

SOME SEQUENCE SPACES AND STATISTICAL CONVERGENCE

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Received 29 January 1999

We introduce the strongly (V, λ) -convergent sequences and give the relation between strongly (V, λ) -convergence and strongly (V, λ) -convergence with respect to a modulus.

2000 Mathematics Subject Classification: 40D25, 40A05, 40C05.

1. Introduction. Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ , and $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

The generalized de la Vallée-Poussin mean is defined by

$$t_n = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \quad (1.1)$$

where $I_n = [n - \lambda_n + 1, n]$. A sequence $x = (x_k)$ is said to be (V, λ) -summable to a number L (see [5]) if $t_n(x) \rightarrow L$ as $n \rightarrow \infty$. If $\lambda_n = n$, then (V, λ) -summability is reduced to $(C, 1)$ -summability. We write

$$[V, \lambda] = \left\{ x = (x_k) : \text{for some } L, \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| = 0 \right\} \quad (1.2)$$

for sets of sequences $x = (x_k)$ which are strongly (V, λ) -summable to L , that is, $x_k \rightarrow L$ $[V, \lambda]$.

We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

- (i) $f(x) = 0$ if and only if $x = 0$;
- (ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$;
- (iii) f is increasing;
- (iv) f is continuous from the right at 0.

It follows that f must be continuous on $[0, \infty)$. A modulus may be bounded or unbounded. Maddox [6] and Ruckle [9] used the modulus f to construct sequence spaces. In this paper, we introduce the strongly (V, λ) -convergent sequences and give the relation between strongly (V, λ) -convergence and strongly (V, λ) -convergence with respect to a modulus.

2. Some sequence spaces

DEFINITION 2.1. Let f be a modulus. We define the spaces,

$$\begin{aligned} [V, \lambda, f] &= \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_k - L|) = 0, \text{ for some } L \right\}, \\ [V, \lambda, f]_0 &= \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_k|) = 0 \right\}. \end{aligned} \quad (2.1)$$

When $\lambda_n = n$ then the sequence spaces defined above become $w_0(f)$ and $w(f)$, respectively, where $w_0(f)$ and $w(f)$ are defined by Maddox [6].

Note that if we put $f(x) = x$, then we have $[V, \lambda, f] = [V, \lambda]$ and $[V, \lambda, f]_0 = [V, \lambda]_0$, where

$$[V, \lambda]_0 = \left\{ x = (x_k) : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k| = 0 \right\}. \quad (2.2)$$

We have the following result.

THEOREM 2.2. *The spaces $[V, \lambda, f]$ and $[V, \lambda, f]_0$ are linear spaces.*

PROOF. We consider only $[V, \lambda, f]$. Suppose that $x_i \rightarrow L$ and $y_j \rightarrow L'$ in $[V, \lambda, f]$ and that α, β are in \mathbb{C} . Then there exists integers T_α and M_β such that $|\alpha| \leq T_\alpha$ and $|\beta| \leq M_\beta$. We therefore have

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k \in I_n} f(|\alpha x_k + \beta y_k - (\alpha L + \beta L')|) \\ & \leq T_\alpha \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_k - L|) + M_\beta \frac{1}{\lambda_n} \sum_{k \in I_n} f(|y_k - L'|). \end{aligned} \quad (2.3)$$

This implies that $\alpha x + \beta y \rightarrow \alpha L + \beta L'$ in $[V, \lambda, f]$. This completes the proof. \square

PROPOSITION 2.3 (see [7]). *Let f be any modulus. Then $\lim_{t \rightarrow \infty} f(t)/t = \beta$ exists.*

PROPOSITION 2.4. *Let f be a modulus and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.*

This can be proved by using the techniques similar to those used in Maddox [6] and hence we omit the proof.

