

COEFFICIENT INEQUALITIES FOR CERTAIN ANALYTIC FUNCTIONS

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For real α ($\alpha > 1$), we introduce subclasses $M(\alpha)$ and $N(\alpha)$ of analytic functions $f(z)$ with $f(0) = 0$ and $f'(0) = 1$ in U . The object of the present paper is to consider the coefficient inequalities for functions $f(z)$ to be in the classes $M(\alpha)$ and $N(\alpha)$. Further, the bounds of α for functions $f(z)$ to be starlike in U are considered.

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1. Introduction. Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $M(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha \quad (z \in U) \quad (1.2)$$

for some α ($\alpha > 1$). And let $N(\alpha)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \alpha \quad (z \in U) \quad (1.3)$$

for some α ($\alpha > 1$). Then, we see that $f(z) \in N(\alpha)$ if and only if $zf'(z) \in M(\alpha)$. We give examples of functions $f(z)$ in the classes $M(\alpha)$ and $N(\alpha)$.

REMARK 1.1. For $1 < \alpha \leq 4/3$, the classes $M(\alpha)$ and $N(\alpha)$ were introduced by Uralegaddi et al. [2].

EXAMPLE 1.2. (i) $f(z) = z(1-z)^{2(\alpha-1)} \in M(\alpha)$.
 (ii) $g(z) = (1/(2\alpha-1))\{1 - (1-z)^{2\alpha-1}\} \in N(\alpha)$.

PROOF. Since $f(z) \in M(\alpha)$ if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < \alpha, \quad (1.4)$$

we can write

$$\frac{\alpha - zf'(z)/f(z)}{\alpha - 1} = \frac{1+z}{1-z}, \quad (1.5)$$

which is equivalent to

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{2(\alpha-1)}{1-z}. \quad (1.6)$$

Integrating both sides of the above equality, we have

$$f(z) = z(1-z)^{2(\alpha-1)} \in M(\alpha). \quad (1.7)$$

Next, since $g(z) \in N(\alpha)$ if and only if $zg'(z) \in M(\alpha)$,

$$zg'(z) = z(1-z)^{2(\alpha-1)}. \quad (1.8)$$

For function $g(z) \in N(\alpha)$, it follows that

$$g(z) = -\frac{1}{2\alpha-1}(1-z)^{2\alpha-1} + \frac{1}{2\alpha-1} = \frac{1}{2\alpha-1}\{1 - (1-z)^{2\alpha-1}\} \in N(\alpha). \quad (1.9)$$

□

2. Coefficient inequalities for the classes $M(\alpha)$ and $N(\alpha)$. We try to derive sufficient conditions for $f(z)$ which are given by using coefficient inequalities.

THEOREM 2.1. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1) \quad (2.1)$$

for some α ($\alpha > 1$), then $f(z) \in M(\alpha)$.

PROOF. Suppose that

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1) \quad (2.2)$$

for $f(z) \in A$.

It suffices to show that

$$\left| \frac{zf'(z)/f(z) - 1}{zf'(z)/f(z) - (2\alpha-1)} \right| < 1 \quad (z \in U). \quad (2.3)$$

We have

$$\begin{aligned} \left| \frac{zf'(z)/f(z) - 1}{zf'(z)/f(z) - (2\alpha-1)} \right| &\leq \frac{\sum_{n=2}^{\infty} (n-1) |a_n| |z|^{n-1}}{2(\alpha-1) - \sum_{n=2}^{\infty} |n-2\alpha+1| |a_n| |z|^{n-1}} \\ &< \frac{\sum_{n=2}^{\infty} (n-1) |a_n|}{2(\alpha-1) - \sum_{n=2}^{\infty} |n-2\alpha+1| |a_n|}. \end{aligned} \quad (2.4)$$

The last expression is bounded above by 1 if

$$\sum_{n=2}^{\infty} (n-1) |a_n| \leq 2(\alpha-1) - \sum_{n=2}^{\infty} |n-2\alpha+1| |a_n| \quad (2.5)$$

which is equivalent to condition (2.1). This completes the proof of the theorem. □

By using [Theorem 2.1](#), we have the following corollary.

COROLLARY 2.2. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n\{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1) \quad (2.6)$$

for some α ($\alpha > 1$), then $f(z) \in N(\alpha)$.

