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STRONG BOUNDEDNESS OF ANALYTIC FUNCTIONS IN TUBES

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ABSTRACT. Certain classes of analytic functions in tube domains $T^C = \mathbb{R}^n + iC$ in n-dimensional complex space, where C is an open connected cone in $\mathbb{R}^n$, are studied. We show that the functions have a boundedness property in the strong topology of the space of tempered distributions $\mathcal{S}'$. We further give a direct proof that each analytic function attains the Fourier transform of its spectral function as distributional boundary value in the strong (and weak) topology of $\mathcal{S}'$.

KEY WORDS AND PHRASES. Analytic Function in Tubes, Strong Boundedness, Tempered Distributions, Distributional Boundary Value.

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1. INTRODUCTION.

Vladimirov [1, p. 230] has defined the spectral function V_t of a function $f(z)$ which is analytic in a tubular domain $T^B = \mathbb{R}^n + iB$ to be the distribution

$V_t \in \mathcal{D}'$, the space of distributions of L. Schwartz [2], which possesses the following properties:

$$e^{-yt} V_t \in \mathcal{S}' \quad \text{for all} \quad y \in B; \tag{1.1}$$

$$f(z) = \langle V_t , e^{izt} \rangle \quad \text{for all} \quad z \in T^B. \tag{1.2}$$

Here $\mathcal{S}'$ is the space of tempered distributions of Schwartz [2] and $\langle V_t , e^{izt} \rangle$ is the Fourier-Laplace transform of the spectral function V_t.

In [3] Vladimirov defined certain classes of analytic functions in tubular cones $T^C = \mathbb{R}^n + iC$, where C is an open cone, and analyzed the spectral functions of these analytic functions corresponding to C being an open connected cone. The results of [3] have been incorporated into the book [1] of Vladimirov [1, section 26.4].

In this paper we add information to the main results of [3] and [1, section 26.4] which are [1, pp. 238-239, Theorems 1 and 2]. We show that the analytic functions considered by Vladimirov in these results have boundedness properties in the strong topology of the space of tempered distributions $\mathcal{S}'$. Further, we give a direct proof by elementary means that each analytic function attains the Fourier transform of its spectral function as distributional boundary value in the strong (and weak) topology of $\mathcal{S}'$, a fact which has been recognized by Vladimirov [1, p. 238] and which is obtained by him as a special case of a more general result.

2. NOTATION AND DEFINITIONS.

Our n-dimensional notation is that of Vladimirov [1, p. 1]. x, y, and t will be points in $\mathbb{R}^n$ in this paper and $z \in \mathbb{C}^n$, n-dimensional complex space. Note the inner products $zt = z_1t_1 + \ldots + z_nt_n$ and $yt = y_1t_1 + \ldots + y_nt_n$ for t and y in $\mathbb{R}^n$ and $z \in \mathbb{C}^n$. Note also the differential operator D^α in [1, p. 1], and we shall write D_z^α or D_t^α to indicate that the differentiation is with respect to z or t, respectively. Here α is an n-tuple of nonnegative integers. The definitions of cone C in $\mathbb{R}^n$, compact subcone of a cone, indicatrix $u_C(t)$ of a cone, and of the number ρ_C, which characterizes the nonconvexity of a cone C, can all be found in [1, section 25.1]. Note that $\rho_C \geq 1$ [1, p. 220] for any cone C. The cone $C^* = \{t \in \mathbb{R}^n : yt \geq 0 , y \in C\}$ is the dual cone of C and C_* will denote $C_* = \mathbb{R}^n \setminus C^*$. $0(C)$ will denote the convex envelope (hull) of the cone C, and we define the tubes T^C and $T^{0(C)}$ by $T^C = \mathbb{R}^n + iC$ and $T^{0(C)} = \mathbb{R}^n + i0(C)$, respectively.

Let C be a cone in $\mathbb{R}^n$. We make the convention throughout this paper that by $z \in T^C(\in T^{0(C)})$ and $y \in C(\in 0(C))$ we mean that $z \in T^{C'}$ and $y \in C'$ for an arbitrary compact subcone $C' \subset C$ $(C' \subset 0(C))$.

