

ON THE FIRST POWER MEAN OF L -FUNCTIONS WITH THE WEIGHT OF GENERAL KLOOSTERMAN SUMS

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The main purpose of this paper is using the estimates for character sums and the analytic method to study the first power mean of Dirichlet L -functions with the weight of general Kloosterman sums, and give an interesting asymptotic formula.

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1. Introduction. Let $q \geq 2$ be an integer, χ denotes a Dirichlet character modulo q . For any integers m and n , we define the general Kloosterman sums $S(m, n, \chi, q)$ as follows:

$$S(m, n, \chi, q) = \sum_{a=1}^q \chi(a) e\left(\frac{ma + n\bar{a}}{q}\right), \quad (1.1)$$

where \bar{a} denotes the inverse of a modulo q and $e(y) = e^{2\pi i y}$. This summation is very important, because it is a generalization of the classical Kloosterman sums. Many authors had studied the properties of $S(m, n, \chi, q)$. For instance, Chowla [1] and Malyšev [3] obtained a sharper upper bound estimation for $S(m, n, \chi, q)$. That is,

$$|S(m, n, \chi, p)| \ll (m, n, p)^{1/2} p^{1/2+\epsilon}, \quad (1.2)$$

where p is a prime, ϵ is any fixed positive number, and (m, n, p) denotes the greatest common divisor of m , n , and p . But for an arbitrary composite number q , we do not know how large $|S(m, n, \chi, q)|$ is. In fact the value of $|S(m, n, \chi, q)|$ is quite irregular if q is not a prime. The main purpose of this paper is to obtain some good distribution properties of $|S(m, n, \chi, q)|$ in some weight mean value problems. For convenience, in this paper we always suppose $q \geq 3$ be an integer and $L(s, \chi)$ denotes the Dirichlet L -function corresponding to character $\chi \bmod q$. Then we can use the estimates for character sums and the analytic method to prove the following main result.

THEOREM 1.1. *For any integers m and n with $(mn, q) = 1$, we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} |S(m, n, \chi, q)|^2 \cdot |L(1, \chi)| = C \cdot \phi^2(q) + O(q^{3/2} \cdot d^2(q) \cdot \ln^2 q), \quad (1.3)$$

where

$$C = \prod_p' \left[1 + \frac{\binom{2}{1}^2}{4^2 \cdot p^2} + \frac{\binom{4}{2}^2}{4^4 \cdot p^4} + \cdots + \frac{\binom{2n}{n}^2}{4^{2n} \cdot p^{2n}} + \cdots \right] \quad (1.4)$$

is a constant, $\sum_{\chi \neq \chi_0}$ denotes the summation over all nonprincipal characters modulo q , \prod'_p denotes the product over all primes p with $(p, q) = 1$, $d(q)$ is the divisor function, $\phi(q)$ is the Euler function, and $\binom{2n}{n} = (2n)!/(n!)^2$.

For general integer $k \geq 2$, whether there exists an asymptotic formula for

$$\sum_{\chi \neq \chi_0} |S(m, n, \chi, q)|^{2k} |L(1, \chi)| \quad (1.5)$$

is an unsolved problem.

2. Some lemmas. In order to complete the proof of [Theorem 1.1](#), we need the following lemmas.

LEMMA 2.1. *For any integer $q \geq 3$, we have the estimate*

$$\sum_{d|q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi \neq \chi_0} \chi(sd+1) |L(1, \chi)| \right| = O(q \cdot d(q) \cdot \ln^2 q). \quad (2.1)$$

PROOF. Let $N = q^{3/2}$, χ be a nonprincipal character mod q and $A(\chi, y) = \sum_{N < n \leq y} \chi(n)$. Then by Abel identity and Pólya-Vinogradov inequality, we have

$$\begin{aligned} L(1, \chi) &= \sum_{n \leq N} \frac{\chi(n)}{n} + \int_N^{+\infty} \frac{A(\chi, y)}{y^2} dy \\ &= \sum_{n \leq N} \frac{\chi(n)}{n} + O\left(\frac{\ln q}{q}\right). \end{aligned} \quad (2.2)$$

So that

$$|L(1, \chi)| = \left| \sum_{n \leq N} \frac{\chi(n)}{n} \right| + O\left(\frac{\ln q}{q}\right). \quad (2.3)$$

On the other hand, let $r(n)$ be a multiplicative function defining by

$$r(p^\alpha) = \frac{\binom{2\alpha}{\alpha}}{4^\alpha}, \quad r(1) = 1, \quad (2.4)$$

where p is a prime and α is any positive integer. For the function $r(n)$, it is easy to prove that

$$\sum_{d|n} r(d) \cdot r\left(\frac{n}{d}\right) = 1, \quad (2.5)$$

$$\begin{aligned} \left(\sum_{n \leq N} \frac{\chi(n)r(n)}{n} \right)^2 &= \sum_{m \leq N} \sum_{n \leq N} \frac{\chi(nm)r(m)r(n)}{mn} \\ &= \sum_{n \leq N} \frac{\chi(n)}{n} + \sum_{N < n \leq N^2} \frac{\chi(n)r(n, N)}{n}, \end{aligned} \quad (2.6)$$

where

$$r(n, N) = \sum_{\substack{d|n \\ d, n/d \leq N}} r(d) \cdot r\left(\frac{n}{d}\right). \quad (2.7)$$

