ON THE OVERCONVERGENCE OF CERTAIN SERIES

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ABSTRACT. In this work, we consider certain class of exponential series with polynomial coefficients and study the properties of convergence of such series. Then we consider a subclass of this class and prove certain theorems on the overconvergence of such a series, which allow us to determine the conditions under which the boundary of the region of convergence of this series is a natural boundary for the function f defined by this series.

KEY WORDS AND PHRASES. LC-Dirichletian element, L-Dirichletian element, Convergence, Overconvergence.

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. 30A16, 30A64.

1. INTRODUCTION.

Let us consider the following LC-dirichletian element

$$\{f\} : \sum_{1}^{\infty} P_n(x) \exp -\lambda_n s \ , \tag{1.1}$$

where $P_n(s) = \sum_{j=0}^{m_n} a_{nj} s^j$, a_{nj} are complex constants with $a_{n,m_n} \neq 0$, $s = \sigma + i\tau$,

$(\sigma, \tau) \in \mathbb{R}^2$ and (λ_n) is a sequence of complex numbers such that $(|\lambda_n|)$ is a D-sequence. That is to say $(|\lambda_n|)$ is a sequence of positive real numbers satisfying

$$0 < |\lambda_1| < |\lambda_2| < \ldots , \lim_{n \to \infty} |\lambda_n| = \infty . \qquad (1.2)$$

Let
$$L = \limsup \left\{ \frac{\log n}{|\lambda_n|} / n \in \mathbb{N}-\{0\} \right\} \qquad (1.3)$$

$$A_n = \text{Max} \left\{ |a_{nj}| / j \in (0,1,\ldots,m_n) \right\} \qquad (1.4)$$

and
$$\beta^* = \limsup \left\{ \frac{m_n}{|\lambda_n|} / n \in \mathbb{N}-\{0\} \right\} . \qquad (1.5)$$

Let $\mathcal{E}_n$ be the set of points of $\mathbb{C}$ which are zeros of $P_n(s)$ and $\mathcal{E} = \bigcup_n \mathcal{E}_n$. Let us denote by $\mathcal{E}^d$ the derived set of $\mathcal{E}$ and $\mathcal{E}_\infty = \left\{ s \in \mathbb{C} \mid \underset{(n_j)}{\exists} P_{n_j}(s) = 0 \right\}$ where (n_j) is an infinite subsequence of $\mathbb{N}-\{0\}$ depending on s ; let $\mathcal{E}^* = \mathcal{E}^d \cup \mathcal{E}_\infty$. $\mathcal{E}^*$ is a closed set. Let us suppose that $\mathbb{C}-\mathcal{E}^*$ is non empty. We put

$$\underset{s \in \mathbb{C}-\mathcal{E}^*}{\forall} \quad \delta(n,s) = - \frac{\log |P_n(s) \exp(-\lambda_n s)|}{|\lambda_n|} , \text{ for sufficiently large } n ,$$

$$\delta_*(s) = \liminf \left\{ \delta_n(s) / n \in \mathbb{N}-\{0\} \right\}$$

$$\underset{\alpha \in \mathbb{R}}{\forall} \quad \mathcal{D}_{*\alpha} = \{ s \in \mathbb{C}-\mathcal{E}^* / \delta_*(s) > \alpha \} .$$

In this paper, using a technique similar to that used by M. Blambert and J. Simeon [2], we prove two lemmas for a LC-dirichletian element which enable us to discuss the properties of absolute convergence and uniform convergence for (1.1) in $\mathbb{C}-\mathcal{E}^*$ exclusively. Then we prove Jentzsch's theorem for a L-dirichletian element that is for element of the type (1.1) where λ_n are positive real numbers satisfying (1.2) $\left((\lambda_n) \text{ is a D-sequence} \right)$ and a theorem on the overconvergence for a L-dirichletian element.

2. MAIN RESULTS.

DEFINITION. - It is said that a function is sub-lipschitzian on an open set, if it is lipschitzian on each compact subset of that open set.

LEMMA 1. - Let $\mathcal{K}$ be any compact subset of $\mathbb{C}$. Then the following assertions are true.

(1) $\underset{\mathcal{K}\subset\mathbb{C}-\mathcal{C}^*}{\forall}\quad\underset{n'}{\exists}\quad\underset{n\geq n'}{\forall}$ the function $\mathcal{K}\ni s\to\delta(n,s)$ is lipschitzian.

(2) If $\beta^* < \infty$, and if there exists a $s_0 \in \mathbb{C}-\mathcal{C}^*$ such that $|\delta_*(s_0)| < \infty$ then the function δ_* is sub-lipschitzian on $\mathbb{C}-\mathcal{C}^*$.

PROOF. Let $\underset{\mathcal{K}\subset\mathbb{C}-\mathcal{C}^*}{\forall}\quad\epsilon_\mathcal{K} = \text{dist}(\mathcal{K},\mathcal{C}^*)$. Then it is easy to see that

$$\underset{s\in\mathcal{K}}{\forall}\quad\underset{\epsilon\in]0,\epsilon_\mathcal{K}[}{\forall}\quad\underset{n'}{\exists}\quad\underset{n\geq n'}{\forall}\quad\{j\in(1,2,\ldots,m_n)\Rightarrow\alpha_{nj}\notin d_{s,\epsilon}\}\ ,$$

where $d_{s,\epsilon}$ is the open disc centred at s and of radius ϵ and (α_{nj}) , $j \in \{1,2,\ldots, m_n\}$, is the sequence of zeros of $P_n(s)$ (with its order of mulplicity is taken into account). More precisely let us show that,

$$\underset{\epsilon\in]0,\epsilon_\mathcal{K}[}{\forall}\quad\underset{n'}{\exists}\quad\underset{n\geq n'}{\forall}\quad\underset{s\in\mathcal{K}}{\forall}\quad\{j\in(1,2,\ldots,m_n)\Rightarrow\alpha_{nj}\notin d_{s,\epsilon}\}\ .$$