The space of functions of rapid decrease $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ are defined and discussed in Schwartz [2, Chapter 7]. The Fourier (inverse Fourier) transform of an $L^1(\mathbb{R}^n)$ function $\phi(t)$, denoted $\mathfrak{F}[\phi(t);x]$ $(\mathfrak{F}^{-1}[\phi(t);x])$, will be as defined in Vladimirov [1, p. 21]. The Fourier transform of a tempered distribution V_t, denoted $\mathfrak{F}[V]$, is defined in Schwartz [2, p. 250, (VII 6; 6)]. All terminology and definitions concerning distributions in this paper, such as support of a distribution, will be that of Schwartz [2].

Let C be an open connected cone. The analytic function $f(z)$, $z \in T^C$, obtains $U \in \mathcal{S}'$ as boundary value in the weak topology of $\mathcal{S}'$ if

$$\lim_{\substack{y \to 0 \\ y \in C}} \langle f(x + iy), \phi(x)\rangle = \langle U,\phi \rangle \tag{2.1}$$

for each $\phi \in \mathcal{S}$. $U \in \mathcal{S}'$ is the boundary value of $f(z)$ in the strong topology of $\mathcal{S}'$ if the convergence (2.1) holds uniformly for ϕ varying over arbitrary bounded sets in $\mathcal{S}$. The set $\{U_y \in \mathcal{S}' : y \in C\}$, where $U_y \in \mathcal{S}'$ in some sense depends on $y \in C$, is said to be a bounded set in the strong topology of $\mathcal{S}'$ if for any bounded set Φ in $\mathcal{S}$, $\{\langle U_y , \phi\rangle : \phi \in \Phi , y \in C\}$ is a bounded set in the complex plane.

3. THE THEOREMS OF VLADIMIROV.

Let C be an open cone. A function $f(z)$ belongs to the class $H_p(a;C)$, where $p \geq 1$ and $a \geq 0$, if $f(z)$ is analytic in the tubular cone T^C and, for an arbitrary compact subcone C' in C, the inequality

$$|f(z)| \leq M(C')\ (1 + |z|)^N\ (1 + |y|^{-K})\ e^{a|y|^p},\ z = x+iy \in T^{C'}, \tag{3.1}$$

is satisfied where $M(C')$ is a constant which depends at most on the compact subcone $C' \subset C$ and N and K are nonnegative real numbers which do not depend on $C' \subset C$. We define

$$H_p(a + \epsilon;C) = \bigcap_{a' > a} H_p(a';C),\ H_0(C) = H_1(0;C).$$

For the convenience of the reader we now state the theorems of Vladimirov with which we are concerned in this paper.

THEOREM 1. [1, p. 238] Let $f(z) \in H_p(a + \epsilon;C)$, where C is an open connected cone, $p > 1$, and $a > 0$. The spectral function V_t of $f(z)$ can be represented in the form of a finite sum of distributional derivatives of continuous functions $g_\alpha(t)$ of power increase,

$$V_t = \sum_\alpha D_t^\alpha(g_\alpha(t)) \tag{3.2}$$

which, for all $t \in C_*'$, where C_*' is an arbitrary compact subcone of $C_* = \mathbb{R}^n \setminus C^*$, and for all $\epsilon > 0$, satisfy

$$|g_\alpha(t)| \leq M_\epsilon'(C_*')\quad \exp[-(a'-\epsilon)(u_C(t))^{p'}] \tag{3.3}$$

where the numbers p and a are connected with p' and a' by the relations

$$\frac{1}{p} + \frac{1}{p'} = 1\ ,\ (p'a')^p(pa)^{p'} = 1. \tag{3.4}$$

Conversely, if V_t satisfies these conditions for certain numbers $a' > 0$, $p' > 1$ and the cone C_*, then all derivatives $D_z^\beta(f(z))$ of its Fourier-Laplace transform $f(z)$ belong to the class $H_p(a\rho_C^p + \epsilon;0(C))$.

Notice that the C^* as printed in [1, p. 239, line 8] should be C_* instead as we have written in Theorem 1.