Note that the triangle inequalities

$$|a| - |b| \leq |a \pm b|, \quad |a \pm b| \leq |a| + |b|, \quad (2.8)$$

from (2.3), (2.6), Cauchy inequality, and the orthogonality relationship for character sums

$$\begin{aligned} \sum_{\chi \bmod q} \chi(n) &= \begin{cases} \phi(q), & \text{if } n \equiv 1 \bmod q; \\ 0, & \text{otherwise,} \end{cases} \\ \sum_{a=1}^q \chi(a) &= \begin{cases} \phi(q), & \text{if } \chi = \chi_0; \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (2.9)$$

we have

$$\begin{aligned} & \sum_{d|q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi \neq \chi_0} \chi(sd+1) |L(1, \chi)| \right| \\ &= \sum_{d|q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi \neq \chi_0} \chi(sd+1) \left| \sum_{n \leq N} \frac{\chi(n)}{n} \right| + O(\ln q) \right| \\ &= \sum_{d|q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi} \chi(sd+1) \left| \sum_{n \leq N} \frac{\chi(n)}{n} \right| \right| + O(q \cdot d(q) \cdot \ln q) \\ &\ll \sum_{d|q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi} \chi(sd+1) \left(\left| \sum_{n \leq N} \frac{\chi(n)}{n} \right| - \left| \sum_{n \leq N} \frac{\chi(n)r(n)}{n} \right|^2 \right) \right| \\ &\quad + \sum_{d|q} d^{1/2} \sum_{s=1}^{q/d-1} \left| \sum_{\chi} \chi(sd+1) \left| \sum_{n \leq N} \frac{\chi(n)r(n)}{n} \right|^2 \right| + q \cdot d(q) \cdot \ln q \\ &\ll \sum_{d|q} d^{1/2} \cdot \frac{q}{\sqrt{d}} \cdot \left[\sum_{\chi} \left(\left| \sum_{n \leq N} \frac{\chi(n)}{n} \right| - \left| \sum_{n \leq N^2} \frac{\chi(n)r(n, N)}{n} \right|^2 \right) \right]^{1/2} \\ &\quad + \phi(q) \sum_{d|q} d^{1/2} \sum_{s=1}^{q/d-1} \sum'_{\substack{m \leq N \\ (sd+1)m \equiv n \bmod q}} \sum'_{n \leq N} \frac{r(m)r(n)}{mn} + q \cdot d(q) \cdot \ln q \\ &\ll q \cdot d(q) \cdot \left[\sum_{\chi} \left| \sum_{N < n \leq N^2} \frac{\chi(n)r(n, N)}{n} \right|^2 \right]^{1/2} + q \cdot d(q) \cdot \ln q \\ &\quad + \phi(q) \sum_{d|q} d^{1/2} \sum'_{\substack{m \leq N \\ m \equiv n \bmod d \\ m \neq n}} \sum'_{\substack{n \leq N \\ m \equiv n \bmod d}} \frac{r(m)r(n)}{mn} \end{aligned}$$

$$\begin{aligned}
&\ll q^{3/2} \cdot d(q) \cdot \left(\sum'_{\substack{N < m \leq N^2 \\ m \equiv n \pmod{q}}} \sum'_{N < n \leq N^2} \frac{r(m, N) \cdot r(n, N)}{mn} \right)^{1/2} + q \cdot d(q) \cdot \ln^2 q \\
&\ll q \cdot d(q) \cdot \ln^2 q.
\end{aligned} \tag{2.10}$$

This proves [Lemma 2.1](#). \square

LEMMA 2.2. *For any integer $q \geq 3$, we have the asymptotic formula*

$$\sum_{\chi \neq \chi_0} |L(1, \chi)| = C \cdot \phi(q) + O(q^{1/2} \cdot \ln q), \tag{2.11}$$

where

$$C = \sum_{n=1}^{\infty} \frac{r^2(n)}{n^2} = \prod_p \left[1 + \frac{\binom{2}{1}^2}{4^2 \cdot p^2} + \frac{\binom{4}{2}^2}{4^4 \cdot p^4} + \cdots + \frac{\binom{2n}{n}^2}{4^{2n} \cdot p^{2n}} + \cdots \right] \tag{2.12}$$

is an absolute constant.

