

ON THE EQUATION $x^2 + 2^a \cdot 3^b = y^n$

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We find all positive integer solutions (x, y, a, b, n) of $x^2 + 2^a \cdot 3^b = y^n$ with $n \geq 3$ and coprime x and y .

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1. Introduction. The Diophantine equation $x^2 + C = y^n$, where x and y are positive integers, $n \geq 3$ and C is a given integer, has received considerable interest. The earliest reference seems to be an assertion by Fermat that he had shown that when $C = 2$, $n = 3$, the only solution is given by $x = 5$, $y = 3$; a proof was published by Euler in 1770. The first result for general n is due to Lebesgue [9] who proved that there are no solutions for $C = 1$. Ljunggren [10] solved this equation for $C = 2$, Nagell [13, 14] solved it for $C = 3, 4$, and 5 and Chao [5] proved that it has no solutions for $C = -1$. For an extensive list of references one should consult Cohn's beautiful paper [6] in which he develops a method by which he finds all solutions of the above equation for 77 of the values of $C \leq 100$. This equation was later solved for two additional values of $C \leq 100$ (namely, $C = 74$ and $C = 86$) by Mignotte and de Weger [12]. It is interesting to mention that the equation $x^2 + 7 = y^n$ is still unsolved.

In recent years, a different form of the above equation has been considered, namely, when C is no longer a fixed integer but a power of a fixed prime. Le [8] investigated the equation $x^2 + 2^m = y^n$. Arif and Muriefah solved the equation $x^2 + 3^m = y^n$ when m is odd (see [2]). They also gave partial results in the case when m is even (see [1]) but the general solution in the case m is even was found by Luca in [11].

For any nonzero integer k , let $P(k)$ be the largest prime dividing k . Let C_1 be any fixed positive constant. It follows, from the work of Bugeaud [4] and Turk [15], that if

$$x^2 + z = y^n \quad \text{with } (x, y) = 1, P(z) < C_1, \quad (1.1)$$

then $\max(|x|, |y|, n)$ is bounded by a constant computable in terms of C_1 alone.

In this paper, we find all solutions of (1.1) when $C_1 = 5$ and $z > 0$. More precisely, we find all solutions of the equation

$$x^2 + 2^a \cdot 3^b = y^n \quad \text{with } a, b \geq 0, n \geq 3, (x, y) = 1. \quad (1.2)$$

The proof uses the new result on the existence of primitive divisors of the Lucas numbers due to Bilu et al. [3] as well as a computational result of de Weger [7].

2. The result

THEOREM 2.1. *All positive solutions of the equation*

$$x^2 + 2^a \cdot 3^b = y^n \quad \text{with } a, b \geq 0, n \geq 3, (x, y) = 1 \quad (2.1)$$

have $n = 4$ or $n = 3$. For $n = 4$, the solutions are

$$(x, y) = (7, 3), (23, 5), (7, 5), (47, 7), (287, 17). \quad (2.2)$$

For $n = 3$, the solutions are

$$(x, y) = (5, 3), (11, 5), (10, 7), (17, 7), (46, 13), (35, 13), \\ (595, 73), (955, 97), (2681, 193), (39151, 1153). \quad (2.3)$$

In the statement of the theorem we have listed only the values of x , y , and n as the values of the parameters a and b that can be determined from the prime factor decomposition of $x^2 - y^n$ once x , y , and n are given.

From Lebesgue's result, we know that the equation $x^2 + 1 = y^n$ has no positive solutions for $n \geq 3$ and from the work of Arif, Muriefah, and Luca, we know that the only positive solutions of the equation $x^2 + 3^m = y^n$ with $(x, y) = 1$ are $(x, y, m, n) = (10, 7, 5, 3)$ and $(46, 13, 4, 3)$. From now on, we assume that $a > 0$. In particular, both x and y are odd.

3. The case $n \neq 3$ or 4. In this section, we show that it suffices to assume that $n \in \{3, 4\}$. Indeed, assume that $n \neq 4$. We may certainly assume that n is an odd prime. If $n \neq 3$, it follows that $n \geq 5$. Write $2^a \cdot 3^b = dz^2$ where $d \in \{1, 2, 3, 6\}$. Equation (2.1) can be written as

$$(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n. \quad (3.1)$$

Since x is odd and dz^2 is even, it follows that the two ideals $[(x + i\sqrt{d}z)]$ and $[(x - i\sqrt{d}z)]$ are coprime in the ring of integers of $\mathbf{Q}(i\sqrt{d})$. Since the class number of $\mathbf{Q}(i\sqrt{d})$ is 1 or 2 and $n \geq 5$ is prime, it follows that there exists an integer u and a root of unity ε in $\mathbf{Q}(i\sqrt{d})$ such that

$$x + i\sqrt{d}z = \varepsilon u^n, \quad x - i\sqrt{d}z = \overline{\varepsilon} \bar{u}^n. \quad (3.2)$$

