

LOCAL COMPLETENESS OF $\ell_p(E)$, $1 \leq p < \infty$

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We study the heredity of local completeness and the strict Mackey convergence property from the locally convex space E to the space of absolutely p -summable sequences on E , $\ell_p(E)$ for $1 \leq p < \infty$.

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1. Introduction. In 1956, Grothendieck [5], introduced the Banach-valued sequence space $\ell_p(E)$, the space of absolutely p -summable sequences on a Banach space E , where he discussed tensor products of ℓ_p and E , with $1 \leq p \leq \infty$. Later, in 1969 Pietsch [8] used Banach-valued sequence spaces $\ell_p(E)$, to study p -summing operators between Banach spaces, also see Diestel et al. [2]. In this paper, we discuss how local completeness and the strict Mackey convergence condition of E imply local completeness and the strict Mackey convergence condition in $\ell_p(E)$ in the case $1 \leq p < \infty$. The case $p = \infty$ was studied in [1].

2. Definitions and notation. Throughout this paper, (E, t) denotes a Hausdorff locally convex space over \mathbb{K} (\mathbb{R} or \mathbb{C}) and $\{\rho_j\}_{j \in J}$ denotes the family of continuous seminorms associated with the topology t on E .

Let $D \subset E$ be a bounded, closed, and absolutely convex set. Denote by $E_D = \cup_{k=1}^{\infty} kD$, and for each $x \in E_D$, $\rho_D(x) = \inf\{r > 0 : x \in rD\}$, the Minkowski seminorm associated with D . Now $E_D \subset E$ and the boundedness of D implies that $i : (E_D, \rho_D) \rightarrow (E, t)$ is continuous, and ρ_D is a norm so that, for every $j \in J$ there exists $r_j \in \mathbb{R}^+$ such that $\rho_j|_{E_D} \leq r_j \rho_D$.

REMARK 2.1. For each $D \subset E$ bounded, closed, and absolutely convex, the family of seminorms $\{\rho_j\}_{j \in J}$ can be replaced by an equivalent family $\{\rho'_j\}_{j \in J}$ such that $\rho'_j \leq \rho_D$. To construct the family $\{\rho'_j\}_{j \in J}$ we know that there exists $r_j > 0$ such that $\rho_j(x) \leq r_j \rho_D(x)$ for every $x \in E_D$ so it suffices to take $\rho'_j = (1/r_j) \rho_j$ if $r_j > 1$, and we will have $\rho'_j \leq \rho_D$, for every $j \in J$. For simplicity we will always work with an equivalent family of seminorms, also denoted by $\{\rho_j\}_{j \in J}$ such that $\rho_j(x) \leq \rho_D(x)$ holds for every $j \in J$ and $x \in E_D$.

A bounded, closed, and absolutely convex set $D \subset E$, called a disk, is a Banach disk if (E_D, ρ_D) is a Banach space. If every bounded set $A \subset E$ is contained in a Banach disk we say that E is locally complete. Let (E, t) satisfies the strict Mackey convergence condition if for every bounded set $A \subset E$, there exists a disk D that contains A such that the topologies of (E, t) and (E_D, ρ_D) agree on A .

Every metrizable space satisfies the strict Mackey convergence condition, [7]. In addition, the strict Mackey convergence condition is preserved under the formation of closed subspaces, countable products, and countable direct sums, [6]. The strict Mackey convergence condition for webbed spaces is studied in [3, 4].

REMARK 2.2. Using the family of seminorms $\{\rho_j\}_{j \in J}$ it is easy to see that the strict Mackey convergence condition is equivalent to: for each D there exists $j_0 \in J$ such that $\rho_{j_0|D} = \rho_D$.

Let p be a real number such that $1 \leq p < \infty$. The space $\ell_p(E)$ of absolutely p -summable sequences on E is

$$\ell_p(E) = \left\{ (x_n) \in E : \sum_{n=1}^{\infty} \rho_j^p(x_n) < \infty, \forall j \in J \right\}. \quad (2.1)$$

The family of seminorms $\rho_{\rho_j}((x_n)) = (\sum_{n=1}^{\infty} \rho_j^p(x_n))^{1/p}$, $j \in J$, induce a topology of locally convex space in $\ell_p(E)$; we will denote by τ this topology.