Let $G_\epsilon = \underset{s\in\mathcal{K}}{\cup}d_{s,\epsilon}$. It is evident that $\overline{G}_\epsilon$ the closure of G_ϵ is a compact subset of $\mathbb{C}-\mathcal{C}^*$. Let $\epsilon' \ni]0,\epsilon_\mathcal{K}-\epsilon[$ where $\epsilon\in]0,\epsilon_\mathcal{K}[$. The set of discs $d_{s,\epsilon'}$ indexed by s on $\overline{G}_\epsilon$ is an open covering of $\overline{G}_\epsilon$. Hence we have a finite subcovering ;

$$\underset{\overline{G}_\epsilon\supset(s_1,\ldots,s_k)}{\exists}\quad\overset{k}{\underset{j=1}{\cup}}d_{s_j,\epsilon'}\supset\overline{G}_\epsilon\ .$$

Let $s \in \mathcal{K}$ and $s' \in d_{s,\epsilon}$; hence $s'\in G_\epsilon$. Then $s' \in \overset{k}{\underset{j=1}{\cup}}d_{s_j,\epsilon'}$ which implies that $\underset{j^*\in(1,\ldots,k)}{\exists}\quad s' \in d_{s_{j^*},\epsilon'}$. Now

$$\underset{j\in(1,\ldots,k)}{\forall}\quad\underset{n'(=n_j)}{\exists}\quad\underset{n\geq n'}{\forall}\quad\underset{s\in d_{s_j,\epsilon'}}{\forall}\quad P_n(s)\neq 0$$

and hence

$$\underset{n\geq\text{Max}\{n_j/j\in(1,\ldots,k)\}}{\forall}\quad\underset{j\in(1,2,\ldots,k)}{\forall}\quad\underset{s\in d_{s_j,\epsilon'}}{\forall}\quad P_n(s)\neq 0\ ,$$

which gives $\underset{n\geq\text{Max}\{n_j/j\in(1,\ldots,k)\}}{\forall}\quad P_n(s')\neq 0$. As s is arbitrary on $\mathcal{K}$ and s' is arbitrary on $d_{s,\epsilon}$ we have

$$\underset{\epsilon \in]0,\epsilon_{\mathcal{H}}[}{\forall} \quad \underset{n'(=n_\epsilon)}{\exists} \quad \underset{n \geq n'}{\forall} \quad \underset{s \in \mathcal{H}}{\forall} \quad \{j \in (1,\ldots,k) \Rightarrow \alpha_{nj} \notin d_{s,\epsilon}\} \ .$$

From which we have

$$\underset{n \geq n'(=n_\epsilon)}{\forall} \quad \underset{(s,s') \in \mathcal{H} \times \mathcal{H}}{\forall} \quad \log|P_n(s)| - \log|P_n(s')| \leq \sum_{j=1}^{m_n} \log\left\{1 + \frac{|s-s'|}{|s'-\alpha_{nj}|}\right\} \leq \sum_{j=1}^{m_n} \log\left\{1 + \frac{|s-s'|}{\epsilon}\right\}.$$

Under the above conditions related to n, s and s' with $s \neq s'$,

$$|\delta(n,s) - \delta(n,s')| \leq |s-s'| + \frac{1}{|\lambda_n|} \sum_{j=1}^{m_n} \log\left\{1 + \frac{|s-s'|}{\epsilon}\right\}$$

$$\leq |s-s'| + \frac{|s-s'|}{\epsilon|\lambda_n|} \sum_{j=1}^{m_n} \left\{\log(1 + \frac{|s-s'|}{\epsilon}) / \frac{|s-s'|}{\epsilon}\right\}$$

$$\leq |s-s'| + \frac{m_n |s-s'|}{\epsilon|\lambda_n|} \sup\left\{\frac{\log(1+x)}{x} / x > 0\right\} \ ;$$

as $\sup\left\{\dfrac{\log(1+x)}{x} / x > 0\right\} = 1$, $\quad |\delta(n,s) - \delta(n,s')| \leq |s-s'| \left\{1 + \dfrac{m_n}{\epsilon|\lambda_n|}\right\}$. Putting

$$\mu_{\epsilon,n} = 1 + \frac{m_n}{\epsilon|\lambda_n|} \ ,$$

$$\underset{\epsilon \in]0,\epsilon_{\mathcal{H}}[}{\forall} \quad \underset{n'}{\exists} \quad \underset{n \geq n'}{\forall} \quad \underset{(s,s') \in \mathcal{H} \times \mathcal{H}}{\forall} \quad |\delta(n,s) - \delta(n,s')| \leq \mu_{\epsilon,n} |s-s'| \ ,$$

which proves the first part of the lemma.

Now let $\mu_\epsilon^* = \underset{n \to \infty}{\lim \sup} \ \mu_{\epsilon,n} = 1 + \beta^*/\epsilon$ with $\epsilon \in]0,\epsilon_{\mathcal{H}}[$; as $\underset{s_0 \in \mathbb{C} - \mathcal{E}^*}{\exists} \ \delta_*(s_0) < \infty$

$$\underset{\epsilon \in]0,\epsilon_{\mathcal{H}}[}{\forall} \quad \underset{(s,s') \in \mathcal{H} \times \mathcal{H}}{\forall} \quad |\delta_*(s) - \delta_*(s')| \leq \mu_\epsilon^* |s-s'|$$

and

$$\underset{(s,s') \in \mathcal{H} \times \mathcal{H}}{\forall} \quad |\delta_*(s) - \delta_*(s')| \leq \mu_{\epsilon_{\mathcal{H}}}^* |s-s'|$$

where

$$\mu_{\epsilon_{\mathcal{H}}}^* = \text{Inf}\{\mu_\epsilon^* / \epsilon \in]0,\epsilon_{\mathcal{H}}[\} = 1 + \frac{\beta^*}{\epsilon_{\mathcal{H}}} \ .$$

Hence

$$\underset{\mathcal{H} \supset \mathbb{C} - \mathcal{E}^*}{\forall} \quad \underset{(s,s') \in \mathcal{H} \times \mathcal{H}}{\forall} \quad |\delta_*(s) - \delta_*(s')| \leq \mu_{\epsilon_{\mathcal{H}}}^* |s-s'| \ ,$$

which completes the proof of the lemma.

Under the condition (2) of Lemma 1, δ_* is continuous on $\mathbb{C} - \mathcal{E}^*$

which implies that $\mathcal{D}_{*\alpha}$ is an open subset of $\mathbb{C}-\mathcal{E}^*$; but $\mathcal{D}_{*\alpha}$ can have several connected components.