THEOREM 2. [1, p. 239] Let $f(z) \in H_1(a + \epsilon;C)$ where C is an open connected cone and $a \geq 0$. Then its spectral function $V_t \in \mathcal{S}'$ and V_t has support in $\{t : u_C(t) \leq a\}$. Conversely, if $V_t \in \mathcal{S}'$ and has support in $\{t : u_C(t) \leq a\}$ for some $a \geq 0$ and some open connected cone C, then all the derivatives $D_z^{\beta}(f(z))$ of the Fourier-Laplace transform $f(z)$ of V_t belong to the class $H_1(a\rho_C;O(C))$.

4. LEMMAS.

As noted in the introduction, we shall add information to Theorems 1 and 2. We shall show that the analytic functions in these theorems have a strong boundedness property in $\mathcal{S}'$. In addition we give a direct proof that the analytic functions attain the Fourier transform of their spectral functions as distributional boundary values in the strong (and weak) topology of $\mathcal{S}'$.

The following lemma is the basis of the boundary value result, and its proof in turn is useful in obtaining our strong boundedness properties. Throughout this section C is an open connected cone.

LEMMA 1. Let $f(z) \in H_p(a + \epsilon;C)$, $p > 1$ and $a > 0$. The spectral function V_t of $f(z)$ is in $\mathcal{S}'$ as is $(e^{-yt}\, V_t)$, $y \in O(C)$, and

$$\lim_{\substack{y \to 0 \\ y \in O(C)}} \mathfrak{F}[e^{-yt}\, V_t] = \mathfrak{F}[V] \tag{4.1}$$

in the strong (and weak) topology of $\mathcal{S}'$.

PROOF. Let C' be an arbitrary compact subcone of $O(C)$. By the sufficiency of Theorem 1, the spectral function V_t of $f(z)$ has the representation (3.2). Since each $g_{\alpha}(t)$ in (3.2) is continuous and of power increase over $\mathbb{R}^n$, we immediately have $V_t \in \mathcal{S}'$. The fact that $(e^{-yt}\, V_t) \in \mathcal{S}'$, $y \in C' \subset O(C)$, follows by the proof of Theorem 1 given in [1, section 26.5]. Let ϕ be an arbitrary element of $\mathcal{S}$. Using the notion of distributional differentiation and the generalized Leibnitz rule, we have for $y \in C' \subset O(C)$ that

$$\langle V_t, (e^{-yt} - 1)\phi(t)\rangle =$$

$$= \sum_{\alpha} (-1)^{|\alpha|} \int_{\mathbb{R}^n} g_\alpha(t) \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D_t^\beta(e^{-yt} - 1)\, D_t^\gamma(\phi(t))\, dt \tag{4.2}$$

$$= \sum_{\alpha} (-1)^{|\alpha|} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} I_y(\alpha,\beta,\gamma)$$

where α, β, and γ are n-tuples of nonnegative integers and

$$I_y(\alpha,\beta,\gamma) = \int_{\mathbb{R}^n} g_\alpha(t)\, ((-1)^{|\beta|} y^\beta e^{-yt} - D_t^\beta(1))\, D_t^\gamma(\phi(t))\, dt. \tag{4.3}$$

For the arbitrary $C' \subset O(C)$ we apply [1, p. 223, Lemma 2] to obtain a number $\delta = \delta(C') > 0$ and an open cone $(C^*)'$, both depending on C', such that $(C^*)'$ contains the cone $C^* = \{t \in \mathbb{R}^n : yt \geq 0,\ y \in C\}$, the dual cone of C, and

$$yt \geq \delta\, |y|\, |t|, \quad y \in C', \quad t \in (C^*)'. \tag{4.4}$$

Put $C'_* = \mathbb{R}^n \setminus (C^*)'$. C'_* is a compact subcone of $C_* = \mathbb{R}^n \setminus C^*$, and we have $C'_* \cap (C^*)' = \emptyset$ and $C'_* \cup (C^*)' = \mathbb{R}^n$. We now write the integral $I_y(\alpha,\beta,\gamma)$ in (4.3) as

$$I_y(\alpha,\beta,\gamma) = I_y^1(\alpha,\beta,\gamma) + I_y^2(\alpha,\beta,\gamma) \tag{4.5}$$

where

$$I_y^1(\alpha,\beta,\gamma) = \int_{(C^*)'} g_\alpha(t)\, ((-1)^{|\beta|} y^\beta e^{-yt} - D_t^\beta(1))\, D_t^\gamma(\phi(t))\, dt$$

$$I_y^2(\alpha,\beta,\gamma) = \int_{C'_*} g_\alpha(t)\, ((-1)^{|\beta|} y^\beta e^{-yt} - D_t^\beta(1))\, D_t^\gamma(\phi(t))\, dt. \tag{4.6}$$