The space $\ell_p(E_D)$ is defined by $\ell_p(E_D) = \{(x_n) \in E_D : \sum_{n=1}^{\infty} \rho_D^p(x_n) < \infty\}$ and endowed with the topology generated by the norm

$$\rho_{\rho_D}((x_n)) = \left[\sum_{n=1}^{\infty} \rho_D^p(x_n) \right]^{1/p}. \quad (2.2)$$

We denote $A_D = \{(x_n) \in \ell_p(E) : (x_n)_{n \in \mathbb{N}} \subset D\}$.

Note that $\rho_{\rho_j}|_{\ell_p(E_D)} \leq \rho_{\rho_D}$ for every $j \in J$ since $\rho_j|_{E_D} \leq \rho_D$.

3. Bounded sets. In this section, we characterize the bounded sets of $\ell_p(E)$ in terms of the bounded sets of E .

LEMMA 3.1. *Let D be a disk in (E, t) ; then*

- (i) $\ell_p(E_D) \subseteq \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD \text{ for some } k \in \mathbb{N}\}$;
- (ii) *if there exists $j_0 \in J$, depending on D , such that $\rho_{j_0|D} = \rho_D$ (i.e., the strict Mackey convergence condition holds), then $\{(x_n) \in \ell_p(E) : \{x_n\} \subset kD \text{ for some } k \in \mathbb{N}\} \subset \ell_p(E_D)$.*

PROOF. (i) Let $(x_n) \in \ell_p(E_D)$. Then $\sum_{n=1}^{\infty} [\rho_D(x_n)]^p < \infty$ so that given $\varepsilon = 1$ there exists $n_0 \in \mathbb{N}$, such that for each $n > n_0$, we have $\rho_D(x_n) \leq (\sum_{n=1}^{\infty} \rho_D^p(x_n))^{1/p} \leq 1$ which means that $x_n \in D$ for every $n > n_0$.

Now for $i = 1, 2, \dots, n_0$ there exists $k_i \geq 0$ such that $x_i \in k_i D$. We take $k = \max\{1, k_1, \dots, k_{n_0}\}$. Then $\{x_n\} \subset kD$ and we have $\ell_p(E_D) \subset \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD \text{ for some } k \in \mathbb{N}\}$.

(ii) Let $(x_n) \in \{(y_n) \in \ell_p(E) : \{y_n\} \subset kD \text{ for some } k \in \mathbb{N}\}$. Thus $x_n \in E_D$ for every $n \in \mathbb{N}$ since $\{x_n\} \subset kD$.

Now observe that $\sum_{n=1}^{\infty} \rho_D^p(x_n) = \sum_{n=1}^{\infty} \rho_{j_0}^p(x_n) < \infty$ since $(x_n) \in \ell_p(E)$. Hence in this case we have the equality $\ell_p(E_D) = \{(x_n) \in \ell_p(E) : \{x_n\} \subset kD \text{ for some } k \in \mathbb{N}\}$.

□

REMARK 3.2. Note that $kA_D = A_{kD}$ for every $k \in \mathbb{N}$.

COROLLARY 3.3. *If E satisfies the strict Mackey convergence condition, then $\ell_p(E)_{A_D} = \ell_p(E_D)$.*

PROOF. It follows from the equality in the proof of [Lemma 3.1\(ii\)](#) that $\ell_p(E)_{A_D} \subset \ell_p(E_D)$. Now let $(x_n) \in \ell_p(E_D)$. Then by [Lemma 3.1\(i\)](#), $(x_n) \subset kD$ for some $k \in \mathbb{N}$ so $\{x_n\} \subset A_{kD} = kA_D$ and $(x_n) \in \ell_p(E)_{A_D}$. \square

REMARK 3.4. If (E, t) satisfies the strict Mackey convergence condition, then

$$\ell_p(E)_{A_D} = \ell_p(E_D) = \{(x_n) \in \ell_p(E) : \{x_n\} \subset A_{kD} \text{ for some } k \in \mathbb{N}\}. \quad (3.1)$$

LEMMA 3.5. (i) $\rho_{A_D}((x_n)) = \sup\{\rho_D(x_n) : n \in \mathbb{N}\}$;
(ii) $\rho_{A_D}((x_n)) \leq \rho_{\rho_D}((x_n))$ for every $(x_n) \in \ell_p(E_D)$.

