

ON SOME NEW GENERALIZATIONS OF OSTROWSKI INTEGRAL INEQUALITY

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In this paper, we give some new generalizations of Ostrowski integral inequality on multivariate.

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1. Introduction. In 1938, Ostrowski proved the following integral inequality [2, page 468].

THEOREM 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , that is, $\|f'\| := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{(x - (a+b)/2)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty \quad (1.1)$$

for all $x \in [a, b]$. Here the constant $1/4$ is the best possible.

Recently, Dragomir [1] gave a generalization of Ostrowski integral inequality for mappings whose derivatives belong to $L_p[a, b]$.

THEOREM 1.2. *Let $I_k : a = x_0 < x_1 < \dots < x_{k-1} < x_k$ be a division of the interval $[a, b]$ and α_i (where $i = 0, \dots, k+1$) be $(k+2)$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ (where $i = 1, \dots, k$), and $\alpha_{k+1} = b$. If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$. Then*

$$\begin{aligned} & \left| \int_a^b f(x) dx - \sum_{i=0}^k (\alpha_{i+1} - \alpha_i) f(\alpha_i) \right| \\ & \leq \frac{1}{(q+1)^{1/q}} \left[\sum_{i=0}^{k-1} [(\alpha_{i+1} - x_i)^{q+1} + (x_{i+1} - \alpha_{i+1})^{q+1}] \right]^{1/q} \|f'\|_p \quad (1.2) \\ & \leq \frac{1}{(q+1)^{1/q}} \left[\sum_{i=0}^{k-1} h_i^{q+1} \right]^{1/q} \|f'\|_p \leq \frac{v(h)(b-a)^{1/q}}{(q+1)^{1/q}} \|f'\|_p, \end{aligned}$$

where $h_i := x_{i+1} - x_i$ (where $i = 0, 1, \dots, k-1$), $v(h) := \max\{h_i \mid i = 0, \dots, k\}$, $p > 1$, $1/p + 1/q = 1$, and $\|\cdot\|_p$ is the usual $L_p[a, b]$ -norm.

In this paper, we show other generalizations of the Ostrowski inequality on multivariate.

2. Main results. First we introduce the lemma in [1, page 609].

LEMMA 2.1. *Let the function $g(t) = (t - a)^{p+1} + (b - t)^{p+1}$, $t \in [a, b]$, $p > 1$. Then*

$$\frac{(b-a)^{p+1}}{2^p} \leq g(t) \leq (b-a)^{p+1}. \quad (2.1)$$

Now we show the main results.

THEOREM 2.2. *Let $I_k : a = x_0 < x_1 < \dots < x_{k-1} < x_k = b$, $c = y_0 < y_1 < \dots < y_{k-1} < y_k = d$ be a division of $[a, b] \times [c, d]$ and α_i, β_j (where $i, j = 0, \dots, k+1$) be $(2k+2)$ points, so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ (where $i = 1, \dots, k$), $\alpha_{k+1} = d$. If $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is continuous on $[a, b] \times [c, d]$, $f''_{x,y} = \partial^2 f / \partial x \partial y$ exist on $(a, b) \times (c, d)$ and $\|f''_{x,y}\|_\infty := \sup_{(x,y) \in (a,b) \times (c,d)} |\partial^2 f(x,y) / \partial x \partial y| < \infty$. Then*

$$\begin{aligned} & \left| \int_a^b \int_c^d f(s,t) ds dt + \sum_{i=0}^k \sum_{j=0}^k (\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j) f(x_i, y_j) \right. \\ & \left. - \sum_{i=0}^k \int_c^d (\alpha_{i+1} - \alpha_i) f(x_i, t) dt - \sum_{j=0}^k \int_a^b (\beta_{j+1} - \beta_j) f(s, y_j) ds \right| \\ & \leq \frac{((b-a)(d-c))^{1/q}}{(p+1)^{2/p}} \|f''_{s,t}\|_\infty \left[\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (x_{i+1} - x_i)^{p+1} (y_{j+1} - y_j)^{p+1} \right]^{1/p} \\ & \leq \frac{u(x)v(y)(b-a)(d-c)}{(p+1)^{2/p}} \|f''_{s,t}\|_\infty, \end{aligned} \quad (2.2)$$

where $u(x) := \max\{x_{i+1} - x_i \mid i = 0, \dots, k-1\}$, $v(y) = \max\{y_{j+1} - y_j \mid j = 0, \dots, k-1\}$, $p > 1$, $1/p + 1/q = 1$.