PROOF. From $f(z) \in N(\alpha)$ if and only if $zf'(z) \in M(\alpha)$, replacing a_n by na_n in [Theorem 2.1](#) we have the corollary. \square

In view of [Theorem 2.1](#) and [Corollary 2.2](#), if $1 < \alpha \leq 3/2$, then $n-2\alpha+1 \geq 0$ for all $n \geq 2$. Thus we have the following corollary.

COROLLARY 2.3. (i) *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha) |a_n| \leq \alpha-1 \quad (2.7)$$

for some α ($1 < \alpha \leq 3/2$), then $f(z) \in M(\alpha)$.

(ii) *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq \alpha-1 \quad (2.8)$$

for some α ($1 < \alpha \leq 3/2$), then $f(z) \in N(\alpha)$.

3. Starlikeness for functions in $M(\alpha)$ and $N(\alpha)$. By Silverman [\[1\]](#), we know that if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n |a_n| \leq 1, \quad (3.1)$$

then $f(z) \in S^*$, where S^* denotes the subclass of A consisting of all univalent and starlike functions $f(z)$ in U . Thus we have the following theorem.

THEOREM 3.1. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} (n-\alpha) |a_n| \leq \alpha-1 \quad (3.2)$$

for some α ($1 < \alpha \leq 4/3$), then $f(z) \in S^ \cap M(\alpha)$, therefore, $f(z)$ is starlike in U . Further, if $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq \alpha-1 \quad (3.3)$$

for some α ($1 < \alpha \leq 3/2$), then $f(z) \in S^ \cap N(\alpha)$, therefore, $f(z)$ is starlike in U .*

PROOF. Consider α such that

$$\sum_{n=2}^{\infty} n |a_n| \leq \sum_{n=2}^{\infty} \frac{n-\alpha}{\alpha-1} |a_n| \leq 1. \quad (3.4)$$

Then we have $f(z) \in S^* \cap M(\alpha)$ by means of [Theorem 2.1](#). This inequality holds true if

$$n \leq \frac{n-\alpha}{\alpha-1} \quad (n = 2, 3, 4, \dots). \quad (3.5)$$

Therefore, we have

$$1 < \alpha \leq 2 - \frac{2}{n+1} \quad (n = 2, 3, 4, \dots), \quad (3.6)$$

which shows that $1 < \alpha \leq 4/3$. Next, considering α such that

$$\sum_{n=2}^{\infty} n |a_n| \leq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{\alpha-1} |a_n| \leq 1, \quad (3.7)$$

we have

$$n \leq \frac{n(n-\alpha)}{\alpha-1} \quad (n = 2, 3, 4, \dots), \quad (3.8)$$

which is equivalent to

$$1 < \alpha \leq \frac{n+1}{2} \quad (n = 2, 3, 4, \dots). \quad (3.9)$$

This implies that $1 < \alpha \leq 3/2$. □

Finally, by virtue of the result for convex functions by Silverman [\[1\]](#), we have, if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n^2 |a_n| \leq 1, \quad (3.10)$$

then $f(z) \in K$, where K denotes the subclass of A consisting of all univalent and convex functions $f(z)$ in U . Using the same method as in the proof of [Theorem 3.1](#), we derive the following theorem.

THEOREM 3.2. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq \alpha-1 \quad (3.11)$$

for some α ($1 < \alpha \leq 4/3$), then $f(z) \in K \cap N(\alpha)$, therefore, $f(z)$ is convex in U .

4. Bounds of α for starlikeness. Note that the sufficient condition for $f(z)$ to be in the class $M(\alpha)$ is given by

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1). \quad (4.1)$$

Since, if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} n |a_n| \leq 1, \quad (4.2)$$

then $f(z) \in S^*$ (cf. [1]). It is interesting to find the bounds of α for starlikeness of $f(z) \in M(\alpha)$. To do this, we have to consider the following inequality:

$$\sum_{n=2}^{\infty} n |a_n| \leq \frac{1}{2(\alpha-1)} \sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 1 \quad (4.3)$$

which is equivalent to

$$\sum_{n=2}^{\infty} \{|n-2\alpha+1| + (3-2\alpha)n\} |a_n| \geq 0. \quad (4.4)$$

We define

$$F(n) = |n-2\alpha+1| + (3-2\alpha)n \quad (n \geq 2). \quad (4.5)$$

Then, if $F(n)$ satisfies

$$\sum_{n=2}^{\infty} F(n) |a_n| \geq 0, \quad (4.6)$$

then $f(z)$ belongs to S^* .

