ON HAUSDORFF COMPACTIFICATIONS OF NON-LOCALLY COMPACT SPACES

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ABSTRACT. Let X be a completely regular, Hausdorff space and let R

be the set of points in X which do not possess compact neighborhoods. Assume

R is compact. If X has a compactification with a countable remainder, then

so does the quotient X/R, and a countable compactification of X/R implies

one for $X-R$. A characterization of when X/R has a compactification with

a countable remainder is obtained. Examples show that the above implications

cannot be reversed.

KEY WORDS AND PHRASES. Countable remainders, compactifications, non-locally
compact spaces, components of $\beta X - X$.

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1. <u>INTRODUCTION</u>.

Let X be a completely regular, Hausdorff topological space. The question
of characterizing when X has a Hausdorff compactification αX, where
αX - X is countably infinite, has been answered for the locally compact case
by Magill [2] and for the case when αX = βX by Okuyama [4] (where βX is
the Stone-Cech compactification of X). In case X is an arbitrary completely
regular space, no such characterization has been given. The purpose of this
paper is to contribute results toward such a characterization.

Let R be the set of points in X which do not possess compact neighbor-
hoods. Then for all compactifications αX of X, $R = Cl_{\alpha X}(\alpha X - X) \cap X$. (See [5].)
Herein we observe that for compact R, a necessary condition for X to have
a countable compactification is that X/R have one. The main theorem of this
paper characterizes when X/R has a countable compactification.

2. <u>CHARACTERIZATION OF α(X/R)</u>.

Throughout this paper all compactifications are Hausdorff compactifications.
Let N denote the natural numbers. If R is a compact, non-empty subset of
a completely regular space X and if X has a coutable compactification γX,
then a countable compactification of X/R can be obtained from γX by iden-
tifying R to a single point. It is readily verified that the resulting space
is Hausdorff.

If α(X/R) is a countable compactification of X/R, then α(X/R) is also
a countable compactification of X - R. Thus, we have the following:

THEOREM 1. If X is completely regular and R is compact, then each of
the following conditions implies the next:

(A) X has a countable compactification;

(B) X/R has a countable compactification;

(C) X - R has a countable compactification.

Examples will be provided to show that none of these implications can be reversed.

If R is non-compact, then (A) no longer implies (C) as in Theorem 1. Let X be the unit disc in the standard plane with a countable dense subset removed from the boundary. The remaining boundary points constitute R. Then, clearly, X has a countable compactification but X - R, the open disc, has no countable compactification.

Let $Y = (\beta X - X) \cup R$.

THEOREM 2. Let X be a completely regular Hausdorff space with R compact and non-empty. Then the following are equivalent:

(A) X/R has a countable compactification.

'B) R is a G_δ-set in Y and components of R are components of Y.

PROOF. (A) implies (B). Take $\{p_n | n \in N\} = \gamma(X/R) - X/R$, where $\gamma(X/R)$ is a countable compactification of X/R, and let t_0 be the canonical mapping of X into $\gamma(X/R)$. Then t_0 has an extension t which maps βX onto $\gamma(X/R)$. We first show that t carries $\beta X - X$ onto $\gamma(X/R) - X/R$. Since the restriction of t to X - R is a homeomorphism and X - R is dense in βX and in $\gamma(X/R)$, t carries Y onto $[\gamma(X/R) - X/R] \cup \{r\}$, where $r = t[R]$ (cf. Lemma 6.11 [1]). If $x \in R$ and $y \in \beta X - X$, then since R is compact there exists a compact neighborhood N_R of R in βX such that $y \notin N_R$. Set $N = N_R \cap X$. Since $R \subseteq N$, $t_0[N]$ is a neighborhood of $t(x) = r$ in X/R. Thus, there is a neighborhood G in $\gamma(X/R)$ for which $t_0[N] = G \cap X/R$. If N_y is any neighborhood of y in βX, choose $z \in N_y \cap (X - N)$. Then $t(z) \notin G$ and it follows from the continuity of t that $t(x) \neq t(y)$. Hence $t[\beta X - X] = \gamma(X/R) - X/R$.

Next, let $K_n = t^{-1}(p_n)$, for each $n \in N$. Evidently, $\beta X - X = \bigcup \{K_n | n \in N\}$. Since each K_n is compact, the sets $Y - K_n$ are open in Y and $R = \bigcap \{Y - K_n | n \in N\}$. Thus R is a G_δ-set in Y.

Let C be a component of R and let C_1 be a component of Y, where $C \subseteq C_1$. If $C \neq C_1$, choose $x \in C_1 - C$. Now there exists a continuous injection f of $\{p_n \in N\}$ $\{r\}$ into the real numbers. (See [3]). But $f \circ t C_1$ must be connected and not a singleton, since $t[R] \neq t(x)$. This contradicts the fact that the image of f is countable. Thus, $C = C_1$, so that components of R are components of Y.

(B) implies (A). First we show that there exist sets $\{U_n | n \in N\}$ which are clopen in Y such that $\bigcap \{U_n | n \in N\} = R$. Note that Y is compact. Let $\{V_n | n \in N\}$ be open subsets of Y satisfying $\bigcap \{V_n | n \in N\} = R$. For each $n \in N$, set $K_n = Y - V_n$. We assume that each $K_n \neq \emptyset$. Let $(x,r) \in K_n \times R$. Since x and r are in distinct quasi-components of Y, there exists a clopen neighborhood $W_n(x,r)$ of r in Y, where $x \notin W_n(x,r)$. Now $\{W_n(x,r) | r \in R\}$ is an open covering of R so that a finite subfamily $\{W_n(x,r_i) | i = 1,\ldots,p(x)\}$ covers R. Take $W_n(x) = \bigcup \{W_n(x,r_i) | i = 1,\ldots,p(x)\}$. Thus $W_n(x)$ is a clopen subset of Y. $R \subseteq W_n(x)$, and $x \notin W_n(x)$. Since $\{Y - W_n(x) | x \in K_n\}$ is an open cover of K_n, there is a finite subcover $\{Y - W_n(x_j) | j = 1,\ldots,q(n)\}$.