LEMMA 2. - When $\beta^* < \infty$, then

$$\underset{\alpha\in\mathbb{R}}{\forall}\left\{\mathcal{D}_{*\alpha}\neq\phi\Rightarrow\underset{\mathcal{K}\subset\mathcal{D}_{*\alpha}}{\forall}\underset{\beta'>\beta^*}{\forall}\underset{n'}{\exists}\underset{n\geq n'}{\forall}\underset{s\in\mathcal{K}}{\forall}\ |P_n(s)\exp(-\lambda_n s)| < \exp(-|\lambda_n|(\alpha-\beta'))\right\}\ .$$

PROOF. Let $\alpha\in\mathbb{R}$ such that $\mathcal{D}_{*\alpha}\neq\phi$ (otherwise the lemma is trivial) and let $\mathcal{K}$ be a compact subset of $\mathcal{D}_{*\alpha}$. We can easily see that

$$\underset{s\in\mathbb{C}-\mathcal{E}^*}{\forall}\underset{\epsilon\in]0,dist(s,\mathcal{E}^*)[}{\forall}\underset{n'(=n_{s,\epsilon})}{\exists}\underset{n\geq n'}{\forall}\underset{s'\in\overline{d_{s,\epsilon}}}{\forall}\ P_n(s')\neq 0$$

where $\overline{d_{s,\epsilon}}$ is the closed disc centred at s and of radius ϵ . Hence

$$\underset{s\in\mathbb{C}-\mathcal{E}^*}{\forall}\underset{\epsilon\in]0,\epsilon_{\mathcal{K}}[}{\forall}\underset{n'(=n_{s,\epsilon})}{\exists}\underset{n\geq n'}{\forall}\underset{s'\in\overline{d_{s,\epsilon}}}{\forall}\ P_n(s')\neq 0\ .$$

Let us consider the compact subset $\mathcal{K}_\epsilon = \overline{\underset{s\in\mathcal{K}}{\cup}d_{s,\epsilon}}$, of $\mathbb{C}-\mathcal{E}^*$. As $dist(\mathcal{K}_\epsilon,\mathcal{E}^*) > 0$, we have from lemma 1,

$$\underset{\epsilon'\in]0,dist(\mathcal{K}_\epsilon,\mathcal{E}^*)[}{\forall}\underset{n'(=n_{\epsilon'})}{\exists}\underset{n\geq n'}{\forall}\underset{(s,s')\in\mathcal{K}_\epsilon\times\mathcal{K}_\epsilon}{\forall}\ |\delta(n,s)-\delta(n,s')| \leq \mu_{\epsilon',n}|s-s'|$$

where $\mu_{\epsilon',n} = 1 + \dfrac{m_n}{\epsilon'|\lambda_n|}$. In particular,

$$\underset{n\geq n_{\epsilon'}}{\forall}\underset{s\in\mathcal{K}}{\forall}\underset{s'\in d_{s,\epsilon}}{\forall}\ |\delta(n,s)-\delta(n,s')| \leq \mu_{\epsilon',n}|s-s'|\ ,$$

and hence

$$\delta(n,s') \geq \delta(n,s) - \mu_{\epsilon',n}|s-s'|\ .$$

Further $\underset{\beta'>\beta^*}{\forall}\underset{n'(=n_{\beta'})}{\exists}\underset{n\geq n'}{\forall}\ \dfrac{m_n}{|\lambda_n|} < \beta'$ and

$$\underset{n\geq\max\{n_{\epsilon'},n_{\beta'}\}=n_1}{\forall}\underset{s\in\mathcal{K}}{\forall}\underset{s'\in d_{s,\epsilon}}{\forall}\ \delta(n,s') \geq \delta(n,s) - |s-s'|(1+\tfrac{\beta'}{\epsilon'})\ .$$

Since $\mathcal{K}$ is a compact subset of $\mathcal{D}_{*\alpha}$, $\underset{s\in\mathcal{K}}{\forall}\underset{n'(=n_s)}{\exists}\underset{n\geq n'}{\forall}\ \delta(n,s') > \alpha$; finally we have

$$\underset{s\in\mathcal{K}}{\forall}\underset{\beta'>\beta^*}{\forall}\underset{\epsilon'\in]0,dist(\mathcal{E}^*,\mathcal{K}_\epsilon)[}{\forall}\underset{n'}{\exists}\underset{n\geq n'}{\forall}\underset{s'\in d_{s,\epsilon}}{\forall}\ \delta(n,s') > \alpha - |s-s'|(1+\tfrac{\beta'}{\epsilon'}) \qquad (2.1)$$

where ϵ is arbitrary in $]0,\epsilon_{\mathcal{K}}[$. The set of discs $d_{s,\epsilon}$ indexed by s on $\mathcal{K}$ is an open covering for $\mathcal{K}$ and hence $\underset{\mathcal{K}\supset(s_1...s_k)}{\exists} \overset{k}{\underset{j=1}{\cup}} d_{s_j,\epsilon} \supset \mathcal{K}$. Further we have $\underset{s\in\mathcal{K}}{\forall} \underset{j'\in(1...k)}{\exists} s \in d_{s_{j'},\epsilon}$. Using (2.1) for the particular pair $(s_{j'},s)$, we have

$$\underset{\beta'>\beta^*}{\forall} \underset{\epsilon'\in]0,dist(\mathcal{C}^*,\mathcal{K}_\epsilon)[}{\forall} \underset{n'(=n_{s_{j'},\beta',\epsilon'})}{\exists} \underset{n\geq n'}{\forall} \delta(n,s) > \alpha - |s-s_{j'}|(1+\frac{\beta'}{\epsilon'}) .$$

Let $n'' = Max\{n_{s_j,\beta',\epsilon'}|j\in(1...k)\}$ and as $|s-s_j| < \epsilon$, we have

$$\underset{\beta'>\beta^*}{\forall} \underset{\epsilon'\in]0,dist(\mathcal{C}^*,\mathcal{K}_\epsilon)[}{\forall} \underset{n''}{\exists} \underset{n\geq n''}{\forall} \delta(n,s) > \alpha - \epsilon(1+\frac{\beta'}{\epsilon'}) .$$