PROOF. (i) Let $s = \sup\{\rho_D(x_n) : n \in \mathbb{N}\}$. Then $\{x_n\} \subset sD$ so $\{x_n\} \subset A_{sD} = sA_D$ and then $\rho_{A_D}((x_n)) \leq s$. Now take $r = \rho_{A_D}((x_n))$. Then $\{x_n\} \subset rA_D = A_{rD}$ and then $\{x_n\} \subset rD$ which means that $r \geq s$.

(ii) $\rho_{\rho_D}((x_n)) = (\sum_{n=1}^{\infty} \rho_D^p(x_n))^{1/p} \geq \rho_D(x_n)$ for every $n \in \mathbb{N}$. Using (i) we have $\rho_{\rho_D}((x_n)) \geq \rho_{A_D}((x_n))$. \square

Note that A_D is not bounded in $\ell_p(E)$; we need to construct a “smaller” set, in the sense of boundedness.

Define for each $j \in J$ and $m \in \mathbb{N}$ the set $A_D(j, m) = \{(x_n)_n \in A_D : \rho_{\rho_j}((x_n)) \leq m\}$ and for each $B \subset \ell_p(E)$, let $B^* = \{x \in E : x \in \{x_n\} \text{ and } (x_n) \in B\}$.

The next proposition gives a way to look at the bounded sets in $\ell_p(E)$.

PROPOSITION 3.6. *If $\beta = \{D_{\lambda}\}_{\lambda \in \Lambda}$ is a fundamental system of bounded disks in E , then $\{C = \cap_{j \in J} A_{D_{\lambda}}(j, m_j)\} : \lambda \in \Lambda, (m_j) \in \mathbb{N}^J\}$ is a fundamental system of τ -bounded sets in $\ell_p(E)$.*

PROOF. Let $B \subset \ell_p(E)$ be a bounded set. Then B^* is bounded in E so $B^* \subset D_{\lambda}$ for some λ . For each $x \in B^*$, if $x \in (x_n)$ then given $j \in J$ there is some s_j such that $\rho_j(x) \leq \rho_{\rho_j}((x_n)) \leq s_j$ so that $\rho_{\rho_j}(B) \leq s_j$. Now let $m_j \in \mathbb{N}$ be such that $s_j \leq m_j$. We have $B \subset C = \cap_{j \in J} A_{D_{\lambda}}(j, m_j)$. \square

REMARK 3.7. (i) If D is bounded in E , then for each $j \in J$, by [Remark 2.1](#) $\rho_{j|E_D} \leq \rho_D$.
(ii) If C is bounded in $\ell_p(E)$, then for each $j \in J$, by [Remark 2.1](#) $\rho_{\rho_j} | \ell_p(E)_C \leq \rho_C$.

4. Main results

PROPOSITION 4.1. *If for some D there exists $j_0 \in J$, such that $\rho_{j_0}|_D = \rho_D$ in E , then $\rho_{\rho_{j_0}|_C} = \rho_C$ where $C = \cap_{j \in J} A_D(j, m_j)$ in $\ell_p(E)$. Equivalently, if E satisfies the strict Mackey convergence condition, then $\ell_p(E)$ also satisfies the strict Mackey convergence condition.*

PROOF. Let $(x_n) \in C$. Then $s = \rho_{\rho_{j_0}}(x_n) = (\sum_{n=1}^{\infty} \rho_{j_0}^p(x_n))^{1/p} = (\sum_{n=1}^{\infty} \rho_D^p(x_n))^{1/p} \geq \rho_D(x_n) \geq \rho_{\rho_j}(x_n)$ for every $j \in J$ and $n \in \mathbb{N}$. So we have $(x_n) \in \cap_{j \in J} A_D(j, s) = s[\cap_{j \in J} A_D(j, 1)] \subset sC$. Thus $\rho_C((x_n)) \leq s = \rho_{\rho_{j_0}}(x_n)$ and since C is bounded in $\ell_p(E)$ we have $\rho_{\rho_j} \leq \rho_C$ for each $j \in J$; now $\rho_{\rho_j}|_C \leq \rho_C$ for every $j \in J$, so for j_0 we have $\rho_{\rho_{j_0}}|_C = \rho_C$.

Notice that if B is a bounded set in $\ell_p(E)$, then $\rho_{\rho_j}(B) \leq m_j$ for all $j \in J$ with $m_j \in N$ and then $B \subset \cap_{j \in J} A_{B^*}(j, m_j)$.