PROOF. Define the mapping $K : [a, b] \times [c, d] \rightarrow \mathbb{R}$ given by

$$K(s,t) = \begin{cases} (s - \alpha_i)(t - \beta_j), & (s,t) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \ (i, j = 1, \dots, k-1), \\ (s - \alpha_i)(t - \beta_k), & (s,t) \in [x_{i-1}, x_i] \times [y_{k-1}, y_k] \ (i = 1, \dots, k-1, j = k), \\ (s - \alpha_k)(t - \beta_j), & (s,t) \in [x_{k-1}, x_k] \times [y_{j-1}, y_j] \ (i = k, j = 1, \dots, k-1), \\ (s - \alpha_k)(t - \beta_k), & (s,t) \in [x_{k-1}, x_k] \times [y_{k-1}, y_k] \ (i = k, j = k). \end{cases}$$

(2.3)

Integrating by parts, we get

$$\begin{aligned}
& \int_a^b \int_c^d K(s,t) f''_{s,t}(s,t) ds dt \\
&= \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} (s - \alpha_{i+1})(t - \beta_{j+1}) f''_{s,t}(s,t) ds dt = \int_a^b \int_c^d f(s,t) ds dt \\
&+ \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \left\{ (\gamma_{j+1} - \beta_{j+1}) \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, \gamma_{j+1}) + (\alpha_{i+1} - x_i) f(x_i, \gamma_{j+1}) \right. \right. \\
&\quad \left. \left. - \int_{x_i}^{x_{i+1}} f(s, \gamma_{j+1}) ds \right] \right. \\
&\quad + (\beta_{j+1} - \gamma_j) \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, \gamma_j) + (\alpha_{i+1} - x_i) f(x_i, \gamma_j) \right. \\
&\quad \left. - \int_{x_i}^{x_{i+1}} f(s, \gamma_j) ds \right] \\
&\quad \left. - \int_{y_j}^{\gamma_{j+1}} [(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, t) + (\alpha_{i+1} - x_i) f(x_i, t)] dt \right\} \\
&= \int_a^b \int_c^d f(s,t) ds dt \\
&+ \sum_{i=0}^{k-1} (\gamma_k - \beta_k) \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, \gamma_k) + (\alpha_{i+1} - x_i) f(x_i, \gamma_k) \right. \\
&\quad \left. - \int_{x_i}^{x_{i+1}} f(s, \gamma_k) ds \right] \\
&+ \sum_{i=0}^{k-1} \sum_{j=1}^{k-1} (\beta_{j+1} - \beta_j) \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, \gamma_j) + (\alpha_{i+1} - x_i) f(x_i, \gamma_j) \right. \\
&\quad \left. - \int_{x_i}^{x_{i+1}} f(s, \gamma_j) ds \right] \\
&+ \sum_{i=0}^{k-1} (\beta_1 - \gamma_0) \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, \gamma_0) + (\alpha_{i+1} - x_i) f(x_i, \gamma_0) - \int_{x_i}^{x_{i+1}} f(s, \gamma_0) ds \right] \\
&- \sum_{j=0}^{k-1} \int_{y_j}^{\gamma_{j+1}} [(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, t) + (\alpha_{i+1} - x_i) f(x_i, t)] dt \\
&= \int_a^b \int_c^d f(s,t) ds dt \\
&+ \sum_{i=0}^{k-1} \sum_{j=0}^k (\beta_{j+1} - \beta_j) \left[(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, \gamma_j) + (\alpha_{i+1} - x_i) f(x_i, \gamma_j) \right. \\
&\quad \left. - \int_{x_i}^{x_{i+1}} f(s, \gamma_j) ds \right] \\
&+ \sum_{j=0}^{k-1} [(x_{i+1} - \alpha_{i+1}) f(x_{i+1}, t) + (\alpha_{i+1} - x_i) f(x_i, t)] dt
\end{aligned}$$

$$\begin{aligned}
&= \int_a^b \int_c^d f(s, t) ds dt + \sum_{i=0}^k \sum_{j=0}^k (\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j) f(x_i, y_j) \\
&\quad - \sum_{i=0}^k \int_c^d (\alpha_{i+1} - \alpha_i) f(x_i, t) dt - \sum_{j=0}^k \int_a^b (\beta_{j+1} - \beta_j) f(s, y_j) ds,
\end{aligned} \tag{2.4}$$

