

## A NOTE ON A PAPER BY BRENNER

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We note that a result of Brenner (1962) follows from a theorem of Lerch (1896) which also extends it.

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Let  $m$  and  $n$  be relatively prime integers with  $n \geq 2$ . Let  $\sim$  be the equivalence relation on the set  $S = (\mathbb{Z}/n\mathbb{Z}) \setminus \{0\}$  given by  $t_1 \sim t_2$  if and only if there exists an integer  $k$  such that  $m^k t_1 = t_2$ . Denote by  $N$  the number of equivalence classes. Brenner proved the following result [1].

**THEOREM 1.** *If  $n$  is odd, then  $(-1)^N$  equals the Jacobi symbol  $(m/n)$ .*

The purpose of this note is to point out that the above result is a consequence of a theorem of Lerch [3] dating back to 1896, which, moreover, extends Theorem 1 to the case of even  $n$ .

**THEOREM 2 (Lerch).** *For relatively prime integers  $m$  and  $n$ , with  $n \geq 2$ , the sign of the permutation  $\pi$  induced by multiplication by  $m$  on  $(\mathbb{Z}/n\mathbb{Z}) \setminus \{0\}$  equals*

- (a) *the Jacobi symbol  $(m/n)$  if  $n$  is odd;*
- (b) *1 if  $n$  is even and not divisible by 4;*
- (c)  *$(-1)^{(m-1)/2}$  if  $n$  is divisible by 4.*

Observe that  $N$  is the number of cycles  $\tau_1, \dots, \tau_N$  in the decomposition of  $\pi$  into a product of disjoint cycles (1-cycles need to be included). Now if  $l_i$  is the length of  $\tau_i$ , then the sign of  $\tau_i$  equals  $(-1)^{l_i-1}$ , so, if  $n$  is odd, the sign of  $\pi$  equals

$$(-1)^{\sum_{i=1}^N (l_i-1)} = (-1)^{n-1-N} = (-1)^N. \quad (1)$$

Thus Theorem 1 follows from Theorem 2, as does the following extension.

**COROLLARY 3.** *For  $n$  even  $(-1)^N$  equals  $-1$ , if  $n \equiv 2 \pmod{4}$ , and  $(-1)^{(m+1)/2}$ , if  $n \equiv 0 \pmod{4}$ .*

Lerch's theorem, which generalizes a result of Zolotareff [4] on the Legendre symbol, considerably simplifies the theory of quadratic residues (see, e.g., [2]) and deserves to be more widely known.

## REFERENCES

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