For any n-tuple β of nonnegative integers we have

$$(-1)^{|\beta|} y^\beta e^{-yt} - D_t^\beta(1) = \begin{cases} e^{-yt} - 1, & \beta = (0,\ldots,0), \\ (-1)^{|\beta|} y^\beta e^{-yt}, & \beta \neq (0,\ldots,0), \end{cases} \tag{4.7}$$

for all $y \in C' \subset O(C)$ and in fact for all $y \in \mathbb{R}^n$; hence for any α in the last sum in (4.2) and any subsequent β and γ, $\beta + \gamma = \alpha$, (4.7) yields

$$\lim_{\substack{y \to 0 \\ y \in O(C)}} g_\alpha(t)\ ((-1)^{|\beta|}\ y^\beta\ e^{-yt} - D_t^\beta(1))\ D_t^\gamma(\phi(t)) = 0 \tag{4.8}$$

for all $t \in \mathbb{R}^n$. (The limit (4.8) actually holds as $y \to 0$, $y \in \mathbb{R}^n$, because (4.7) holds for all $y \in \mathbb{R}^n$.)

Recall that we desire a convergence result in this lemma as $y \to 0$, $y \in O(C)$. Hence to obtain (4.1) it suffices to consider $y \in O(C)$ such that $|y| \le Q$ for $Q > 0$ fixed. Now consider the integrand of the integral $I_y^1(\alpha,\beta,\gamma)$ in (4.6) for $t \in (C^*)'$. Since each $g_\alpha(t)$ in (3.2) is of power increase over $\mathbb{R}^n$, we have the existence of a polynomial $P_\alpha(t)$ corresponding to each $g_\alpha(t)$ such that

$$|g_\alpha(t)| \le P_\alpha(|t|) \quad , \quad t \in \mathbb{R}^n . \tag{4.9}$$

Using (4.9) and (4.4) we get

$$\begin{aligned} &|g_\alpha(t)\ ((-1)^{|\beta|}\ y^\beta\ e^{-yt} - D_t^\beta(1))\ D_t^\gamma(\phi(t))| \le \\ &\le P_\alpha(|t|)\ (1 + |y|^{|\beta|}\ \exp(-\delta|y|\,|t|))\ |D_t^\gamma(\phi(t))| \\ &\le P_\alpha(|t|)\ (1 + Q^{|\beta|})\ |D_t^\gamma(\phi(t))| \end{aligned} \tag{4.10}$$

for $t \in (C^*)'$ and $y \in C' \subset O(C)$ such that $|y| \le Q$. Since $\phi \in \mathcal{S}$, the right side of the last inequality in (4.10) is an L^1 function over $\mathbb{R}^n$ which is independent of $y \in C' \subset O(C)$ such that $|y| \le Q$. Using this fact, (4.8), and the Lebesgue dominated convergence theorem we obtain

$$\lim_{\substack{y \to 0 \\ y \in O(C)}} I_y^1(\alpha,\beta,\gamma) = 0 \tag{4.11}$$

for any α in (4.2) and any subsequent β and γ, $\beta + \gamma = \alpha$.

We now consider the integrand of the integral $I_y^2(\alpha,\beta,\gamma)$ in (4.6) for $t \in C_*'$. For such t each $g_\alpha(t)$ in (3.2) satisfies (3.3). Using (3.3), the relations (3.4), the facts

$$-yt \le |y|\ u_{O(C)}(t) \ , \ u_{O(C)}(t) \le \rho_C\ u_C(t) \ , \quad t \in C_* \ , \quad y \in O(C) \tag{4.12}$$

contained in [1, section 25.1], and analysis as in [1, p. 244], we have for $t \in C'_* \subset C_*$ and $y \in C' \subset O(C)$ such that $|y| \leq Q$ that