and then we get the integral equality

$$\begin{aligned}
\int_a^b \int_c^d f(s, t) ds dt &= \int_a^b \int_c^d K(s, t) f''_{s,t}(s, t) ds dt - \sum_{i=0}^k \sum_{j=0}^k (\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j) f(x_i, y_j) \\
&\quad + \sum_{i=0}^k \int_c^d (\alpha_{i+1} - \alpha_i) f(x_i, t) dt + \sum_{j=0}^k \int_a^b (\beta_{j+1} - \beta_j) f(s, y_j) ds.
\end{aligned} \tag{2.5}$$

On the other hand, we have

$$\begin{aligned}
\left| \int_a^b \int_c^d K(s, t) f''_{s,t}(s, t) ds dt \right| &= \left| \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} (s - \alpha_{i+1})(t - \beta_{j+1}) f''_{s,t}(s, t) ds dt \right| \\
&\leq \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |(s - \alpha_{i+1})(t - \beta_{j+1})| |f''_{s,t}(s, t)| ds dt.
\end{aligned} \tag{2.6}$$

Using Hölder's integral inequality, we get

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |(s - \alpha_{i+1})(t - \beta_{j+1})| |f''_{s,t}(s, t)| ds dt \\
&\leq \left(\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |(s - \alpha_{i+1})(t - \beta_{j+1})|^p ds dt \right)^{1/p} \left(\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |f''_{s,t}(s, t)|^q ds dt \right)^{1/q}.
\end{aligned} \tag{2.7}$$

But

$$\begin{aligned}
&\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} |(s - \alpha_{i+1})(t - \beta_{j+1})|^p ds dt \\
&= \left[\int_{x_i}^{\alpha_{i+1}} (\alpha_{i+1} - s)^p ds + \int_{\alpha_{i+1}}^{x_{i+1}} (s - \alpha_{i+1})^p ds \right] \\
&\quad \times \left[\int_{y_j}^{\beta_{j+1}} (\beta_{j+1} - t)^p dt + \int_{\beta_{j+1}}^{y_{j+1}} (t - \beta_{j+1})^p dt \right] \\
&= \frac{1}{(p+1)^2} \left[(\alpha_{i+1} - x_i)^{p+1} + (x_{i+1} - \alpha_{i+1})^{p+1} \right] \\
&\quad \times \left[(\beta_{j+1} - y_j)^{p+1} + (y_{j+1} - \beta_{j+1})^{p+1} \right]
\end{aligned} \tag{2.8}$$

and using relations (2.6), (2.7), (2.8), and Hölder's discrete inequality, we get

$$\begin{aligned}
 & \left| \int_a^b \int_c^d K(s,t) f''_{s,t}(s,t) ds dt \right| \\
 & \leq \frac{1}{(p+1)^{2/p}} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \left\{ [(\alpha_{i+1} - x_i)^{p+1} + (x_{i+1} - \alpha_{i+1})^{p+1}] \right. \\
 & \quad \times [(\beta_{j+1} - \gamma_j)^{p+1} + (\gamma_{j+1} - \beta_{j+1})^{p+1}] \left. \right\}^{1/p} \\
 & \quad \times \left(\int_{x_i}^{x_{i+1}} \int_{\gamma_j}^{\gamma_{j+1}} |f''_{s,t}(s,t)|^q ds dt \right)^{1/q} \\
 & \leq \frac{1}{(p+1)^{2/p}} \left[\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \left(\left\{ [(\alpha_{i+1} - x_i)^{p+1} + (x_{i+1} - \alpha_{i+1})^{p+1}] \right. \right. \right. \\
 & \quad \left. \left. \left. \times [(\beta_{j+1} - \gamma_j)^{p+1} + (\gamma_{j+1} - \beta_{j+1})^{p+1}] \right\}^{1/p} \right)^p \right]^{1/p} \quad (2.9) \\
 & \quad \times \left[\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \left(\left(\int_{x_i}^{x_{i+1}} \int_{\gamma_j}^{\gamma_{j+1}} |f''_{s,t}(s,t)|^q ds dt \right)^{1/q} \right)^q \right]^{1/q} \\
 & \leq \frac{1}{(p+1)^{2/p}} \left(\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} [(\alpha_{i+1} - x_i)^{p+1} + (x_{i+1} - \alpha_{i+1})^{p+1}] \right. \\
 & \quad \left. \times [(\beta_{j+1} - \gamma_j)^{p+1} + (\gamma_{j+1} - \beta_{j+1})^{p+1}] \right)^{1/p} \\
 & \quad \times [(b-a)(d-c)]^{1/q} \|f''_{s,t}\|_{\infty}.
 \end{aligned}$$

By Lemma 2.1, the first inequality in (2.2) is obtained.

