

# ON THE EQUATION $x^2 + 2^a \cdot 3^b = y^n$

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We find all positive integer solutions  $(x, y, a, b, n)$  of  $x^2 + 2^a \cdot 3^b = y^n$  with  $n \geq 3$  and coprime  $x$  and  $y$ .

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**1. Introduction.** The Diophantine equation  $x^2 + C = y^n$ , where  $x$  and  $y$  are positive integers,  $n \geq 3$  and  $C$  is a given integer, has received considerable interest. The earliest reference seems to be an assertion by Fermat that he had shown that when  $C = 2$ ,  $n = 3$ , the only solution is given by  $x = 5$ ,  $y = 3$ ; a proof was published by Euler in 1770. The first result for general  $n$  is due to Lebesgue [9] who proved that there are no solutions for  $C = 1$ . Ljunggren [10] solved this equation for  $C = 2$ , Nagell [13, 14] solved it for  $C = 3, 4$ , and 5 and Chao [5] proved that it has no solutions for  $C = -1$ . For an extensive list of references one should consult Cohn's beautiful paper [6] in which he develops a method by which he finds all solutions of the above equation for 77 of the values of  $C \leq 100$ . This equation was later solved for two additional values of  $C \leq 100$  (namely,  $C = 74$  and  $C = 86$ ) by Mignotte and de Weger [12]. It is interesting to mention that the equation  $x^2 + 7 = y^n$  is still unsolved.

In recent years, a different form of the above equation has been considered, namely, when  $C$  is no longer a fixed integer but a power of a fixed prime. Le [8] investigated the equation  $x^2 + 2^m = y^n$ . Arif and Muriefah solved the equation  $x^2 + 3^m = y^n$  when  $m$  is odd (see [2]). They also gave partial results in the case when  $m$  is even (see [1]) but the general solution in the case  $m$  is even was found by Luca in [11].

For any nonzero integer  $k$ , let  $P(k)$  be the largest prime dividing  $k$ . Let  $C_1$  be any fixed positive constant. It follows, from the work of Bugeaud [4] and Turk [15], that if

$$x^2 + z = y^n \quad \text{with } (x, y) = 1, P(z) < C_1, \quad (1.1)$$

then  $\max(|x|, |y|, n)$  is bounded by a constant computable in terms of  $C_1$  alone.

In this paper, we find all solutions of (1.1) when  $C_1 = 5$  and  $z > 0$ . More precisely, we find all solutions of the equation

$$x^2 + 2^a \cdot 3^b = y^n \quad \text{with } a, b \geq 0, n \geq 3, (x, y) = 1. \quad (1.2)$$

The proof uses the new result on the existence of primitive divisors of the Lucas numbers due to Bilu et al. [3] as well as a computational result of de Weger [7].

## 2. The result

**THEOREM 2.1.** *All positive solutions of the equation*

$$x^2 + 2^a \cdot 3^b = y^n \quad \text{with } a, b \geq 0, n \geq 3, (x, y) = 1 \quad (2.1)$$

*have  $n = 4$  or  $n = 3$ . For  $n = 4$ , the solutions are*

$$(x, y) = (7, 3), (23, 5), (7, 5), (47, 7), (287, 17). \quad (2.2)$$

*For  $n = 3$ , the solutions are*

$$(x, y) = (5, 3), (11, 5), (10, 7), (17, 7), (46, 13), (35, 13), \\ (595, 73), (955, 97), (2681, 193), (39151, 1153). \quad (2.3)$$

In the statement of the theorem we have listed only the values of  $x$ ,  $y$ , and  $n$  as the values of the parameters  $a$  and  $b$  that can be determined from the prime factor decomposition of  $x^2 - y^n$  once  $x$ ,  $y$ , and  $n$  are given.