Since ε is a root of unity belonging to a quadratic extension of \mathbf{Q} , it follows that $\varepsilon^k = 1$ for some $k \in \{1, 2, 3, 4, 6\}$. Since $n \geq 5$ is prime, it follows that up to a substitution one may assume that $\varepsilon = 1$ in system (3.2). From (3.2) with $\varepsilon = 1$, it follows that

$$2i\sqrt{d}z = u^n - \bar{u}^n. \quad (3.3)$$

Since certainly

$$\frac{u^n - \bar{u}^n}{u - \bar{u}} \in \mathbf{Z}, \quad (3.4)$$

we have that

$$P\left(\frac{u^n - \bar{u}^n}{u - \bar{u}}\right) < 5. \quad (3.5)$$

From (3.5), we find that the Lucas number given by formula (3.4) has no primitive divisor. From [3], it follows that there are at most 10 pairs (u, n) satisfying inequality (3.5) and all of them appear in [3, Table 1]. A quick investigation reveals that none of the u 's from [3, Table 1] belongs to $\mathbb{Q}(i\sqrt{d})$ for some $d \in \{1, 2, 3, 6\}$, which is the desired contradiction.

4. The case $n = 4$. Let $S = \{k \mid P(k) < 5\}$. Then, we have the following preliminary result.

LEMMA 4.1. *All solutions of the equation*

$$x^2 = k \pm l \quad \text{with } k, l > 0, \ k, l \in S, \ (k, l) = 1 \quad (4.1)$$

are

$$\begin{aligned} (x, k, l) = & (1, 2, 1), (2, 3, 1), (3, 8, 1), (5, 24, 1), (7, 48, 1), \\ & (17, 288, 1), (1, 4, 3), (1, 9, 8), (5, 16, 9), (5, 27, 2), (7, 81, 32). \end{aligned} \quad (4.2)$$

PROOF OF LEMMA 4.1. This lemma is a particular case of a result of de Weger [7, Chapter 7]. \square

THE PROOF OF THE THEOREM FOR $n = 4$. Rewrite (2.1) as

$$(y^2 - x)(y^2 + x) = 2^a \cdot 3^b. \quad (4.3)$$

Since $a > 0$ and $(x, y) = 1$, it follows that $(y^2 - x, y^2 + x) = 2$. Thus,

$$y^2 - x = k, \quad y^2 + x = l, \quad \text{with } k, l > 0, \ k, l \in S, \ (k, l) = 2. \quad (4.4)$$

Hence,

$$y^2 = \frac{k}{2} + \frac{l}{2}, \quad (4.5)$$

where $k/2, l/2 \in S$ are positive and coprime. By Lemma 4.1, we obtain that (4.5) has only 6 solutions. Five of them lead to solutions (2.2) of (2.1). One of the solutions of (4.5) leads to

$$2^2 + 2^2 \cdot 3 = 2^4, \quad (4.6)$$

which is not a convenient solution of (2.1) because $x = 2$ and $y = 2$ are not coprime.

The case $n = 4$ is therefore settled.

5. The case $n = 3$. We begin with another lemma.

LEMMA 5.1. *The only solutions of the equation*

$$3x^2 = k \pm l \quad \text{with } k, l > 0, \ k, l \in S, \ (k, l) \in \{1, 3\} \quad (5.1)$$

are

$$(x, k, l) = (1, 2, 1), (1, 4, 1), (1, 6, 3), (2, 9, 3), \\ (3, 24, 3), (5, 72, 3), (7, 144, 3), (17, 864, 3), \\ (1, 12, 9), (1, 27, 24), (5, 48, 27), (5, 81, 6), (7, 243, 96). \quad (5.2)$$

PROOF OF LEMMA 5.1. This lemma too is a particular instance of the more general computation of de Weger [7, Chapter 7]. \square

THE PROOF OF THE THEOREM FOR $n = 3$. Write again $2^a \cdot 3^b = dz^2$ where $d \in \{1, 2, 3, 6\}$. From arguments employed in Section 3, we know that there exist u and ε in $\mathbb{Q}(i\sqrt{d})$ such that $y = |u|^2$, ε is a root of unity and

$$x + i\sqrt{d}z = \varepsilon u^3, \quad x - i\sqrt{d}z = \overline{\varepsilon u^3}. \quad (5.3)$$

Clearly,

$$2i\sqrt{d}z = \varepsilon u^3 - \overline{\varepsilon u^3}. \quad (5.4)$$

We distinguish two cases.