Choosing $\epsilon = \epsilon' < \frac{\epsilon_\mathcal{K}}{2}$ we have $\frac{dist(\mathcal{K},\mathcal{C}^*)}{2} < dist(\mathcal{C}^*,\mathcal{K}_\epsilon)$ and

$$\underset{\beta'>\beta^*}{\forall} \underset{\epsilon\in]0,\frac{\epsilon_\mathcal{K}}{2}[}{\forall} \underset{n''}{\exists} \underset{n\geq n''}{\forall} \delta(n,s) > \alpha - \epsilon - \beta'$$

where s is any arbitrary point of $\mathcal{K}$ and n'' does not depend on s . Hence

$$\underset{\beta'>\beta^*}{\forall} \underset{\epsilon\in]0,\frac{\epsilon_\mathcal{K}}{2}[}{\forall} \underset{n''}{\exists} \underset{n\geq n''}{\forall} \underset{s\in\mathcal{K}}{\forall} \delta(n,s) > \alpha-\epsilon-\beta' .$$

As β' is arbitrary and strictly greater than β^* , we have

$$\underset{\mathcal{K}\subset\mathcal{D}_{*\alpha}}{\forall} \underset{\beta'>\beta^*}{\forall} \underset{n'}{\exists} \underset{n\geq n'}{\forall} \underset{s\in\mathcal{K}}{\forall} \delta(n,s) > \alpha-\beta'$$

and hence

$$\underset{\mathcal{K}\subset\mathcal{D}_{*\alpha}}{\forall} \underset{\beta'>\beta^*}{\forall} \underset{n'}{\exists} \underset{n\geq n'}{\forall} \underset{s\in\mathcal{K}}{\forall} |P_n(s)\exp-\lambda_n s| < \exp(-|\lambda_n|(\alpha-\beta')) .$$

THEOREM 1. - When $\beta^* < \infty$, $L < \infty$, the LC-dirichletian element $\{f\}$ converges absolutely on $\mathcal{D}_{*,L+\beta^*}$ and uniformly on any compact subset of $\mathcal{D}_{*,L+\beta^*}$.

PROOF. Let us suppose that $\mathcal{D}_{*,L+\beta^*}$ is non empty. Let $\mathcal{K}_0$ be a compact subset of $\mathcal{D}_{*,L+\beta^*}$. We know that $\underset{\alpha>L+\beta^*}{\exists} \mathcal{K}_0 \subset \mathcal{D}_{*\alpha}$. Let

$\beta' \in]\beta^*, \alpha - L[$. From Lemma 2 we have,

$$\underset{n'}{\exists} \quad \underset{n \geq n'}{\forall} \quad \underset{s \in \mathcal{K}_o}{\forall} \quad |P_n(s) \exp(-\lambda_n s)| < \exp\{-|\lambda_n|(\alpha - \beta')\} \ ,$$

where $\alpha - \beta' > L$. Hence

$$\sum_{n=n'}^{\infty} |P_n(s) \exp(-\lambda_n s)| < \sum_{n=n'}^{\infty} \exp\{-|\lambda_n|(\alpha - \beta')\}$$

and the series on the right hand side is convergent which proves that $\{f\}$ converges absolutely and uniformly on $\mathcal{K}_o$. Since $\mathcal{K}_o$ is any arbitrary compact subset of $\mathcal{B}_{*,L+\beta^*}$, $\{f\}$ converges uniformly on any compact subset of $\mathcal{B}_{*,L+\beta^*}$ and absolutely on $\mathcal{B}_{*,L+\beta^*}$.

REMARK 1. By the following method, we obtain a bigger set of absolute convergence for $\{f\}$. Let $\mathcal{B}_{*L}$ be supposed to be non-empty and $L < \infty$. Then $\underset{s \in \mathcal{B}_{*L}}{\forall} \underset{\epsilon_s > 0}{\exists} \delta_*(s) > L + \epsilon_s$; $\underset{n'_s}{\exists} \underset{n \geq n'_s}{\forall} \delta(n,s) > L + \epsilon_s$ and

$\underset{n \geq n'_s}{\forall} -\text{Log}|P_n(s) \exp(-\lambda_n s)| > (L + \epsilon_s)|\lambda_n|$. Hence

$$\sum_{n=n'_s}^{\infty} |P_n(s) \exp - \lambda_n s| < \sum_{n=n'_s}^{\infty} \exp\{-(L + \epsilon_s)|\lambda_n|\}$$

and as the series on the right hand side converges, the series (1.1) converges absolutely on $\mathcal{B}_{*L}$. In this result, we have no restriction on β^* .

REMARK 2. $\{f\}$ diverges on $\mathbb{C} - \mathcal{E}^* - \overline{\mathcal{B}}_{*o}$. If $s \in \mathbb{C} - \mathcal{E}^* - \overline{\mathcal{B}}_{*o}$, then $\delta_*(s) < 0$ and $\underset{\alpha \in \mathbb{R}_o^+}{\exists} \delta_*(s) < -\alpha$. Hence $\underset{s \in \mathbb{C} - \mathcal{E}^* - \overline{\mathcal{B}}_{*o}}{\forall} \underset{\alpha > 0}{\exists} \underset{(n_j)}{\exists} \delta(n_j, s) < -\alpha$ where (n_j) is an infinite subsequence of $\mathbb{N} - \{0\}$. Therefore

$$|P_{n_j}(s) \exp(-\lambda_{n_j} s)| > \exp(\alpha|\lambda_{n_j}|) > 1$$

and which shows that $\{f\}$ diverges on $\mathbb{C} - \mathcal{E}^* - \overline{\mathcal{B}}_{*o}$. When $L = 0$, we have convergence of the series (1.1) in $\mathcal{B}_{*o} \subset \mathbb{C} - \mathcal{E}^*$ and divergence in $\mathbb{C} - \mathcal{E}^* - \overline{\mathcal{B}}_{*o}$. We do not discuss the property of convergence of the series in $\mathcal{E}^*$.

From here onwards we consider a L-dirichletian element,

$$\{f\} : \sum_1^\infty P_n(s) \exp(-\lambda_n s) \tag{2.2}$$

where (λ_n) is a D-sequence (here λ_n are positive real numbers).

DEFINITION. It is said that a D-sequence (λ_n) is of the type (Λ) if the following conditions are satisfied :

i) the Dirichlet series $\sum_{j=1}^\infty \exp(-\lambda_n s)$ converges on $P_o = \{s \in \mathbb{C} \mid \sigma > 0\}$.