From Lebesgue's result, we know that the equation  $x^2 + 1 = y^n$  has no positive solutions for  $n \geq 3$  and from the work of Arif, Muriefah, and Luca, we know that the only positive solutions of the equation  $x^2 + 3^m = y^n$  with  $(x, y) = 1$  are  $(x, y, m, n) = (10, 7, 5, 3)$  and  $(46, 13, 4, 3)$ . From now on, we assume that  $a > 0$ . In particular, both  $x$  and  $y$  are odd.

**3. The case  $n \neq 3$  or 4.** In this section, we show that it suffices to assume that  $n \in \{3, 4\}$ . Indeed, assume that  $n \neq 4$ . We may certainly assume that  $n$  is an odd prime. If  $n \neq 3$ , it follows that  $n \geq 5$ . Write  $2^a \cdot 3^b = dz^2$  where  $d \in \{1, 2, 3, 6\}$ . Equation (2.1) can be written as

$$(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n. \quad (3.1)$$

Since  $x$  is odd and  $dz^2$  is even, it follows that the two ideals  $[(x + i\sqrt{d}z)]$  and  $[(x - i\sqrt{d}z)]$  are coprime in the ring of integers of  $\mathbf{Q}(i\sqrt{d})$ . Since the class number of  $\mathbf{Q}(i\sqrt{d})$  is 1 or 2 and  $n \geq 5$  is prime, it follows that there exists an integer  $u$  and a root of unity  $\varepsilon$  in  $\mathbf{Q}(i\sqrt{d})$  such that

$$x + i\sqrt{d}z = \varepsilon u^n, \quad x - i\sqrt{d}z = \overline{\varepsilon} \bar{u}^n. \quad (3.2)$$

Since  $\varepsilon$  is a root of unity belonging to a quadratic extension of  $\mathbf{Q}$ , it follows that  $\varepsilon^k = 1$  for some  $k \in \{1, 2, 3, 4, 6\}$ . Since  $n \geq 5$  is prime, it follows that up to a substitution one may assume that  $\varepsilon = 1$  in system (3.2). From (3.2) with  $\varepsilon = 1$ , it follows that

$$2i\sqrt{d}z = u^n - \bar{u}^n. \quad (3.3)$$

Since certainly

$$\frac{u^n - \bar{u}^n}{u - \bar{u}} \in \mathbf{Z}, \quad (3.4)$$

we have that

$$P\left(\frac{u^n - \bar{u}^n}{u - \bar{u}}\right) < 5. \quad (3.5)$$

From (3.5), we find that the Lucas number given by formula (3.4) has no primitive divisor. From [3], it follows that there are at most 10 pairs  $(u, n)$  satisfying inequality (3.5) and all of them appear in [3, Table 1]. A quick investigation reveals that none of the  $u$ 's from [3, Table 1] belongs to  $\mathbf{Q}(i\sqrt{d})$  for some  $d \in \{1, 2, 3, 6\}$ , which is the desired contradiction.

**4. The case  $n = 4$ .** Let  $S = \{k \mid P(k) < 5\}$ . Then, we have the following preliminary result.

**LEMMA 4.1.** *All solutions of the equation*

$$x^2 = k \pm l \quad \text{with } k, l > 0, k, l \in S, (k, l) = 1 \quad (4.1)$$

are

$$\begin{aligned} (x, k, l) = & (1, 2, 1), (2, 3, 1), (3, 8, 1), (5, 24, 1), (7, 48, 1), \\ & (17, 288, 1), (1, 4, 3), (1, 9, 8), (5, 16, 9), (5, 27, 2), (7, 81, 32). \end{aligned} \quad (4.2)$$

**PROOF OF LEMMA 4.1.** This lemma is a particular case of a result of de Weger [7, Chapter 7].  $\square$

**THE PROOF OF THE THEOREM FOR  $n = 4$ .** Rewrite (2.1) as

$$(y^2 - x)(y^2 + x) = 2^a \cdot 3^b. \quad (4.3)$$

Since  $a > 0$  and  $(x, y) = 1$ , it follows that  $(y^2 - x, y^2 + x) = 2$ . Thus,

$$y^2 - x = k, \quad y^2 + x = l, \quad \text{with } k, l > 0, k, l \in S, (k, l) = 2. \quad (4.4)$$

Hence,

$$y^2 = \frac{k}{2} + \frac{l}{2}, \quad (4.5)$$

where  $k/2, l/2 \in S$  are positive and coprime. By Lemma 4.1, we obtain that (4.5) has only 6 solutions. Five of them lead to solutions (2.2) of (2.1). One of the solutions of (4.5) leads to

$$2^2 + 2^2 \cdot 3 = 2^4, \quad (4.6)$$

which is not a convenient solution of (2.1) because  $x = 2$  and  $y = 2$  are not coprime.