PROOF. Let $N = q^{3/2}$. Then from (2.3), (2.6), the orthogonality relationship for character sums, and the method of proving [Lemma 2.1](#), we have

$$\begin{aligned}
\sum_{\chi \neq \chi_0} |L(1, \chi)| &= \sum_{\chi \neq \chi_0} \left| \sum_{n \leq N} \frac{\chi(n)}{n} \right| + O(\ln q) \\
&= \sum_{\chi} \left| \sum_{n \leq N} \frac{\chi(n)r(n)}{n} \right|^2 + O(\ln q) \\
&\quad + \sum_{\chi} \left(\left| \sum_{n \leq N} \frac{\chi(n)}{n} \right| - \left| \sum_{n \leq N} \frac{\chi(n)r(n)}{n} \right|^2 \right) \\
&= \phi(q) \cdot \sum'_{\substack{m \leq N \\ m \equiv n \pmod{q}}} \sum'_{\substack{n \leq N \\ m \equiv n \pmod{q}}} \frac{r(m) \cdot r(n)}{mn} + O(\ln q) \\
&\quad + O \left(\sum_{\chi} \left| \sum_{N < n \leq N^2} \frac{\chi(n) \cdot r(n, N)}{n} \right| \right) \\
&= \phi(q) \cdot \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{r^2(n)}{n^2} + O(\ln^2 q) \\
&\quad + O \left(q^{1/2} \left(\sum_{\chi} \left| \sum_{N < n \leq N^2} \frac{\chi(n) \cdot r(n, N)}{n} \right|^2 \right)^{1/2} \right) \\
&= \phi(q) \cdot \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{r^2(n)}{n^2} + O(q^{1/2} \cdot \ln q) \\
&= C \cdot \phi(q) + O(q^{1/2} \cdot \ln q).
\end{aligned} \tag{2.13}$$

This proves [Lemma 2.2](#). \square

LEMMA 2.3. *Let m, n , and q be integers with $q \geq 3$. Then we have the estimates*

$$S(m, n, q) = \sum_{a=1}^q e\left(\frac{ma + n\bar{a}}{q}\right) \ll (m, n, q)^{1/2} q^{1/2} d(q), \quad (2.14)$$

where \sum'_a denotes the summation over all a such that $(a, q) = 1$.

PROOF. See [2]. □

3. Proof of Theorem 1.1. In this section, we will complete the proof of Theorem 1.1. First we have

$$\begin{aligned} & \sum_{\chi \neq \chi_0} |L(1, \chi)| \cdot |S(m, n, \chi, q)|^2 \\ &= \sum_{r=1}^q \sum_{s=1}^q e\left(\frac{(r-s)m + (\bar{r}-\bar{s})n}{q}\right) \sum_{\chi \neq \chi_0} \chi(r\bar{s}) |L(1, \chi)| \\ &= \sum_{r=1}^q \sum_{s=1}^q e\left(\frac{r(1-\bar{s})m + \bar{r}(1-s)n}{q}\right) \sum_{\chi \neq \chi_0} \chi(s) |L(1, \chi)| \\ &= \phi(q) \sum_{\chi \neq \chi_0} |L(1, \chi)| + \sum_{s=2}^q \sum_{r=1}^q e\left(\frac{r(1-\bar{s})m + \bar{r}(1-s)n}{q}\right) \sum_{\chi \neq \chi_0} \chi(s) |L(1, \chi)|. \end{aligned} \quad (3.1)$$

From (3.1) and Lemmas 2.2 and 2.3 we get

$$\begin{aligned} & \sum_{\chi \neq \chi_0} |L(1, \chi)| \cdot |S(m, n, \chi, q)|^2 \\ &= C \cdot \phi^2(q) + O(q^{3/2} \cdot \ln q) \\ &+ O\left(q^{1/2} d(q) \sum_{s=2}^q ((s-1)m, (\bar{s}-1)n, q)^{1/2} \left| \sum_{\chi \neq \chi_0} \chi(s) |L(1, \chi)| \right| \right). \end{aligned} \quad (3.2)$$

Note that $(m, q) = (n, q) = (s, q) = 1$, and $(1-s, q) = (s\bar{s}-s, q) = (s(\bar{s}-1), q) = (\bar{s}-1, q)$, so we have $(s-1, \bar{s}-1, q) = (s-1, q)$. Thus from (3.2) and Lemma 2.1 we obtain the asymptotic formula

$$\begin{aligned} & \sum_{\chi \neq \chi_0} |L(1, \chi)| \cdot |S(m, n, \chi, q)|^2 \\ &= C \cdot \phi^2(q) + O(q^{3/2} \cdot \ln q) \\ &+ O\left(q^{1/2} d(q) \sum_{s=2}^q (s-1, q)^{1/2} \left| \sum_{\chi \neq \chi_0} \chi(s) |L(1, \chi)| \right| \right) \\ &= C \cdot \phi^2(q) + O(q^{3/2} \cdot \ln q) \\ &+ O\left(q^{1/2} d(q) \sum_{d|q} \sum_{t=1}^{q/d-1} d^{1/2} \left| \sum_{\chi \neq \chi_0} \chi(td+1) |L(1, \chi)| \right| \right) \\ &= C \cdot \phi^2(q) + O(q^{3/2} \cdot d^2(q) \cdot \ln^2 q). \end{aligned} \quad (3.3)$$

This completes the proof of the theorem.

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