For the second inequality in (2.2) we only remark that

$$\begin{aligned}
 & \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (x_{i+1} - x_i)^{p+1} (\gamma_{j+1} - \gamma_j)^{p+1} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (x_{i+1} - x_i) (\gamma_{j+1} - \gamma_j) \\
 & = (b-a)(d-c)(u(x)v(\gamma))^p.
 \end{aligned} \tag{2.10}$$

The theorem is completely proved. □

Under the assumption that the points of the division I_k are fixed, the best inequality which is obtained from Theorem 2.2 is embodied in the following corollaries.

COROLLARY 2.3. *Let $I_k : a = x_0 < x_1 < \dots < x_{k-1} < x_k = b$, $c = \gamma_0 < \gamma_1 < \dots < \gamma_{k-1} < \gamma_k = d$ be a division of $[a, b] \times [c, d]$. If f is as above, then*

$$\begin{aligned}
& \left| \int_a^b \int_c^d f(s,t) ds dt + \frac{1}{4} \left\{ (x_1 - a) \sum_{j=1}^{k-1} (\mathcal{Y}_{j+1} - \mathcal{Y}_{j-1}) f(a, \mathcal{Y}_j) + (x_1 - a)(\mathcal{Y}_1 - c) f(a, c) \right. \right. \\
& \quad + (\mathcal{Y}_1 - c) \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i, c) + (b - x_{k-1})(\mathcal{Y}_1 - c) f(b, c) \\
& \quad + \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (x_{i+1} - x_{i-1})(\mathcal{Y}_{j+1} - \mathcal{Y}_{j-1}) f(x_i, \mathcal{Y}_j) \\
& \quad \quad + (b - x_{k-1})(d - \mathcal{Y}_{k-1}) f(b, d) \\
& \quad + (d - \mathcal{Y}_{k-1}) \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i, d) \\
& \quad + (b - x_{k-1}) \sum_{j=1}^{k-1} (\mathcal{Y}_{j+1} - \mathcal{Y}_{j-1}) f(b, \mathcal{Y}_j) \\
& \quad - 2(x_1 - a) \int_c^d f(a, t) dt - 2 \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) \int_c^d f(x_i, t) dt \\
& \quad - 2(b - x_{k-1}) \int_c^d f(b, t) dt - 2(\mathcal{Y}_1 - c) \int_a^b f(s, c) ds \\
& \quad \left. - 2 \sum_{j=1}^{k-1} (\mathcal{Y}_{j+1} - \mathcal{Y}_{j-1}) \int_a^b f(s, \mathcal{Y}_j) ds - 2(d - \mathcal{Y}_{k-1}) \int_a^b f(s, d) ds \right\} \Big| \\
& \leq \frac{(b-a)(d-c)^{1/q}}{4(p+1)^{2/p}} \|f''_{s,t}\|_{\infty} \left[\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} ((x_{i+1} - x_i)(\mathcal{Y}_{j+1} - \mathcal{Y}_j))^{p+1} \right]^{1/p} \\
& \leq \frac{u(x)v(y)(b-a)(d-c)}{4(p+1)^{2/p}} \|f''_{s,t}\|_{\infty}.
\end{aligned} \tag{2.11}$$

PROOF. In [Theorem 2.2](#), let

$$\begin{aligned}
\alpha_0 &= a, & \alpha_1 &= \frac{a+x_1}{2}, & \alpha_2 &= \frac{x_1+x_2}{2}, \dots, \\
\alpha_{k-1} &= \frac{x_{k-2}+x_{k-1}}{2}, & \alpha_k &= \frac{x_{k-1}+x_k}{2}, & \alpha_{k+1} &= b, \\
\beta_0 &= c, & \beta_1 &= \frac{c+\mathcal{Y}_1}{2}, & \beta_2 &= \frac{\mathcal{Y}_1+\mathcal{Y}_2}{2}, \dots, \\
\beta_{k-1} &= \frac{\mathcal{Y}_{k-2}+\mathcal{Y}_{k-1}}{2}, & \beta_k &= \frac{\mathcal{Y}_{k-1}+\mathcal{Y}_k}{2}, & \beta_{k+1} &= d.
\end{aligned} \tag{2.12}$$