$$|g_\alpha(t)((-1)^{|\beta|} y^\beta e^{-yt} - D_t^\beta(1))\, D_t^\gamma(\phi(t))| \leq$$

$$\leq M'_\epsilon(C'_*) \exp[-(a'-\epsilon)(u_C(t))^{p'}]\, (1 + |y|^{|\beta|} e^{-yt})\, |D_t^\gamma(\phi(t))|$$

$$\leq M'_\epsilon(C'_*) \exp[-(a'-\epsilon)(u_C(t))^{p'}]\, (1 + |y|^{|\beta|} \exp[|y|\, \rho_C\, u_C(t)])\, |D_t^\gamma(\phi(t))| \quad (4.13)$$

$$\leq M'_\epsilon(C'_*)\, (1 + |y|^{|\beta|}) \exp[-(a'-\epsilon)(u_C(t))^{p'} + |y|\, \rho_C\, u_C(t)]\, |D_t^\gamma(\phi(t))|$$

$$\leq M'_\epsilon(C'_*)\, (1 + Q^{|\beta|}) \exp[\frac{1}{p}(\frac{1}{p'(a'-2\epsilon)})^{p/p'} \rho_C^p\, |y|^p]\, |D_t^\gamma(\phi(t))|.$$

(3.3) holds for all $\epsilon > 0$. In particular (3.3), and hence (4.13), holds for $\epsilon > 0$ fixed such that $(a' - 2\epsilon) > 0$ for the fixed a' in (3.4). For $\epsilon > 0$ fixed in this way in obtaining (4.13), we now conclude from (4.13) that

$$|g_\alpha(t)((-1)^{|\beta|} y^\beta e^{-yt} - D_t^\beta(1))\, D_t^\gamma(\phi(t))| \leq$$
$$\leq M'_\epsilon(C'_*)\, (1 + Q^{|\beta|}) \exp[\frac{1}{p}(\frac{1}{p'(a'-2\epsilon)})^{p/p'} \rho_C^p\, Q^p]\, |D_t^\gamma(\phi(t))| \quad (4.14)$$

for all $t \in C'_* \subset C_*$ and $y \in C' \subset O(C)$ such that $|y| \leq Q$. Since $\phi \in \mathcal{S}$ the right side of (4.14) is an L^1 function over $\mathbb{R}^n$ and is independent of $y \in C' \subset O(C)$ such that $|y| \leq Q$. Thus by (4.14), (4.8), and the Lebesgue dominated convergence theorem we have

$$\lim_{\substack{y\to 0\\ y\in O(C)}} I_y^2(\alpha,\beta,\gamma) = 0 \quad (4.15)$$

for each relevant α, β, and γ. Combining (4.5), (4.11), and (4.15) we get

$$\lim_{\substack{y\to 0\\ y\in O(C)}} I_y(\alpha,\beta,\gamma) = 0 \quad (4.16)$$

for each α in (4.2) and each β and γ, $\beta + \gamma = \alpha$. Since ϕ is an arbitrary element of $\mathcal{S}$, we combine (4.2) and (4.16) to yield

$$\lim_{\substack{y \to 0 \\ y \in O(C)}} e^{-yt}\, V_t = V_t \tag{4.17}$$

in the weak topology of $\mathcal{S}'$. But $\mathcal{S}$ is a Montel space ([1, p. 21] and [4, p. 510].) Hence by Edwards [4, p. 510, Corollary 8.4.9] the convergence (4.17) is in the strong topology of $\mathcal{S}'$ also. Since the Fourier transform on $\mathcal{S}'$ [2, Chapter 7] is a strongly continuous mapping of $\mathcal{S}'$ onto $\mathcal{S}'$, the desired convergence (4.1) now follows in the strong (and weak) topology of $\mathcal{S}'$. The proof is complete.

The next lemma is the basis of our strong boundedness results concerning the analytic functions $H_p(a + \epsilon; C)$, $p > 1$ and $a > 0$.