(this gives that $\underset{n \in \mathbb{N}-\{0\}}{\forall} \sum_{j=n}^\infty \exp(-s(\lambda_j - \lambda_n))$ converges on P_o . Let $\theta_n(s)$ be its sum at the point s);

ii) $\forall \eta > 0$ the sequence of functions (θ_n) where $\theta_n : P_o \ni s \to \theta_n(s)$ is bounded on $\overline{P}_\eta = \{s \in \mathbb{C} / \sigma \geq \eta\}$;

iii) $\forall \eta > 0$ the sequence of functions (θ_n^*) where

$\theta_n^* : P_o \ni s \to \sum_{j=1}^n \exp(-s(\lambda_n - \lambda_j))$ is bounded on $\overline{P}_\eta$.

EXAMPLE.- If (λ_n) is a D-sequence and $\underset{\mu > 0}{\exists} \operatorname{Inf}(\lambda_{n+1} - \lambda_n) = \mu$, then it is easy to see that (λ_n) is of the type (Λ) .

If the D-sequence (λ_n) is of the type (Λ) , then we can easily show that $L = 0$.

Now let us prove Jentzsch's theorem for L-dirichletian element. This theorem for Dirichlet series with complex exponents was proved by T.M. Gallie [3]. First let us consider the associated Dirichlet series of $\{f\}$.

$$\{f_A\} : \sum_1^\infty A_n \exp(-\lambda_n s)$$

where A_n is defined by (1.4). Let

$$\sigma_p^{f_A} = \operatorname{Inf}\{\sigma \in \mathbb{R} / \lim |A_n \exp(-\lambda_n s)| = 0 , n \to \infty\}$$

be the abscisse of pseudo convergence of $\{f_A\}$. Then we know that

$$\sigma_p^{f_A} = \limsup_{n \to \infty} \{\frac{\log A_n}{\lambda_n}\} ;$$

when $L = 0$, $\sigma_p^{f_A}$ is the same as $\sigma_c^{f_A}$, the abscisse of convergence of $\{f_A\}$.

Let n and n' be two natural numbers such that $n' \geq n$. Let $E_{n,n'}$ denote the set, indexed by (n,n'), of points of $\mathbb{C}$ which are zeros of the LC-dirichletian polynomial

$$S_{n,n'}(S) = \sum_{j=n}^{n'} P_j(s) \exp(-s\lambda_j) \; ;$$

let E denote the union of all sets $E_{n,n'}$ corresponding to all pairs (n,n') and E_∞ be the set formed by the points which are zeros for an infinity of polynomials $S_{n,n'}(s)$. Let us put $E^* = E^d \cup E_\infty$ where E^d is the derived set of E. E^* is a closed subset of $\mathbb{C}$. It is evident that $E \supset \mathcal{E}$ and $E_\infty \supset \mathcal{E}_\infty$ and hence $E^* \supset \mathcal{E}^*$. We suppose in what follows that $\mathbb{C}-E^* \neq \phi$ (which implies $\mathbb{C}-\mathcal{E}^* \neq \phi$). Then we have

THEOREM 2. - When the D-sequence (λ_n) is of the type (Λ), $\sigma_c^{f_A} < \infty$ and $\beta^* < \infty$, then we have $(Fr(\mathcal{B}_{*0}) \cap \mathbb{C}-\mathcal{E}^*) \subset E^*$.

PROOF. Let us suppose that the theorem is not true. Then there exists a point $b \in (Fr(\mathcal{B}_{*0}) \cap \mathbb{C}-\mathcal{E}^*)$ and a disc $d(b,\rho)$ centred at b of radius $\rho > 0$, included in $\mathbb{C}-\mathcal{E}^*$ such that

$$\underset{n_0}{\exists} \; \underset{n' \geq n \geq n_0}{\forall} \; \underset{s \in d(b,\rho)}{\forall} \; S_{n,n'}(s) \neq 0 \; .$$

We have $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+|s|)^{m_n} |\exp(-\lambda_n s)|$ and

$$\underset{n}{\forall} \; \underset{}{\exists} \; \underset{}{\forall} \; (m_n/\lambda_n) < \beta' \; .$$ Let us take a certain $\beta' > \beta^*$ and put $\beta' > \beta^* \; n_0'(=n_{\beta'}) \; n \geq n_0'$

$\omega = \beta' \, Log[1+\sup\{|s|/s \in d(b,\rho)\}] - Inf\{\sigma/s \in d(b,\rho)\}$ and hence

$$\underset{n \geq n_0'}{\forall} \; \underset{s \in d(b,\rho)}{\forall} \; |P_n(s)\exp(-\lambda_n s)| < A_n \exp(\omega\lambda_n) \; .$$ From the definition of $\sigma_c^{f_A}$ we

have $\underset{\sigma' > \sigma_c^{f_A}}{\forall} \; \underset{n_0''(=n_{\sigma'})}{\exists} \; \underset{n \geq n_0''}{\forall} \; A_n < \exp(\sigma'\lambda_n)$. Hence putting $n_1 = Max(n_0, n_0', n_0'')$,

we get $\underset{n \geq n_1}{\forall} \; \underset{s \in d(b,\rho)}{\forall} \; |P_n(s)\exp(-\lambda_n s)| < \exp((\omega+\sigma')\lambda_n)$.

Let $S_{n_1,n}(s) = \sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s)$ and $\underset{s \in d(b,\rho)}{\forall} \; T_{n_1,n}(s) = (S_{n_1,n}(s))^{1/\lambda_n}$;

$[S_{n_1,n}(s)]^{1/\lambda_n}$ is defined to be equal to $\exp((1/\lambda_n) \, Log \, S_{n_1,n}(s))$ where

Im Log $S_{n_1,n}(s) \in]-\pi, \pi]$. For each integer $n \geq n_1$ the function

$T_{n_1,n} : d(b,\rho) \ni s \to T_{n_1,n}(s)$ is holomorphic on $d(b,\rho)$. We have

$$\underset{s \in d(b,\rho)}{\forall} |T_{n_1,n}(s)| = |(\sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s))^{1/\lambda_n}| \leq \{\exp(\lambda_n(\omega+\sigma') + \log n)\}^{1/\lambda_n}$$

$$= \exp(\omega+\sigma')\exp(\frac{\log n}{\lambda_n}) .$$

Since (λ_n) is of the type (Λ) which implies $L = 0$, we have $\underset{n \to \infty}{\lim}\exp(\frac{\log n}{\lambda_n}) = 1$.
Hence the sequence of functions $(T_{n_1,n})$, $n \geq n_1$, is bounded and hence normal on $d(b,\rho)$.