CASE 1 ($\varepsilon = 1$). Equation (5.4) reads

$$2i\sqrt{d}z = u^3 - \bar{u}^3. \quad (5.5)$$

Assume first that $u = a + ib\sqrt{d}$ with a and b integers. Then, we get

$$2i\sqrt{d}z = (a + ib\sqrt{d})^3 - (a - ib\sqrt{d})^3 \quad (5.6)$$

or

$$2i\sqrt{d}z = 2i\sqrt{d}b(3a^2 - db^2). \quad (5.7)$$

Hence, $b \mid z$ and

$$3a^2 = db^2 \pm \frac{z}{b}. \quad (5.8)$$

Let $k = db^2$ and $l = z/b$. Notice that $k, l \in S$. Moreover, notice that $(k, l) \in \{1, 3\}$. Indeed, if $(k, l) \notin \{1, 3\}$, it follows that there exists a prime p such that $p \mid (k, l, a)$. In particular, $p \mid db^2$ and $p \mid a$, therefore $p \mid a^2 + db^2 = y$. Since $p \mid z$ and $2^a \cdot 3^b = dz^2$, we come to $p \mid 2^a \cdot 3^b$. It follows now that $p \mid (y^3 - 2^a \cdot 3^b) = x^2$ and therefore $p \mid x$. This contradicts the fact that x and y are coprime. Now all solutions of (5.8) are given by Lemma 5.1. For example, the solution

$$3 \cdot 1^2 = 2^1 + 1 \quad (5.9)$$

gives either $a = 1$, $d = 2$, $b = 1$, and $z = 1$ or $a = 1$, $d = 1$, $b = 1$, and $z = 2$. The first possibility yields $y = a^2 + db^2 = 1 + 2 = 3$ and $dz^2 = 2$, which leads to the solution $3^3 = 2 + 5^2$ of (2.1). The second possibility gives $y = a^2 + db^2 = 2$ and $dz^2 = 4$, which leads to the solution $2^3 = 2^2 + 2^2$ of (2.1). This is not an acceptable solution, since $x = 2$ and $y = 2$ are not coprime.

All the solutions of (2.1) for the case $n = 3$ except for $(x, y) = (10, 7)$ are obtained in this way by identifying a, b, d , and z from (5.8) via [Lemma 5.1](#).

When $d = 3$, we also need to investigate the case in which

$$u = \frac{a + i\sqrt{3}b}{2} \quad (5.10)$$

for some odd integers a and b . From (5.5), we simply get that

$$16i\sqrt{3}z = (a + i\sqrt{3}b)^3 - (a - i\sqrt{3}b)^3 \quad (5.11)$$

or

$$16i\sqrt{3}z = 2i\sqrt{3}b(3a^2 - 3b^2). \quad (5.12)$$

It follows that b divides z and

$$3a^2 = 3b^2 \pm \frac{8z}{b}. \quad (5.13)$$

From [Lemma 5.1](#), we derive that (5.13) has only two convenient solutions, namely, $3 \cdot 1^2 = 3 \cdot 3^2 - 8 \cdot 3$ and $3 \cdot 7^2 = 3 \cdot 3^4 - 8 \cdot 12$. These lead to the solutions $(x, y) = (10, 7)$ and $(595, 73)$ of (2.1).

CASE 2 ($\varepsilon \neq 1$). It is easy to see that the only case in which one may not be able to set $\varepsilon = 1$ in system (5.4) is when $d = 3$. In this case, one may assume that $\varepsilon = (1 + i\sqrt{3})/2$ and that $u = (a + i\sqrt{3}b)/2$ for some integers a and b such that $a \equiv b \pmod{2}$. Then (5.4) becomes

$$2i\sqrt{3}z = \left(\frac{1+i\sqrt{3}}{2}\right) \cdot \left(\frac{a+i\sqrt{3}b}{2}\right)^3 - \left(\frac{1-i\sqrt{3}}{2}\right) \cdot \left(\frac{a-i\sqrt{3}b}{2}\right)^3. \quad (5.14)$$

This equation is equivalent to

$$16z = a^3 + 3a^2b - 9ab^2 - 3b^3. \quad (5.15)$$

Assume first that both a and b are odd. Then, from (5.15), it follows that

$$16z = (a^3 - ab^2) + (3a^2b - 3b^3) - 8ab^2 = (a^2 - b^2)(a + 3b) - 8ab^2. \quad (5.16)$$

Since a and b are both odd, we obtain that $16 \mid (a^2 - b^2)(a + 3b)$. Equation (5.16) forces $16 \mid 8ab^2$, which is impossible.

Assume now that both a and b are even. Since $y = (a/2)^2 + 3(b/2)^2$ is odd, it follows that exactly one of the numbers $a/2$ and $b/2$ is even. Equation (5.15) now implies that

$$2z = \left(\frac{a}{2}\right)^3 + 3\left(\frac{a}{2}\right)^2\left(\frac{b}{2}\right) - 9\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)^2 - 3\left(\frac{b}{2}\right)^3. \quad (5.17)$$

However, (5.17) is now impossible, because precisely one of the numbers $a/2$ and $b/2$ is even and the other one is odd. Hence, this case can never occur.

The theorem is therefore completely proved. \square

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