LEMMA 2. Let $p > 1$ and $a > 0$. Let C be an open connected cone. Let V_t be any generalized function of the form (3.2) where the $g_\alpha(t)$ satisfy the conditions stated in Theorem 1. Then $V_t \in \mathcal{S}'$, $(e^{-yt}\, V_t) \in \mathcal{S}'$ for all $y \in O(C)$, and $\{\mathfrak{F}[e^{-yt}\, V_t] \in \mathcal{S}' : y \in O(C)\ ,\ |y| \le Q\}$ is a strongly bounded set in $\mathcal{S}'$ for $Q > 0$ being arbitrary but fixed.

PROOF. Let C' be an arbitrary compact subcone of $O(C)$. The facts that $V_t \in \mathcal{S}'$ and $(e^{-yt}\, V_t) \in \mathcal{S}'$ for all $y \in C' \subset O(C)$ follow as at the beginning of the proof of Lemma 1. The locally convex topology of $\mathcal{S}$ is defined by the norms

$$||\phi||_k = \sup_{\substack{|\alpha| \le k \\ t \in \mathbb{R}^n}} (1 + |t|)^k\, |D_t^\alpha(\phi(t))|\ , \quad k = 1,2,3,\dots\ . \tag{4.18}$$

Let Φ be an arbitrary bounded set in $\mathcal{S}$. For the arbitrary $C' \subset O(C)$ we apply [1, p. 223, Lemma 2] as in the proof of Lemma 1 and obtain a number $\delta = \delta(C') > 0$ and an open cone $(C^*)'$, both depending on C', such that $(C^*)'$ contains the cone C^* and (4.4) holds. We then put $C_*' = \mathbb{R}^n \setminus (C^*)'$, and C_*' is a compact subcone of $C_* = \mathbb{R}^n \setminus C^*$ as in the proof of Lemma 1. Using the form of V_t in (3.2) and the generalized Leibnitz rule we obtain for any $\phi \in \Phi$ and $y \in C' \subset O(C)$ that

$$\langle e^{-yt} V_t , \phi(t)\rangle = \sum_{\alpha} (-1)^{|\alpha|} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (-1)^{|\beta|} y^{\beta} (I_y^1(\alpha,\gamma) + I_y^2(\alpha,\gamma)) \tag{4.19}$$

where

$$I_y^1(\alpha,\gamma) = \int_{(C^*)'} g_\alpha(t)\, e^{-yt}\, D_t^\gamma(\phi(t))\, dt$$
$$I_y^2(\alpha,\gamma) = \int_{C'_*} g_\alpha(t)\, e^{-yt}\, D_t^\gamma(\phi(t))\, dt. \tag{4.20}$$

Using (4.4), (4.18), and the fact that each $g_\alpha(t)$ satisfies (4.9) for some polynomial $P_\alpha(t)$, we have

$$|I_y^1(\alpha,\gamma)| \le \int_{(C^*)'} P_\alpha(|t|) \exp[-\delta|y||t|]\, |D_t^\gamma(\phi(t))|\, dt$$

$$\le \int_{(C^*)'} P_\alpha(|t|)\, (1 + |t|)^{n+1}\, |D_t^\gamma(\phi(t))|\, (1 + |t|)^{-n-1}\, dt \tag{4.21}$$

$$\le R_\alpha \|\phi\|_{k_\alpha} \int_{\mathbb{R}^n} (1 + |t|)^{-n-1}\, dt$$

where R_α is a constant and k_α is a positive integer with both depending on α; and (4.21) holds for each α and γ, $\alpha = \beta + \gamma$, in (4.19). Also recall that each $g_\alpha(t)$ satisfies (3.3). Using (3.3), (4.12), and analysis as in (4.21), (4.13), and (4.14) we have for $y \in C' \subset 0(C)$ that

$$|I_y^2(\alpha,\gamma)| \le M'_\epsilon(C'_*) \int_{C'_*} \exp[-(a'-\epsilon)(u_C(t))^{p'}]\, \exp[|y|\, \rho_C\, u_C(t)]\, |D_t^\gamma(\phi(t))|\, dt$$

$$\le M''_\epsilon(C'_*)\, \|\phi\|_{k'_\alpha} \int_{C'_*} \exp[-(a'-\epsilon)(u_C(t))^{p'} + |y|\, \rho_C\, u_C(t)]\, (1 + |t|)^{-n-1}\, dt \tag{4.22}$$