Let $\varkappa$ be a compact subset of $d(b,\rho)$ such that Int $\varkappa \cap \mathcal{B}_{*0} \neq \varphi$.
From any extracted subsequence of $(T_{n_1,n})$ we can extract a subsequence which converges uniformly on $\varkappa$ and the limit function is holomorphic on the Int $\varkappa$.

Let $\varkappa_1$ be a compact subset of $d(b,\rho) \cap \mathcal{B}_{*0}$ such that Int $\varkappa \cap$ Int $\varkappa_1 \neq \varphi$.
Then we have $\underset{s \in \varkappa_1}{\forall} \underset{n \to \infty}{\lim} T_{n_1,n} = 1$. Now $\varkappa \cup \varkappa_1$ is a compact subset of
$d(b,\rho)$. Then the subsequence extracted from the arbitrarily extracted subsequence of $(T_{n_1,n})$ converges uniformly on $\varkappa \cup \varkappa_1$ to a limit function holomorphic in Int$(\varkappa \cup \varkappa_1)$ and continuous on the boundary of $\varkappa \cup \varkappa_1$ and takes the value one at each point of $\varkappa_1$. Hence the limit function takes the value one at each point of $\varkappa \cup \varkappa_1$. This results that the sequence $(T_{n_1,n})$ converges to the same limit function on $\varkappa \cup \varkappa_1$.

As $\varkappa$ is any arbitrary compact subset of $d(b,\rho)$ and $\varkappa_1$ is any arbitrary compact subset of $d(b,\rho) \cap \mathcal{B}_{*0}$ such that Int $\varkappa \cap$ Int $\varkappa_1 \neq \varphi$, we have

$$\underset{s \in d(b,\rho)}{\forall} \underset{n \to \infty}{\lim} T_{n_1,n}(s) = 1 .$$

Let $s_0 \in d(b,\rho) \cap (\mathbb{C}-\varepsilon^*-\overline{\mathcal{B}}_{*0})$. Then

$$\underset{\varepsilon > 0}{\forall} \underset{n'_1(=n_{s_0,\varepsilon}) \geq n_1}{\exists} \underset{n \geq n'_1}{\forall} |\sum_{j=n_1}^{n} P_j(s)\exp(-\lambda_j s)| < (1+\varepsilon)^{\lambda_n}$$

and hence

$$\underset{n>n_1'}{\forall} \quad |P_n(s_o)\exp(-\lambda_n s_o)| = |S_{n_1,n}(s_o) - S_{n_1,n-1}(s_o)| < 2(1+\epsilon)^{\lambda_n}$$

which gives

$$-\frac{\text{Log}\,|P_n(s_o)\exp(-\lambda_n s_o)|}{\lambda_n} > \frac{-\text{Log}\,2}{\lambda_n} - \text{Log}(1+\epsilon) \; ;$$

$\delta_*(s_o) \geq 0$ as ϵ is arbitrary. Hence we arrive at a contradiction that $s_o \in \overline{\mathcal{D}}_{*o} \cap \mathbb{C}\text{-}\mathcal{E}^*$ which establishes the result.

Finally, let us prove a theorem on the overconvergence of $\{f\}$ defined by (7). Before proving the theorem let us note that

REMARK 3. Let $\overline{\Delta}$ be any compact subset of $\mathbb{C}\text{-}\mathcal{E}^*$ and (λ_n) be a D-sequence of the type (Λ). We have $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+|s|)^{m_n}\exp(-\sigma\lambda_n)$. If $s \in \overline{\Delta}$, then $|P_n(s)\exp(-\lambda_n s)| \leq A_n(1+m_\Delta)^{m_n}\exp(m_\Delta\lambda_n)$ where $m_\Delta = \sup\{|s|/s \in \overline{\Delta}\}$. As $\overline{\Delta}$ is a compact set, m_Δ is finite ; for sufficiently large n we have

$$\frac{\text{Log}\,|P_n(s)\exp(-\lambda_n s)|}{\lambda_n} \leq \frac{\text{Log}\,A_n}{\lambda_n} + \frac{m_n}{\lambda_n}\text{Log}(1+m_\Delta)+m_\Delta \; ;$$

$$\delta_*(s) \geq -\sigma_c^{f_A} - \beta^*\,\text{Log}(1+m_\Delta) - m_\Delta \; .$$

Hence $\underset{\epsilon>0}{\forall}\; \overline{\Delta} \subset \mathcal{D}_{*,\alpha_o-\epsilon}$ with $\alpha_o = -\sigma_c^{f_A} - \beta^*\,\text{Log}(1+m_\Delta) - m_\Delta$. If

$$\beta^* < \frac{-\sigma_c^{f_A} - m_\Delta}{1+\text{Log}(1+m_\Delta)} \; , \text{ we have } \overline{\Delta} \subset \mathcal{D}_{*\beta^*} \; .$$

THEOREM 3. - When (λ_n) is a D-sequence of the type (Λ), $\beta^* < \infty$ and $\mathcal{D}_{*\beta^*} \neq \phi$ if there exist an infinite subsequence (n_ν), $\nu \in \mathbb{N}$, of $\mathbb{N}-\{0\}$ and a sequence of strictly positive numbers (θ_ν) such that

$$\lim_{\nu\to\infty} \theta_\nu = +\infty$$

and

$$\underset{\nu\in\mathbb{N}}{\forall}\quad \lambda_{n_\nu+1} > (1+\theta_\nu)\lambda_{n_\nu} \tag{2.3}$$

then the sequence $\{S_{n_\nu}(s)\}$, $\nu \in \mathbb{N}$, where $S_{n_\nu}(s) = \sum_{j=1}^{n_\nu} P_j(s) \exp(-\lambda_j s)$, conver-ges at each point s of any open simply connected subset (whose intersection with $\mathcal{b}_{*,\beta*}$ in non empty) of an open set included in $\mathbb{C}-\mathcal{E}^*$ in which the function f defined by $\{f\}$ is holomorphic.