The case  $n = 4$  is therefore settled.

**5. The case  $n = 3$ .** We begin with another lemma.

**LEMMA 5.1.** *The only solutions of the equation*

$$3x^2 = k \pm l \quad \text{with } k, l > 0, k, l \in S, (k, l) \in \{1, 3\} \quad (5.1)$$

are

$$\begin{aligned} (x, k, l) = & (1, 2, 1), (1, 4, 1), (1, 6, 3), (2, 9, 3), \\ & (3, 24, 3), (5, 72, 3), (7, 144, 3), (17, 864, 3), \\ & (1, 12, 9), (1, 27, 24), (5, 48, 27), (5, 81, 6), (7, 243, 96). \end{aligned} \quad (5.2)$$

**PROOF OF LEMMA 5.1.** This lemma too is a particular instance of the more general computation of de Weger [7, Chapter 7].  $\square$

**THE PROOF OF THE THEOREM FOR  $n = 3$ .** Write again  $2^a \cdot 3^b = dz^2$  where  $d \in \{1, 2, 3, 6\}$ . From arguments employed in Section 3, we know that there exist  $u$  and  $\varepsilon$  in  $\mathbf{Q}(i\sqrt{d})$  such that  $y = |u|^2$ ,  $\varepsilon$  is a root of unity and

$$x + i\sqrt{d}z = \varepsilon u^3, \quad x - i\sqrt{d}z = \overline{\varepsilon} \overline{u}^3. \quad (5.3)$$

Clearly,

$$2i\sqrt{d}z = \varepsilon u^3 - \overline{\varepsilon} \overline{u}^3. \quad (5.4)$$

We distinguish two cases.

**CASE 1** ( $\varepsilon = 1$ ). Equation (5.4) reads

$$2i\sqrt{d}z = u^3 - \overline{u}^3. \quad (5.5)$$

Assume first that  $u = a + ib\sqrt{d}$  with  $a$  and  $b$  integers. Then, we get

$$2i\sqrt{d}z = (a + ib\sqrt{d})^3 - (a - ib\sqrt{d})^3 \quad (5.6)$$

or

$$2i\sqrt{d}z = 2i\sqrt{d}b(3a^2 - db^2). \quad (5.7)$$

Hence,  $b \mid z$  and

$$3a^2 = db^2 \pm \frac{z}{b}. \quad (5.8)$$

Let  $k = db^2$  and  $l = z/b$ . Notice that  $k, l \in S$ . Moreover, notice that  $(k, l) \in \{1, 3\}$ . Indeed, if  $(k, l) \notin \{1, 3\}$ , it follows that there exists a prime  $p$  such that  $p \mid (k, l, a)$ . In particular,  $p \mid db^2$  and  $p \mid a$ , therefore  $p \mid a^2 + db^2 = y$ . Since  $p \mid z$  and  $2^a \cdot 3^b = dz^2$ , we come to  $p \mid 2^a \cdot 3^b$ . It follows now that  $p \mid (y^3 - 2^a \cdot 3^b) = x^2$  and therefore  $p \mid x$ . This contradicts the fact that  $x$  and  $y$  are coprime. Now all solutions of (5.8) are given by Lemma 5.1. For example, the solution

$$3 \cdot 1^2 = 2^1 + 1 \quad (5.9)$$

gives either  $a = 1$ ,  $d = 2$ ,  $b = 1$ , and  $z = 1$  or  $a = 1$ ,  $d = 1$ ,  $b = 1$ , and  $z = 2$ . The first possibility yields  $y = a^2 + db^2 = 1 + 2 = 3$  and  $dz^2 = 2$ , which leads to the solution  $3^3 = 2 + 5^2$  of (2.1). The second possibility gives  $y = a^2 + db^2 = 2$  and  $dz^2 = 4$ , which leads to the solution  $2^3 = 2^2 + 2^2$  of (2.1). This is not an acceptable solution, since  $x = 2$  and  $y = 2$  are not coprime.