This gives the property we need to characterize the bounded sets in $\ell_p(E)$. \square

THEOREM 4.2. *If E is locally complete and satisfies the strict Mackey convergence condition, then $(\ell_p(E)_C, \rho_C)$ where $C = \cap_{j \in J} A_D(j, m_j)$ in $\ell_p(E)$, is a Banach space so $\ell_p(E)$ is locally complete.*

PROOF. Let D be a bounded closed disk such that (E_D, ρ_D) is a Banach space and let $C = \cap_{j \in J} A_D(j, m_j)$. By Remark 2.1 there is a $j_0 \in J$ such that $\rho_{j_0}|_D = \rho_D$. We will show that $(\ell_p(E)_C, \rho_C)$ is a Banach space. By Corollary 3.3 we have $\ell_p(E)_{A_D} = \ell_p(E_D)$ and since $C \subset A_D$, $\ell_p(E) \subset \ell_p(E)_{A_D}$. Let $(x_n^k)_{k \in \mathbb{N}} \subset \ell_p(E)_C$ be a ρ_C -Cauchy sequence. Thus for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n, m \geq N$ we have $\rho_C((x_n^k) - (x_m^k)) < \varepsilon$. Using Remark 3.7(ii) we have that $\rho_{\rho_j} \mid \ell(E)_C \leq \rho_C$. Hence (x_n^k) is also a ρ_{ρ_j} -Cauchy sequence and then a $\rho_{\rho_{j_0}}$ -Cauchy sequence. Thus $\rho_D(x_n^k - x_m^k) = \rho_{j_0}(x_n^k - x_m^k) \leq \rho_{\rho_{j_0}}((x_n^k) - (x_m^k))$, then the sequence $(x_n^k)_{k \in \mathbb{N}}$ for every $n \in \mathbb{N}$ is also a ρ_D -Cauchy sequence in (E_D, ρ_D) which is a Banach space, so there exists z^k in E_D such that (x_n^k) converges to z^k with respect to the norm ρ_D . Using Remark 3.7(i) we have $\rho_{j|E_D} \leq \rho_D$. Hence, we have the following claims.

CLAIM 1. We have that (x_n^k) converges to z^k with respect to the seminorm ρ_j for every $j \in J$.

CLAIM 2. Consider the sequence formed by the $(z^k)_{k \in \mathbb{N}} \in \ell_p(E_D)$. We compute

$$\begin{aligned}
\sum_{k=1}^{\infty} (\rho_D(z^k))^p &= \lim_{m \rightarrow \infty} \sum_{k=1}^m (\rho_D(z^k))^p \\
&= \lim_{m \rightarrow \infty} \sum_{k=1}^m (\rho_{j_0}(z^k))^p \\
&= \lim_{m \rightarrow \infty} \sum_{k=1}^m \rho_{j_0} \left(\lim_{n \rightarrow \infty} x_n^k \right)^p \\
&= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{k=1}^m \rho_{j_0}(x_n^k)^p \\
&\leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \rho_{j_0}(x_n^k)^p \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \rho_{j_0}(x_n^k)^p \\
&\leq \lim_{n \rightarrow \infty} \rho_{\rho_{j_0}}((x_n)) \\
&\leq \varepsilon + \rho_{\rho_{j_0}}((x_N)) < \infty, \quad \text{for some } N \in \mathbb{N}.
\end{aligned} \tag{4.1}$$

In this last inequality we used $x_n = (x_n^k)_{k \in \mathbb{N}}$ and since it is a $\rho_{\rho_{j_0}}$ -Cauchy sequence, given $\varepsilon > 0$, $\rho_{\rho_{j_0}}(x_n^k) - \rho_{\rho_{j_0}}(x_m^k) \leq \rho_{\rho_{j_0}}((x_n^k) - (x_m^k)) < \varepsilon$ for every $n, m > N$, so $\rho_{\rho_{j_0}}((x_n)) \leq \varepsilon + \rho_{\rho_{j_0}}((x_N))$. Notice that (x_n) is a ρ_{ρ_j} -Cauchy sequence for every $j \in J$.