PROOF. Let us choose 3 bounded domains Δ_1, Δ_2 and Δ_3 in the following manner : $\bar{\Delta}_1 \subset \Delta_2$, $\bar{\Delta}_2 \subset \Delta_3$, $\bar{\Delta}_3 \subset \mathbb{C}-\mathcal{E}^*$, $\bar{\Delta}_1 \subset \mathcal{b}_{*\beta*}$ and $\bar{\Delta}_3$ is included in an open subset of $\mathbb{C}-\mathcal{E}^*$ in which the function f defined by $\{f\}$ is holomorphic. Further let $Fr(\Delta_1)$, $Fr(\Delta_2)$ and $Fr(\Delta_3)$ satisfy a condition of Hadamard's type, namely

$$\underset{b\in]0,1[}{\exists} \quad Log\, M_2 \le b\, Log\, M_1 + (1-b)\, Log\, M_3$$

where $\underset{i\in(1,2,3)}{\forall} \quad M_i = Max\{\,|f(s)|\,/s\in Fr(\Delta_i)\}$.

It is easy to see that $\underset{\alpha>\beta*}{\exists} \quad \bar{\Delta}_1 \subset \mathcal{b}_{*\alpha}$. Let us consider the set $I_1 = \{\alpha/\mathcal{b}_{*\alpha} \supset \bar{\Delta}_1\}$. I_1 is non empty and is an interval. Let $\alpha_{\Delta_1} = \sup I_1$. Then $\alpha_{\Delta_1} > \beta^*$ and $\underset{\epsilon>0}{\forall} \mathcal{b}_{*,\alpha_{\Delta_1}-\epsilon} \supset \bar{\Delta}_1$. We can easily show that, $\alpha_{\Delta_1} = Inf\{\delta_*(s)|s\in\bar{\Delta}_1\}$ which implies that α_{Δ_1} is a finite number. Hence from lemma 2,

$$\underset{\beta'\in]\beta*,\alpha_{\Delta_1}[}{\forall} \quad \underset{n_1}{\exists} \quad \underset{n\ge n_1}{\forall} \quad \underset{s\in Fr(\Delta_1)}{\forall} \quad |P_n(s)\exp(-\lambda_n s)| < \exp(-\lambda_n(\alpha_{\Delta_1}-\beta')) \; ;$$

hence for $n \ge n_1$

$$\sum_{j=n+1}^{\infty} |P_j(s)\exp(-\lambda_j s)| < \sum_{j=n+1}^{\infty} \exp\{-\lambda_j(\alpha_{\Delta_1}-\beta^*)\}$$

$$= \exp\{-\lambda_{n+1}(\alpha_{\Delta_1}-\beta')\} \sum_{j=n+1}^{\infty} \exp\{-(\alpha_{\Delta_1}-\beta')(\lambda_j-\lambda_{n+1})\} \; .$$

Since (λ_n) is a D-sequence of the type (Λ) and $\alpha_{\Delta_1}-\beta' > 0$, there exists a finite number strictly positive $B(\beta')$ such that

$$\underset{n\in\mathbb{N}}{\forall} \sum_{j=n+1}^{\infty} |\exp\{-(\lambda_j-\lambda_{n+1})s| \le B(\beta') \quad \text{where} \quad \Re e s \ge \alpha_{\Delta_1}-\beta' \; ;$$

thus we have for each $n \ge n_1$

$$\sum_{j=n+1}^{\infty} |P_j(s)\exp(-\lambda_j s)| < B(\beta')\exp\{-\lambda_{n+1}(\alpha_{\Delta_1} - \beta')\} \ . \tag{2.4}$$

Now let $I_2 = \{\alpha \in \mathbf{R} \mid \mathcal{D}_{*\alpha} \supset \bar{\Delta}_3\}$. We have

$$\underset{s\in\bar{\Delta}_3}{\forall} \quad \underset{n'(=n_s)}{\exists} \quad \underset{n\geq n'}{\forall} \quad \delta(n,s) \geq \frac{-\operatorname{Log} A_n}{\lambda_n} - \frac{m_n}{\lambda_n}\operatorname{Log}(1+|s|) + \sigma \ .$$

Let $m_{\Delta_3} = \sup\{|s| / s \in \bar{\Delta}_3\}$. Then

$$\underset{n\geq n'}{\forall} \quad \delta(n,s) \geq \frac{-\operatorname{Log} A_n}{\lambda_n} - \frac{m_n}{\lambda_n}\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3}$$

$$\delta_*(s) \geq -\sigma_c^{f_A} - \beta^*\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3} \ ,$$

which shows that $\bar{\Delta}_3 \subset \mathcal{D}_{*\alpha}$ with $\alpha < -\sigma_c^{f_A} - \beta^*\operatorname{Log}(1+m_{\Delta_3}) - m_{\Delta_3}$, and hence $I_2 \neq \phi$ and is an interval in $\mathbf{R}$. Let $\alpha_{\Delta_3} = \sup I_2$. Then $\underset{\epsilon>0}{\forall} \mathcal{D}_{*,\alpha_{\Delta_3}-\epsilon} \supset \bar{\Delta}_3$. We can easily show that $\alpha_{\Delta_3} = \operatorname{Inf}\{\delta_*(s) \mid s \in \bar{\Delta}_3\}$ which implies that α_{Δ_3} is a finite number. Once again, from lemma 2, we get

$$\underset{\beta'>\beta^*}{\forall} \quad \underset{n_2}{\exists} \quad \underset{n\geq n_2}{\forall} \quad \underset{s\in\operatorname{Fr}(\Delta_3)}{\forall} \quad |P_n(s)\exp(-\lambda_n s)| < \exp\{-\lambda_n(\alpha_{\Delta_3} - \beta')\}$$

which gives

$$\underset{s\in\operatorname{Fr}(\Delta_3)}{\forall} \quad \overset{n(\geq n_2)}{\underset{j=1}{\sum}} |P_j(s)\exp(-\lambda_j s)| = \overset{n_2-1}{\underset{j=1}{\sum}} |P_j(s)\exp(-\lambda_j s)| + \overset{n}{\underset{j=n_2}{\sum}} |P_j(s)\exp(-\lambda_j s)|$$

$$\leq \max\left\{ \overset{n_2-1}{\underset{j=1}{\sum}} |P_j(s)\exp(-\lambda_j s)| / s\in\operatorname{Fr}(\Delta_3) \right\} + \overset{n}{\underset{j=n_2}{\sum}} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) \ .$$

Let us choose $\beta' > \beta^*$ such that $\alpha_{\Delta_3} - \beta' \neq 0$. Now we examine the two cases.