THEOREM 2.5. *Let f be any modulus. If $\lim_{t \rightarrow \infty} f(t)/t = \beta > 0$, then $[V, \lambda, f] = [V, \lambda]$.*

PROOF. If $x \in [V, \lambda]$, then

$$s_n = \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - L| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \text{ for some } L. \quad (2.4)$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for every t with $0 \leq t \leq \delta$. We can write

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_k - L|) &= \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - L| \leq \delta} f(|x_k - L|) + \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - L| > \delta} f(|x_k - L|) \\ &\leq \frac{1}{\lambda_n} (\lambda_n \cdot \varepsilon) + 2f(1)\delta^{-1}s_n, \end{aligned} \quad (2.5)$$

by Proposition 2.4, as $n \rightarrow \infty$. Therefore $x \in [V, \lambda, f]$. It is trivial that $[V, \lambda, f] \subset [V, \lambda]$ and this completes the proof. \square

3. λ -statistical convergence. In [3], Fast introduced the idea of statistical convergence, which is closely related to the concept of natural density or asymptotic density of subsets of the positive integers \mathbb{N} . In recent years, statistical convergence has been studied by several authors [1, 2, 4, 8, 10].

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0, \quad (3.1)$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $s - \lim x = L$ or $x_k \rightarrow L(s)$ and s denotes the set of all statistically convergent sequences.

In this section, we introduce and study the concept of λ -statistical convergence and find its relation with $[V, \lambda, f]$ and s_λ .

DEFINITION 3.1. A sequence $x = (x_k)$ is said to be λ -statistically convergent or s_λ -convergent to L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0. \quad (3.2)$$

In this case, we write $s_\lambda - \lim x = L$ or $x_k \rightarrow L(s_\lambda)$ and $s_\lambda = \{x : \text{for some } L, s_\lambda - \lim x = L\}$. Note that if $\lambda_n = n$, then s_λ is same as s .

The following definition was introduced by Connor [2] as an extension of the original definition of statistical convergence which appeared in [3].

DEFINITION 3.2. Let A be a nonnegative regular summability method and let x be a sequence. Then x is said to be A -statistically convergent to L if $\chi_{S(x-L; \varepsilon)}$ is contained in $w_0(A)$ for every $\varepsilon > 0$, where

$$w_0(A) = \left\{ x : \lim_n \sum a_{n,k} |x_k| = 0 \right\}. \quad (3.3)$$

In the above definition, if we define the matrix by

$$a_{n,k} = \begin{cases} \frac{1}{\lambda_n}, & \text{if } n \in I_n, \\ 0, & \text{if } n \notin I_n \end{cases} \quad (3.4)$$

we get λ -statistical convergence as a special case of A -statistical convergence.

Let ∇ denote the set of all nondecreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$.

We have the following result.

THEOREM 3.3. Let $\lambda \in \nabla$ and f be any modulus. Then $[V, \lambda, f] \subset (s_\lambda)$.

PROOF. Suppose that $\varepsilon > 0$ and $x \in [V, \lambda, f]$. Since,

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_k - L|) &\geq \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - L| \geq \varepsilon} f(|x_k - L|) \\ &\geq \frac{1}{\lambda_n} f(\varepsilon) \cdot |\{k \in I_n : |x_k - L| \geq \varepsilon\}| \end{aligned} \quad (3.5)$$

from which it follows that $x \in (s_\lambda)$. This completes the proof. \square

THEOREM 3.4. $(s_\lambda) = [V, \lambda, f]$ if and only if f is bounded.

PROOF. Suppose that f is bounded and that $x \in (s_\lambda)$. Since f is bounded, there is a constant M such that $f(x) \leq M$ for all $x \geq 0$. Given $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k \in I_n} f(|x_k - L|) &\leq \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - L| \geq \varepsilon} f(|x_k - L|) + \frac{1}{\lambda_n} \sum_{k \in I_n, |x_k - L| < \varepsilon} f(|x_k - L|) \\ &\leq \frac{M}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| + f(\varepsilon). \end{aligned} \quad (3.6)$$

Taking the limit as $\varepsilon \rightarrow 0$, the result follows. Conversely, suppose that f is unbounded so that there is a positive sequence $0 < t_1 < t_2 < \dots < t_i < \dots$ such that $f(t_i) \geq \lambda_i$. Define the sequence $x = (x_i)$ by putting $x_{k_i} = t_i$ for $i = 1, 2, \dots$ and $x_i = 0$ otherwise. Then we have $x \in (s_\lambda)$, but $x \notin [V, \lambda, f]$. \square

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