THEOREM 4.1. *Let $f(z) \in A$ satisfy*

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1) \quad (4.7)$$

for some $\alpha > 1$. Further, let δ_k be defined by

$$\delta_k = \sum_{n=k}^{\infty} F(n) |a_n|. \quad (4.8)$$

Then,

- (i) if $1 < \alpha \leq 3/2$, then $f(z) \in S^*$,
- (ii) if $3/2 \leq \alpha \leq \min(13/8, (3+\delta_3)/2)$, then $f(z) \in S^*$,
- (iii) if $8/3 \leq \alpha \leq \min(17/10, (12-\delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48})/12)$, then $f(z) \in S^*$.

PROOF. For $1 < \alpha \leq 3/2$, we know that

$$n-2\alpha+1 \geq 3-2\alpha \geq 0 \quad (n \geq 2), \quad (4.9)$$

that is, $F(n) \geq 0$ ($n \geq 2$). Therefore, we have

$$\sum_{n=2}^{\infty} F(n) |a_n| \geq 0. \quad (4.10)$$

If $3/2 \leq \alpha \leq 13/8$, then $F(2) = 3-2\alpha \leq 0$ and

$$F(n) = 2n(2-\alpha) + 1 - 2\alpha \geq 13-8\alpha \geq 0 \quad (4.11)$$

for $n \geq 3$. Further, we know that

$$|a_n| \leq \frac{2(\alpha-1)}{(n-1)+|n-2\alpha+1|} \quad (n \geq 2), \quad (4.12)$$

then $|a_2| \leq 1$. Therefore, we obtain that

$$\sum_{n=2}^{\infty} F(n) |a_n| = F(2) |a_2| + \sum_{n=3}^{\infty} F(n) |a_n| \geq 3 - 2\alpha + \delta_3 \geq 0 \quad (4.13)$$

for

$$\frac{3}{2} \leq \alpha \leq \min\left(\frac{13}{8}, \frac{3+\delta_3}{2}\right). \quad (4.14)$$

Furthermore, if $13/8 \leq \alpha \leq 17/10$, then

$$\begin{aligned} F(2) &= 3 - 2\alpha \leq 0, \\ F(3) &= |4 - 2\alpha| + 3(3 - 2\alpha) = 13 - 8\alpha \leq 0, \\ F(n) &= |n - 2\alpha + 1| + (3 - 2\alpha)n = 4n + 1 - 2(n+1)\alpha \geq \frac{3(n-4)}{5} \geq 0 \end{aligned} \quad (4.15)$$

for $n \geq 4$. Noting that $|a_2| \leq 1$ and $|a_3| \leq (\alpha-1)/(3-\alpha)$, we conclude that

$$\begin{aligned} \sum_{n=2}^{\infty} F(n) |a_n| &= F(2) |a_2| + F(3) |a_3| + \sum_{n=4}^{\infty} F(n) |a_n| \\ &\geq (3 - 2\alpha) + (13 - 8\alpha) \frac{\alpha-1}{3-\alpha} + \delta_4 \geq 0, \end{aligned} \quad (4.16)$$

for α that satisfies

$$6\alpha^2 - (12 - \delta_4)\alpha + 4 - 3\delta_4 \leq 0. \quad (4.17)$$

This shows that

$$\frac{8}{3} \leq \alpha \leq \min\left(\frac{17}{10}, \frac{12 - \delta_4 + \sqrt{\delta_4^2 + 48\delta_4 + 48}}{12}\right). \quad (4.18)$$

This completes the proof of [Theorem 4.1](#). \square

Finally, by virtue of [Theorem 4.1](#), we may suppose that if $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \{(n-1) + |n-2\alpha+1|\} |a_n| \leq 2(\alpha-1) \quad (4.19)$$

for some $1 < \alpha < 2$, then $f(z) \in S^*$.

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