Then we get

$$\begin{aligned}
& \sum_{i=0}^k \sum_{j=0}^k (\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j) f(x_i, \mathcal{Y}_j) - \sum_{i=0}^k \int_c^d (\alpha_{i+1} - \alpha_i) f(x_i, t) dt \\
& \quad - \sum_{j=0}^k \int_a^b (\beta_{j+1} - \beta_j) f(s, \mathcal{Y}_j) ds \\
& = \frac{1}{4} \left\{ (x_1 - a) \sum_{j=1}^{k-1} (\mathcal{Y}_{j+1} - \mathcal{Y}_{j-1}) f(a, \mathcal{Y}_j) + (x_1 - a)(\mathcal{Y}_1 - c) f(a, c) \right.
\end{aligned}$$

$$\begin{aligned}
 & + (\gamma_1 - c) \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i, c) + (b - x_{k-1})(\gamma_1 - c) f(b, c) \\
 & + \sum_{i=1}^{k-1} \sum_{j=1}^{k-1} (x_{i+1} - x_{i-1})(\gamma_{j+1} - \gamma_{j-1}) f(x_i, \gamma_j) + (b - x_{k-1})(d - \gamma_{k-1}) f(b, d) \\
 & + (d - \gamma_{k-1}) \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) f(x_i, d) + (b - x_{k-1}) \sum_{j=1}^{k-1} (\gamma_{j+1} - \gamma_{j-1}) f(b, \gamma_j) \\
 & - 2(x_1 - a) \int_c^d f(a, t) dt - 2 \sum_{i=1}^{k-1} (x_{i+1} - x_{i-1}) \int_c^d f(x_i, t) dt \\
 & - 2(b - x_{k-1}) \int_c^d f(b, t) dt - 2(\gamma_1 - c) \int_a^b f(s, c) ds \\
 & - 2 \sum_{j=1}^{k-1} (\gamma_{j+1} - \gamma_{j-1}) \int_a^b f(s, \gamma_j) ds - 2(d - \gamma_{k-1}) \int_a^b f(s, d) ds \Big\}.
 \end{aligned}
 \tag{2.13}$$

Applying inequality (2.2), we obtain the desired result. □

COROLLARY 2.4. *Let $I_k : x_i = a + i((b - a)/k)$, $\gamma_j = c + j((d - c)/k)$ (where $i, j = 0, \dots, k$) be a division of $[a, b] \times [c, d]$. If f is as above, then*

$$\begin{aligned}
 & \left| \int_a^b \int_c^d f(s, t) ds dt + \frac{(b - a)(d - c)}{4k^2} \left[f(a, c) + f(a, d) + f(b, c) + f(b, d) \right. \right. \\
 & \quad \left. \left. + 2 \sum_{i=1}^{k-1} \sum_{j=0}^k f\left(\frac{(k - i)a + ib}{k}, \frac{(k - j)c + jd}{k}\right) \right. \right. \\
 & \quad \left. \left. + 2 \sum_{i=0}^k \sum_{j=1}^{k-1} f\left(\frac{(k - i)a + ib}{k}, \frac{(k - j)c + jd}{k}\right) \right] \right. \\
 & \quad \left. - \frac{b - a}{2k} \left[\int_c^d f(a, t) dt + 2 \sum_{i=1}^{k-1} \int_c^d f\left(\frac{(k - i)a + ib}{k}, t\right) dt + \int_c^d f(b, t) dt \right] \right. \\
 & \quad \left. - \frac{d - c}{2k} \left[\int_a^b f(s, c) ds + 2 \sum_{j=1}^{k-1} \int_a^b f\left(s, \frac{(k - j)c + jd}{k}\right) ds + \int_a^b f(s, d) ds \right] \right| \\
 & \leq \frac{((b - a)(d - c))^2}{4k^2(p + 1)^{2/p}} \|f''_{s,t}\|_{\infty}.
 \end{aligned}
 \tag{2.14}$$

Similarly, we can obtain the following theorem.