Therefore for j_0 and consequently for ρ_{ρ_D} , then for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $\rho_D(x_n^k - z^k) = \rho_D(x_n^k - \lim_{m \rightarrow \infty} x_m^k) = \lim_{m \rightarrow \infty} \rho_D(x_n^k - x_m^k) < \varepsilon$.

CLAIM 3. The sequence (x_n^k) converges to $(z^k)_{k \in \mathbb{N}}$ in $\ell_p(E_D)$. Since

$$\begin{aligned} \rho_{\rho_D}(x_n^k - (z^k)_k) &= \left[\sum_{k=1}^{\infty} \rho_D^p(x_n^k - z^k) \right]^{1/p} \\ &\leq \left[\sum_{k=1}^N \rho_D^p(x_n^k - z^k) + \frac{\varepsilon^p}{2} \right]^{1/p} \\ &\leq \left(\underbrace{\frac{\varepsilon^p}{2N} + \dots + \frac{\varepsilon^p}{2N}}_{N \text{ factors}} + \frac{\varepsilon^p}{2} \right)^{1/p} \\ &= \varepsilon, \quad \text{for } n > N. \end{aligned} \tag{4.2}$$

In the first inequality we used [Claim 2](#). This completes the proof of the convergence.

CLAIM 4. We have $(z^k)_{k \in \mathbb{N}} \in \ell_p(E)_C$. $(x_n^k)_{k \in \mathbb{N}}$ is a ρ_C -Cauchy sequence so it is bounded and there is an $s \in \mathbb{N}$ such that $(x_n^k) \subset sC$. Using [Claim 3](#), (x_n^k) converges to (z^k) in $\ell_p(E)_C$ with respect to ρ_{ρ_D} and since $\rho_{\rho_j}|_{\ell_p(E_D)} \leq \rho_{\rho_D}$ for every $j \in J$ the sequence (x_n^k) is τ -convergent to (z^k) , it is convergent for each ρ_{ρ_j} . Now for each $\varepsilon > 0$ there exists N_j such that $\rho_{\rho_j}((z^k)) \leq \rho_{\rho_j}((z^k) - (x_n^k)) + \rho_{\rho_j}((x_n^k)) < \varepsilon + sm_j$ for every $j \in J$ and $n \geq N_j$, this means that $(z^k) \in sC \subset \ell_p(E)_C$.

CLAIM 5. The sequence (x_n^k) converges to $(z^k)_{k \in \mathbb{N}}$ in $\ell_p(E)_C$. Let $\varepsilon > 0$, since (x_n^k) is a ρ_C -Cauchy sequence, there is $N \in \mathbb{N}$ such that $(x_n^k) - (x_m^k) \in \varepsilon C$ for every $n, m \geq N$. C is τ -closed so $(x_n^k) - (\tau - \lim(x_m^k)) \in \varepsilon C$; then $(x_n^k) - (z^k) \in \varepsilon C$ for every $n \geq N$ which means $\rho_C((x_n^k) - (z^k)) \leq \varepsilon$ for every $n \geq N$.

Notice that this is true for every $1 \leq p < \infty$. The case $p = \infty$ also follows from this and we get the characterization given in [1], although under a stronger hypothesis. Here we need E to satisfy the strict Mackey convergence condition. \square

LEMMA 4.3. If $D \subset E$ is t -complete and the net $\{x_\lambda\}_\Lambda$ is a τ -Cauchy net bounded with respect to ρ_C , that is if there exists $s \in \mathbb{N}$ such that $\{x_\lambda\}_\Lambda \subset sC$ then there exists $z \in 2sC$ such that x_λ converges to z with respect to the τ topology in $\ell_p(E)$.

PROOF. Let $\{x_\lambda\}_\Lambda$ be a τ -Cauchy net, $x_\lambda = (x_\lambda^1, x_\lambda^2, \dots)$, then for every $\varepsilon > 0$ there exists $\lambda_j \in \Lambda$ such that for every $j \in J$, $\rho_j(x_\lambda^k - x_{\lambda'}^k) \leq \rho_{\rho_j}(x_\lambda - x_{\lambda'}) < \varepsilon$ for every $\lambda, \lambda' \geq \lambda_j$ and $k \in \mathbb{N}$. So $\{x_\lambda^k\}_\Lambda \subset D$ is t -Cauchy for each $k \in \mathbb{N}$, and since D is complete there is a z^k such that x_λ^k converges to z^k with respect to the topology t for each $k \in \mathbb{N}$. Let $z = \{z^1, z^2, \dots\}$. Then $z \subset D$, and for each $j \in J$ and $k \in \mathbb{N}$ we have $\rho_j(x_\lambda^k - z^k) = \rho_j(x_\lambda^k - (\rho_j - \lim_{\lambda'} x_{\lambda'}^k)) = \lim_{\lambda'} \rho_j(x_\lambda^k - x_{\lambda'}^k)$, so raising to the p th power and adding with respect to k we have