$$\le M''_\epsilon(C'_*)\, \|\phi\|_{k'_\alpha} \exp[\frac{1}{p}(\frac{1}{p'(a'-2\epsilon)})^{p/p'} \rho_C^p\, |y|^p] \int_{\mathbb{R}^n} (1 + |t|)^{-n-1}\, dt$$

where $M''_\epsilon(C'_*)$ is a constant and k'_α is a positive integer depending on α. Because of (3.3), we can assume that $\epsilon > 0$ in (4.22) is fixed such that $(a' - 2\epsilon) > 0$. Since (4.22) holds for each α and γ, $\beta + \gamma = \alpha$, in (4.19) and since Φ is a bounded set in $\mathcal{S}$, it follows from the combination of (4.19), (4.20), (4.21), and (4.22) that $\{\langle e^{-yt} V_t, \phi(t)\rangle : \phi \in \Phi , y \in O(C) , |y| \leq Q\}$ is a bounded set in the complex plane for $Q > 0$ arbitrary but fixed. Since Φ was assumed to be an arbitrary bounded set in $\mathcal{S}$, this proves that $\{e^{-yt} V_t : y \in O(C) , |y| \leq Q\}$ is a strongly bounded set in $\mathcal{S}'$; hence $\{\mathfrak{F}[e^{-yt} V_t] \in \mathcal{S}' : y \in O(C) , |y| \leq Q\}$ is a strongly bounded set in $\mathcal{S}'$ since the Fourier transform in $\mathcal{S}'$ [2, Chapter 7] is a strongly continuous mapping from $\mathcal{S}'$ onto $\mathcal{S}'$. The proof is complete.

5. ADDITIONS TO THEOREMS 1 AND 2.

Let us now consider Theorem 1. Let C be an open connected cone. Let $f(z) \in H_p(a + \epsilon;C)$, $p > 1$ and $a > 0$. By the sufficiency of Theorem 1 we have that the spectral function V_t of $f(z)$ has the form (3.2) and

$$f(z) = \langle V_t , e^{izt}\rangle , \quad z \in T^C . \tag{5.1}$$

(Recall (1.2).) Further note that $V_t \in \mathcal{S}'$ and $(e^{-yt} V_t) \in \mathcal{S}'$ for all $y \in O(C)$ as obtained in the proofs of Lemmas 1 and 2. For any fixed $y \in C$, $f(x + iy) \in \mathcal{S}'$ as a function of $x \in \mathbb{R}^n$ because of the growth (3.1) defining the $H_p(a + \epsilon;C)$ spaces. Let $\Psi \in \mathcal{S}$ and let $\phi \in \mathcal{S}$ be that unique element of $\mathcal{S}$ such that $\phi(t) = \mathfrak{F}[\Psi(x);t]$ [2, Chapter 7]. Using (5.1), (3.2), distributional differentiation, a change of order of integration, and differentiation under the integral sign we get

$$\begin{aligned}\langle f(z) , \Psi(x)\rangle &= \sum_\alpha (-1)^{|\alpha|} i^{|\alpha|} \int_{\mathbb{R}^n} z^\alpha \Psi(x) \int_{\mathbb{R}^n} g_\alpha(t) e^{izt} dt\, dx \\ &= \sum_\alpha (-1)^{|\alpha|} \int_{\mathbb{R}^n} g_\alpha(t) (D_t^\alpha \int_{\mathbb{R}^n} \Psi(x) e^{izt} dx)\, dt.\end{aligned} \tag{5.2}$$

But if $\phi(t) = \mathfrak{F}[\Psi(x);t]$ then

$$e^{-yt} \phi(t) = \int_{\mathbb{R}^n} \Psi(x) e^{izt} dx. \tag{5.3}$$

Putting (5.3) into (5.2) and using the Fourier transform on $\mathcal{S}'$ [2, Chapter 7] we have

$$\begin{aligned}\langle f(z)\ ,\ \Psi(x)\rangle &= \sum_{\alpha} (-1)^{|\alpha|} \int_{\mathbb{R}^n} g_\alpha(t)\ (D_t^\alpha(e^{-yt}\ \phi(t)))\, dt \\ &= \langle e^{-yt}\ V_t\ ,\ \phi(t)\rangle = \langle \mathfrak{F}[e^{-yt}\ V_t],\ \Psi(x)\rangle \end{aligned} \tag{5.4}$$