THEOREM 2.5. *Let $I_k : a = x_0 < x_1 < \dots < x_{k-1} < x_k = b$, $c = \gamma_0 < \gamma_1 < \dots < \gamma_{k-1} < \gamma_k = d$, $l = z_0 < z_1 < \dots < z_{k-1} < z_k = m$ be a division of $[a, b] \times [c, d] \times [l, m]$ and $\alpha_i, \beta_j, \gamma_h$ ($i, j, h = 0, \dots, k + 1$) be $(3k + 3)$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$*

($i = 1, \dots, k$), $\alpha_{k+1} = b$, $\beta_0 = c$, $\beta_j \in [\gamma_{j-1}, \gamma_j]$ ($j = 1, \dots, k$), $\beta_{k+1} = d$, $\gamma_0 = l$, $\gamma_h \in [z_{h-1}, z_h]$ ($h = 1, \dots, k$), $\gamma_{k+1} = m$. If $f : [a, b] \times [c, d] \times [l, m] \rightarrow \mathbb{R}$ is continuous on $[a, b] \times [c, d] \times [l, m]$, $f''_{x,y,z} = \partial^3 f / \partial x \partial y \partial z$ exist on $(a, b) \times (c, d) \times (l, m)$ and $\|f''_{x,y,z}\|_\infty := \sup_{(x,y,z) \in (a,b) \times (c,d) \times (l,m)} |\partial^3 f(x, y, z) / \partial x \partial y \partial z| < \infty$. Then

$$\begin{aligned}
& \left| \int_a^b \int_c^d \int_l^m f(r, s, t) dr ds dt - \sum_{i=0}^k \sum_{j=0}^k \sum_{h=0}^k (\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j)(\gamma_{h+1} - \gamma_h) f(x_i, \gamma_j, z_h) \right. \\
& + \sum_{i=0}^k \sum_{h=0}^k (\alpha_{i+1} - \alpha_i)(\gamma_{h+1} - \gamma_h) \int_c^d f(x_i, s, z_h) ds \\
& + \sum_{j=0}^k \sum_{h=0}^k (\beta_{j+1} - \beta_j)(\gamma_{h+1} - \gamma_h) \int_a^b f(r, \gamma_j, z_h) dr \\
& + \sum_{i=0}^k \sum_{j=0}^k (\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j) \int_l^m f(x_i, \gamma_j, t) dt \\
& - \sum_{i=0}^k (\alpha_{i+1} - \alpha_i) \int_c^d \int_l^m f(x_i, s, t) ds dt \\
& - \sum_{l=0}^k (\gamma_{h+1} - \gamma_h) \int_a^b \int_c^d f(r, s, z_h) dr ds \\
& \left. - \sum_{j=0}^k (\beta_{j+1} - \beta_j) \int_a^b \int_l^m f(r, \gamma_j, t) dr dt \right| \\
& \leq \frac{((b-a)(d-c)(m-l))^{1/q}}{(p+1)^{3/p}} \\
& \quad \times \left[\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \sum_{h=0}^{k-1} ((x_{i+1} - x_i)(\gamma_{j+1} - \gamma_j)(z_{h+1} - z_h))^{p+1} \right]^{1/p} \|f''_{r,s,t}\|_\infty \\
& \leq \frac{u(x)v(\gamma)w(z)(b-a)(d-c)(m-l)}{(p+1)^{3/p}} \|f''_{r,s,t}\|_\infty,
\end{aligned} \tag{2.15}$$

where $u(x) := \max\{x_{i+1} - x_i \mid i = 0, \dots, k-1\}$, $v(\gamma) = \max\{\gamma_{j+1} - \gamma_j \mid j = 0, \dots, k-1\}$, $w(z) := \max\{z_{h+1} - z_h \mid h = 0, \dots, k-1\}$, $p > 1$, $1/p + 1/q = 1$.

