1)/2)_n$. Upon replacing t by $-t$ and z by $-y$, we get

$$\left(\frac{x-t}{x}\right)^y \left(\frac{1-(x-t)^{\delta}}{1-x}\right) F\left(\frac{1}{(x-t)^{1/2}}, -\frac{yt}{(1-(x-t))}\right) = \sum_{r=0}^{\infty} \left(\frac{-t}{1-x}\right)^r \phi(r) \cdot$$

$$\cdot P_{2r}(x^{-1/2}) b_r(-y), (2.6)$$

where

$$F\left(\frac{1}{x^{1/2}}, \frac{t}{1-x}\right) = \sum_{m=0}^{\infty} a_m\left(\frac{t}{1-x}\right)^m P_{2m}(x^{-1/2})$$

and

$$b_r(-y) = \sum_{m=0}^{\infty} \frac{a_m(-y)^m}{\phi(m)(r-m)!}. \quad (2.7)$$

Now replacing $x^{-1/2}$ by s and $t/(1-x)$ by t in (2.6), we are led to the following bilateral generating function for gerneralized even Legendre polynomials:

COROLLARY.   1:   If

$$F(x,t) = \sum_{m=0}^{\infty} a_m t^m P_{2m}(x),$$

then

$$[1-(x^2-1)t]^y \ (1+t)^\delta \ F\left(\frac{x}{(1-t(x^2-1))^{1/2}} \ , \ \frac{yt}{1+t}\right) = \sum_{r=o}^{\infty} (-t)^r \phi\ (r) \cdot$$

$$\cdot P_{2r}(x)\ b_r\ (-y),$$

where $b_r\ (-y)$ is given by (2.7).

In the same way, let $Q_n\ (x) = P_{2n+1}\ (x;\ \alpha-2n,\ \beta) \equiv P_{2n+1}\ (x)$, and set

$y = -\ (\alpha+2)/2,\ \ \delta = (\alpha-\beta\ )/2$. Then (2.1) holds with $f(x) = x^y\ (1-x)^\delta$,

$g(x) = (1-x)^{-1}\ ,\ \chi\ (x) = x^{-1/2}$ and $c_n = \psi(n) = (-n-(\alpha+1)/2)_n$. Replacing t by $-t$ and

z by $-y$ and making the same substitution as before in (2.7), we are led to the

following bilateral generating function for generalized odd Legendre polynomials.

CGROLLARY 2: If

$$F\ (x,t) = \sum_{m=o}^{\infty} a_m t^m P_{2m+1}\ (x),$$

then

$$[\ 1-(x^2-1)t]^y\ (1-t)^\delta\ F\left(\frac{x}{(1-t(x^2-1))^{1/2}},\ \frac{ty}{1+t}\right) = \sum_{r=o}^{\infty} (-t)^r\ \psi\ (r) \cdot$$

$$\cdot P_{2r+1}\ (x)\ c_r\ (-y),$$

where

$$c_r\ (-y) = \sum_{m=o}^{r} \frac{a_m\ (-y)^m}{\Psi(m)\ (r-m)!}\ .$$

Taking $\alpha = \beta$ in Corollary 1 and 2, we can obtain bilateral generating functions

for generalized Legendre polynomials due to Dutta and More (3).

Next, we note that (2),

$$P_{2m}\ (x;o,o) = \frac{(-1)^m\ m!\ P_{2m}\ (x)}{(-2m + \frac{1}{2})_m},\tag{2.8}$$

and

$$P_{2m}\ (x;o,o) = \frac{(-1)^m\ m!\ P_{2m+1}\ (x)}{(-2m- \frac{1}{2})_m},\tag{2.9}$$

where $P_{2m}\ (x)$ and $P_{2m+1}\ (x)$ are even and odd Legendre polynomials. Therefore, by

(2.8), (2.9) and the above two corollories we can obtain bilateral generating

functions for Legendre polynomials.

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