Case 1. - If $\alpha_{\Delta_3} - \beta' > 0$, then

$$\overset{n}{\underset{j=n_2}{\sum}} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) = \exp(\lambda_n(\alpha_{\Delta_3} - \beta'))\overset{n}{\underset{j=n_2}{\sum}} \exp\{-(\alpha_{\Delta_3} - \beta')(\lambda_j + \lambda_n)\} < B''(\beta')\exp(\lambda_n(\alpha_{\Delta_3} - \beta'))$$

where $B''(\beta')$ is the sum of the series $\overset{\infty}{\underset{j=0}{\sum}} \exp(-2(\alpha_{\Delta_3} - \beta')\lambda_j)$.

<u>Case 2</u>. - If $\alpha_{\Delta_3} - \beta' < 0$, then

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) = \sum_{j=n_2}^{n} \exp(\lambda_j |\alpha_{\Delta_3} - \beta'|) = \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|) \sum_{j=n_2}^{n} \exp(-(\lambda_n - \lambda_j) |\alpha_{\Delta_3} - \beta'|)$$

Since the D-sequence (λ_n) is of the type (Λ) there exists a finite number strictly positive $B'(\beta')$ such that

$$\underset{n \in \mathbb{N} - \{0\}}{\forall} \sum_{j=1}^{n} \exp\{-(\lambda_n - \lambda_j) |\alpha_{\Delta_3} - \beta'|\} \leq B'(\beta')$$

which implies that

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) \leq B'(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|) \ .$$

On putting $B'''(\beta') = \text{Max}\{B'(\beta'), B''(\beta')\}$ we have

$$\sum_{j=n_2}^{n} \exp(-\lambda_j(\alpha_{\Delta_3} - \beta')) \leq B'''(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|) \ . \qquad (2.5)$$

Using the generalized form of Hadamard three circle theorem [4] we have

$$\underset{b \in]0,1[}{\exists} \quad \text{Log } M_{2,\nu} \leq b \, \text{Log } M_{1,\nu} + (1-b) \text{Log } M_{3,\nu} \qquad (2.6)$$

where

$$M_{i,\nu} = \text{Max}\{\, |R_{n_\nu}(s)| \, / \, s \in \text{Fr}(\Delta_i)\} \ , \quad i = 1,2,3$$

with

$$R_{n_\nu}(s) = f(s) - \sum_{j=1}^{n_\nu} P_j(s) \exp(-\lambda_j s) \ .$$

From (2.4) we have for $n_\nu \geq n_1$

$$M_{1,\nu} \leq B(\beta') \exp\{-(\lambda_{n_\nu + 1})(\alpha_{\Delta_1} - \beta')\} < B(\beta') \exp\{-(1 + \theta_\nu)\lambda_{n_\nu}(\alpha_{\Delta_1} - \beta')\} \quad (2.7)$$

because of (2.3) . On putting

$$B_o = \text{Max}\{\, |f(s)| / s \in \text{Fr}(\Delta_3)\} + \text{Max}\left\{\sum_{j=1}^{n_2 - 1} |P_j(s) \exp(-\lambda_j s)| \, / s \in \text{Fr}(\Delta_3)\right\}$$

we have from (2.5) for $n_\nu \geq n_2$,

$$M_{3,\nu} \leq B_o + B'''(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|) \ .$$

Let $B'_o(\beta') = \text{Max}(B_o, B'''(\beta'))$. Then for $n_\nu \geq n_2$,

$$M_{3,\nu} \leq B'_0(\beta') \exp(\lambda_n |\alpha_{\Delta_3} - \beta'|) . \tag{2.8}$$

Then using (2.7) and (2.8) in (2.6) we get, for $n_\nu \geq \max\{n_1, n_2\}$

$$\text{Log } M_{2,\nu} \leq b \text{ Log } B(\beta') + (1-b) \text{Log } B'_0(\beta') + \{-b(1+\theta_\nu)(\alpha_{\Delta_1} - \beta') + (1-b) |\alpha_{\Delta_3} - \beta'|\} \lambda_{n_\nu}$$

Since $\lim_{\nu \to \infty} \theta_\nu = \infty$, we have $\lim -b(1+\theta_\nu)(\alpha_{\Delta_1} - \beta') + (1-b) |\alpha_{\Delta_3} - \beta'| = -\infty$, $\nu \uparrow \infty$

and hence $\lim_{\nu \to \infty} \text{Log } M_{2,\nu} = -\infty$ which proves the theorem.

When the polynomial $P_n(s)$ reduces to a complex number $a_{n,o}$, we get the famous Ostrowaski's theorem [1] for Dirichlet series. Our theorem contains G.L. Lunt'z theorem [5] as a particular case when $P_n(s) = a_n s^{m_n}$.

COROLLARY. - In theorem 3 if we replace (2.3) by the condition that there exists a sequence (θ_n) of strictly positive numbers such that $\lim_{n \to \infty} \theta_n = \infty$ and $\exists_{n'} \forall_{n \geq n'} \lambda_{n+1} > (1+\theta_n)\lambda_n$, then each point of $(\text{Fr } \mathcal{B}_{*0}) \cap \mathbb{C}-\mathcal{E}^*$ is a singular point for f defined by (2.2). In particular if $(\text{Fr } \mathcal{B}_{*0}) \subset \mathbb{C}-\mathcal{E}^*$ then $\text{Fr } \mathcal{B}_{*0}$ is a natural boundary for f .

PROOF. Let us suppose that the corollary is false. Then there exists a point $b \in (\text{Fr } \mathcal{B}_{*0}) \cap \mathbb{C}-\mathcal{E}^*$ and a disc $d(b,\rho)$ centred at b and of radius $\rho > 0$ on which f is holomorphic. As a result of theorem 3 the sequence (S_n) converges on $d(b,\rho)$. From remark 2 $\{f\}$ diverges on $\mathbb{C}-\mathcal{E}^*-\overline{\mathcal{B}}_{*0}$. There exists necessarily points common to $\mathbb{C}-\mathcal{E}^*-\overline{\mathcal{B}}_{*0}$ and $d(b,\rho)$. For these points there is a contradiction which establishes the corollary.

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