$$\begin{aligned} \sum_{k=1}^{\infty} \rho_j(x_\lambda^k - z^k)^p &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \rho_j(x_\lambda^k - z^k)^p \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \lim_{\lambda'} \rho_j(x_\lambda^k - x_{\lambda'}^k)^p \end{aligned}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \lim_{\lambda'} \sum_{k=1}^n \rho_j(x_\lambda^k - z^k)^p \\
&\leq \lim_{\lambda'} \sum_{k=1}^\infty \rho_j(x_\lambda^k - z^k)^p \\
&= \lim_{\lambda'} \rho_{\rho_j}(x_\lambda - x_{\lambda'}) < \varepsilon^p,
\end{aligned} \tag{4.3}$$

for every $\lambda \geq \lambda_j$.

So we have $\rho_{\rho_j}(x_\lambda - z)^p = \sum_{k=1}^\infty \rho_j(x_\lambda^k - z^k)^p < \varepsilon^p$, for every $\lambda \geq \lambda_j$. This means that x_λ converges to z with respect to the topology τ . We still need to prove that $z \in \ell_p(E)$

$$\begin{aligned}
\rho_{\rho_j}(z)^p &= \sum_{k=1}^\infty \rho_j(z^k)^p \\
&= \sum_{k=1}^\infty \rho_j(z^k + x_\lambda^k - x_\lambda^k)^p \\
&\leq \sum_{k=1}^\infty 2^p [\rho_j(z^k - x_\lambda^k)^p + \rho_j(x_\lambda^k)^p] \\
&= 2^p \sum_{k=1}^\infty \rho_j(z^k - x_\lambda^k)^p + 2^p \sum_{k=1}^\infty \rho_j(x_\lambda^k)^p \\
&< 2^p \varepsilon^p + 2^p \rho_{\rho_j}(x_\lambda)^p \\
&\leq 2^p \varepsilon^p + 2^p m_j
\end{aligned} \tag{4.4}$$

($x_\lambda \in C = \cap_{j \in J} A_D(j, m_j)$), then if we let $\varepsilon \rightarrow 0$ we get $\rho_{\rho_j}(z) \leq 2m_j$, and finally $z \in 2C \subset \ell_p(E)$. \square

THEOREM 4.4. *If D is t -complete, then $\ell_p(E)_C$ is ρ_C -complete.*

PROOF. Let (x_n^k) be a ρ_C -Cauchy sequence; it is clearly ρ_C -bounded and τ -Cauchy, so $(x_n^k) \subset sC$ for some $s \in \mathbb{N}$. Then by [Lemma 4.3](#), there is a $z = (z^k) \in 2sC \subset \ell_p(E)_C$ such that the sequence (x_n^k) converges to z with respect to the topology τ . Note that A_D is τ -closed so $A_D(j, m)$ is also τ -closed for every $j \in J$ and $m \in \mathbb{N}$; then $C = \cap_{j \in J} A_D(j, m_j)$ is τ -closed. For $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $(x_n^k) - (x_m^k) \in \varepsilon C$ for every $n, m \geq N$, and since C is τ -closed $(x_n^k) - (\tau\text{-}\lim(x_m^k)) \in \varepsilon C$ then $(x_n^k) - (z^k) \in \varepsilon C$ for every $n \geq N$. This means that (x_n^k) converges to (z^k) with respect to ρ_C . \square

THEOREM 4.5. *If E is t -complete, then $\ell_p(E)$ is τ -complete.*

PROOF. The proof of [Lemma 4.3](#) can be repeated here to get the τ -completeness of $\ell_p(E)$. \square

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Special Issue on Boundary Value Problems on Time Scales

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The study of dynamic equations on a time scale goes back to its founder Stefan Hilger (1988), and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics; moreover, it often reveals the reasons for the discrepancies between two theories.

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- Comparison principles
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