for all $y = \mathrm{Im}(z) \varepsilon C$ which proves that

$$f(z) = \mathfrak{F}[e^{-yt}\ V_t]\ ,\ z = x + iy \varepsilon T^C\ , \tag{5.5}$$

with this equality holding in $\mathcal{S}'$. Thus by combining (5.5) and Lemma 2 we can also conclude in the sufficiency of Theorem 1 that $\{f(z) : y = \mathrm{Im}(z) \varepsilon C,\ |y| \leq Q\}$ is a strongly bounded set in $\mathcal{S}'$ for $Q > 0$ being arbitrary but fixed. Further, by combining (5.5) and Lemma 1 we have obtained a direct proof of the fact that

$$\lim_{\substack{y\to 0 \\ y\varepsilon C}} f(x + iy) = \mathfrak{F}[V] \tag{5.6}$$

in the strong (and weak) topology of $\mathcal{S}'$.

In the converse of Theorem 1 Vladimirov proves that if V_t has the form (3.2) then all derivatives $D_z^\beta(f(z))$ of the Fourier-Laplace transform $f(z) = \langle V_t\ ,\ e^{izt}\rangle$ of V_t belong to the class $H_p(a\ \rho_C^p + \epsilon; O(C))$, C being an open connected cone. By the analysis in (5.2), (5.3), and (5.4) we conclude that (5.5) holds in this converse also for $z = x + iy \varepsilon T^{O(C)}$. Then combining this fact with Lemmas 1 and 2 we add the conclusions to the converse of Theorem 1 that $\{f(x) : y = \mathrm{Im}(z) \varepsilon O(C)\ ,\ |y| \leq Q\}$ is a strongly bounded set in $\mathcal{S}'$, where $Q > 0$ is arbitrary but fixed, and (5.6), with C replaced by $O(C)$, holds in the strong (and weak) topology of $\mathcal{S}'$.

We now consider Theorem 2. For the element $f(z) \varepsilon H_1(a + \epsilon; C)$ ($\varepsilon H_1(a\ \rho_C; O(C))$ in the converse), $a \geq 0$, and its corresponding spectral function $V_t \varepsilon \mathcal{S}'$ in both the sufficiency and necessity of this theorem, we can prove lemmas like Lemmas 1 and 2. Then using techniques as in our preceding additions to Theorem 1 we have the conclusions in both the sufficiency and necessity of Theorem 2 that

$$f(z) = \mathfrak{F}[e^{-yt} V_t] \text{ , } z = x + iy \in T^C \text{ } (\in T^{O(C)} \text{ in the converse}),$$

with this equality holding in $\mathcal{S}'$; $\{f(z) : y = \text{Im}(z) \in C$ ($\in O(C)$ in the converse), $|y| \leq Q\}$ is a strongly bounded set in $\mathcal{S}'$ for $Q > 0$ being arbitrary but fixed; and (5.6) holds in the strong (and weak) topology of $\mathcal{S}'$ with $O(C)$ replacing C in the converse. The now evident details are left to the interested reader.

Let us also note the generalization of Theorems 1 and 2 given by Vladimirov in [1, section 26.7] concerning functions $f(z) \in H_p(a + \epsilon;C)$ which are analytic in tubular cones T^C where C is an open cone that is the union of a finite number of open connected component cones C_k , $k=1,2,\ldots,r$. By our analysis in this paper one can also conclude our strong boundedness property in $\mathcal{S}'$ for the analytic function $f(z) \in H_p(a + \epsilon;C)$ in [1, p. 247, Theorem] in each of the connected components T^{C_k} , $k=1,2,\ldots,r$, of T^C and for the analytic extension function $f(z)$ in the conclusion of this result of Vladimirov for $z \in T^{O(C)}$.

The Theorems 1 and 2 of Vladimirov have recently motivated this author to define more general spaces of analytic functions in tubes than the $H_p(a;C)$ and $H_p(a + \epsilon;C)$ spaces. The associated spectral functions are distributions of exponential growth, a class of distributions which contains the tempered distributions $\mathcal{S}'$. Our analysis will appear in [5].

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