THEOREM 2.6. Let $I_k : a_{i1} = x_{i,0} < x_{i,1} < \dots < x_{i,k-1} < x_{i,k} = a_{i2}$, ($i = 1, \dots, k$) be a division of $[a_{11}, a_{12}] \times \dots \times [a_{n1}, a_{n2}]$ and α_{ij} ($i = 1, \dots, k$; $j = 0, \dots, k+1$) be " $nk + n$ " points so that $\alpha_{i0} = a_{i1}$, $\alpha_{ij} \in [x_{i,j-1}, x_{i,j}]$ ($j = 1, \dots, k$), $\alpha_{i,k+1} = a_{i2}$. If $f : [a_{11}, a_{12}] \times \dots \times [a_{n1}, a_{n2}] \rightarrow \mathbb{R}$ is continuous on $[a_{11}, a_{12}] \times \dots \times [a_{n1}, a_{n2}]$, $f_{s_1, \dots, s_n}^{(n)}$ = $\partial^n f / \partial s_1 \dots \partial s_n$ exist on $(a_{11}, a_{12}) \times \dots \times (a_{n1}, a_{n2})$ and

$$\|f_{s_1, \dots, s_n}^{(n)}\|_\infty := \sup_{(s_1, \dots, s_n) \in (a_{11}, a_{12}) \times \dots \times (a_{n1}, a_{n2})} \left| \frac{\partial^n f(s_1, \dots, s_n)}{\partial s_1 \dots \partial s_n} \right| < \infty. \tag{2.16}$$

Then,

$$\begin{aligned}
 & \left| \int_{a_{11}}^{a_{12}} \int_{a_{21}}^{a_{22}} \cdots \int_{a_{n1}}^{a_{n2}} f(s_1, s_2, \dots, s_n) ds_1 ds_2 \cdots ds_n \right. \\
 & + (-1)^n \sum_{i_1=0}^k \sum_{i_2=0}^k \cdots \sum_{i_n=0}^k (\alpha_{1,i_1+1} - \alpha_{1,i_1})(\alpha_{2,i_2+1} - \alpha_{2,i_2}) \cdots \\
 & \quad (\alpha_{n,i_n+1} - \alpha_{n,i_n}) f(x_{1,i_1}, x_{2,i_2}, \dots, x_{n,i_n}) \\
 & + (-1)^{n-1} \sum_{C(i_{j_1} i_{j_2} \cdots i_{j_{n-1}})} \sum_{i_{j_1}, i_{j_2}, \dots, i_{j_{n-1}}=0}^k (\alpha_{j_1, i_{j_1}+1} - \alpha_{j_1, i_{j_1}}) \cdots (\alpha_{j_{n-1}, i_{j_{n-1}}+1} - \alpha_{j_{n-1}, i_{j_{n-1}}}) \\
 & \times \int_{a_{j_{n1}}}^{a_{j_{n2}}} f(x_{j_1, i_{j_1}}, \dots, s, \dots, x_{j_{n-1}, i_{j_{n-1}}}) ds + \cdots \\
 & - \sum_{h=1}^n \sum_{i_h=0}^k \int_{a_{11}}^{a_{12}} \cdots \int_{a_{h-1,1}}^{a_{h-1,2}} \int_{a_{h+1,1}}^{a_{h+1,2}} \cdots \\
 & \quad \times \int_{a_{n1}}^{a_{n2}} f(s_1, \dots, s_{h-1}, x_{i_h}, s_{h+1}, \dots, s_n) ds_1 \cdots ds_{h-1} ds_{h+1} \cdots ds_n \left. \right| \\
 & \leq \frac{((a_{12} - a_{11}) \cdots (a_{n2} - a_{n1}))^{1/q}}{(p+1)^{n/p}} \|f_{s_1, \dots, s_n}^{(n)}\|_{\infty} \\
 & \quad \times \left[\sum_{i_1=0}^{k-1} \sum_{i_2=0}^{k-1} \cdots \sum_{i_n=0}^{k-1} ((x_{1,i_1+1} - x_{1,i_1}) \cdots (x_{n,i_n+1} - x_{n,i_n}))^{p+1} \right]^{1/p} \\
 & \leq \frac{(u_1(x_1) \cdots u_n(x_n))(a_{12} - a_{11}) \cdots (a_{n2} - a_{n1})}{(p+1)^{n/p}} \|f_{s_1, \dots, s_n}^{(n)}\|_{\infty},
 \end{aligned}
 \tag{2.17}$$

where $C(i_{j_1} i_{j_2} \cdots i_{j_{n-1}})$ is combination dependent on $i_{j_1} i_{j_2} \cdots i_{j_{n-1}}$, $\sum_{C(i_{j_1} i_{j_2} \cdots i_{j_{n-1}})}$ is combinatorial sum dependent on $C(i_{j_1} i_{j_2} \cdots i_{j_{n-1}})$, and $u_i(x_j) := \max\{x_{j,i_j+1} - x_{j,i_j} \mid j = 0, \dots, k-1\}$.

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