For each $n \in N$, let $U_n = \bigcap \{W_n(x_j) | j = 1,\ldots,q(n)\}$. Then each U_n is a clopen subset of Y, $R \subseteq U_n$ and $K_n \subseteq Y - U_n$. Hence $R = \bigcap \{U_n | n \in N\}$.

Let $C_1 = Y - U_1$, and for $n > 1$, take $C_n = [Y - \bigcap \{U_i | i = 1,\ldots,n\}] - \bigcup \{C_i | i = 1,\ldots,n - 1\}$. Then each C_n is a clopen subset of Y and $\beta X - X = \bigcup \{C_n | n \in N\}$.

Let $\sim$ be the equivalence relation in βX which identifies each C_n to a point and R to a point. The projection of βX onto $\beta X / \sim$ is denoted by Π.

For each $n \in N$, consider the point $\Pi[C_n]$ in $\beta X/\sim$. Now $\{C_n, Y - C_n\}$ is a partition of Y into disjoint open sets. Thus, C_n and $Y - C_n$ can be separated by open sets U and V in βX. Evidently, $\Pi[U]$ and $\Pi[V]$ are disjoint open subsets of $\beta X/\sim$. This shows that $\Pi[C_n]$ can be separated from any other point of $\beta X/\sim$. Since points of $\beta X - Y$ have compact βX - neighborhoods in $\beta X - Y$, it follows that $\beta X/\sim$ is a compact Hausdorff space.

It remains to show that X/R can be embedded in $\beta X/\sim$ in the desired manner. Let i be the natural embedding of X in βX and let p be the projection of X onto X/R. Since i is relation preserving, a continuous mapping j of X/R into $\beta X/\sim$ is induced such that $j \circ p = \Pi \circ i$. It follows that j is also a closed mapping, hence an embedding of X/R into $\beta X/\sim$ as desired. This completes the proof.

In [2] Magill shows that a locally compact space X has a countable compactification if and only if $\beta X - X$ has infinitely many components. As an application of the proof of Theorem 2, the following is proven.

COROLLARY 3. Let X be completely regular with R compact. If X has a countable compactification, then $\beta X - X$ has infinitely many components.

PROOF. Let t be a continuous mapping of βX onto $\alpha(X/R)$ which carries $\beta X - X$ onto $\alpha(X/R) - X/R$. Since the subspace $K = (\alpha(X/R) - X/R) \cup \{t(R)\}$ is compact and countable, it contains an open countable discrete subspace. Since $\alpha(X/R) - X/R$ contains infinitely many components of K, Y must contain infinitely many components.

The converse of Corollary 3 is false when X is not locally compact. Example (A) shows that X/R can have a countable compactification, so that $\beta X - X$ has infinitely many components, but X has no countable compactification. Example (A) also shows that condition (B) of Theorem 1 is not sufficient to insure that X has a countable compactification when R is compact.

EXAMPLE (A). Let S be the closed unit square in R^2, I be the unit interval, $L_0 = I \times \{0\}$, and, for $n \in N$, $L_n = I \times \{\frac{1}{n+1}\}$. For $X = S - \bigcup_{n \in N} L_n$, it is clear that X is not rim compact, and hence does not have a countable compactification (cf. [6]). Furthermore, $R = L_0$ and S is a compactification of X. The existence of a continuous surjection from βX onto S which leaves X fixed and which carries $\beta X - X$ onto $S - X$ guarantees that condition (B) of Theorem 2 is satisfied. Hence X/R has a countable compactification.

The following example shows that for R non-empty and compact the implication of (C) by (B) of Theorem 1 cannot be reversed. It suffices to exhibit X, with R a singleton, where $X - R$ has a countable compactification but X does not.

EXAMPLE (B). In the plane R^2 take
$$X = [\{(x,y) \mid -1 < x < 1; -1 < y < 1\} \cup \{(1,0)\}] - \{(\frac{-n}{n+1}, 0) \mid n \in N\}. \text{ Then}$$
$R = \{(1,0)\}$. Since X is not rim compact, it has no countable compactification. However, a countable compactification for $X - R$ is obtained by adjoining the points $(\frac{-n}{n+1}, 0)$, for each $n \in N$, and taking the one-point compactification of the resulting space.

REFERENCES

1. Gillman, L. and Jerison, M. Rings of continous functions, The University Series in Higher Math., Princeton, N.J., 1960.

2. Magill, K. D., Jr. Countable compactifications, Canad. J. Math. 18 (1966), 616-620.

3. Mrowka, S. Continuous functions on countable subspaces, Port. Math. 29 (1970), 177-180.

4. Okuyama, A. A characterization of a space with countable infinity, Proc. A.M.S. 28 (1971), 595-597.

5. Rayburn, M. On Hausdorff compactifications, Pac. J. of Math. 44 (1973), 707-714.

6. Zippin, L. On semicompact spaces, Amer. J. Math. 57 (1935), 327-341.

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