All the solutions of (2.1) for the case  $n = 3$  except for  $(x, y) = (10, 7)$  are obtained in this way by identifying  $a, b, d$ , and  $z$  from (5.8) via Lemma 5.1.

When  $d = 3$ , we also need to investigate the case in which

$$u = \frac{a + i\sqrt{3}b}{2} \quad (5.10)$$

for some odd integers  $a$  and  $b$ . From (5.5), we simply get that

$$16i\sqrt{3}z = (a + i\sqrt{3}b)^3 - (a - i\sqrt{3}b)^3 \quad (5.11)$$

or

$$16i\sqrt{3}z = 2i\sqrt{3}b(3a^2 - 3b^2). \quad (5.12)$$

It follows that  $b$  divides  $z$  and

$$3a^2 = 3b^2 \pm \frac{8z}{b}. \quad (5.13)$$

From Lemma 5.1, we derive that (5.13) has only two convenient solutions, namely,  $3 \cdot 1^2 = 3 \cdot 3^2 - 8 \cdot 3$  and  $3 \cdot 7^2 = 3 \cdot 3^4 - 8 \cdot 12$ . These lead to the solutions  $(x, y) = (10, 7)$  and  $(595, 73)$  of (2.1).

**CASE 2** ( $\varepsilon \neq 1$ ). It is easy to see that the only case in which one may not be able to set  $\varepsilon = 1$  in system (5.4) is when  $d = 3$ . In this case, one may assume that  $\varepsilon = (1 + i\sqrt{3})/2$  and that  $u = (a + i\sqrt{3}b)/2$  for some integers  $a$  and  $b$  such that  $a \equiv b \pmod{2}$ . Then (5.4) becomes

$$2i\sqrt{3}z = \left(\frac{1 + i\sqrt{3}}{2}\right) \cdot \left(\frac{a + i\sqrt{3}b}{2}\right)^3 - \left(\frac{1 - i\sqrt{3}}{2}\right) \cdot \left(\frac{a - i\sqrt{3}b}{2}\right)^3. \quad (5.14)$$

This equation is equivalent to

$$16z = a^3 + 3a^2b - 9ab^2 - 3b^3. \quad (5.15)$$

Assume first that both  $a$  and  $b$  are odd. Then, from (5.15), it follows that

$$16z = (a^3 - ab^2) + (3a^2b - 3b^3) - 8ab^2 = (a^2 - b^2)(a + 3b) - 8ab^2. \quad (5.16)$$

Since  $a$  and  $b$  are both odd, we obtain that  $16 \mid (a^2 - b^2)(a + 3b)$ . Equation (5.16) forces  $16 \mid 8ab^2$ , which is impossible.

Assume now that both  $a$  and  $b$  are even. Since  $y = (a/2)^2 + 3(b/2)^2$  is odd, it follows that exactly one of the numbers  $a/2$  and  $b/2$  is even. Equation (5.15) now implies that

$$2z = \left(\frac{a}{2}\right)^3 + 3\left(\frac{a}{2}\right)^2\left(\frac{b}{2}\right) - 9\left(\frac{a}{2}\right)\left(\frac{b}{2}\right)^2 - 3\left(\frac{b}{2}\right)^3. \quad (5.17)$$

However, (5.17) is now impossible, because precisely one of the numbers  $a/2$  and  $b/2$  is even and the other one is odd. Hence, this case can never occur.

The theorem is therefore completely proved.  $\square$

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