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ON ALMOST PERIODIC SOLUTIONS OF THE DIFFERENTIAL EQUATION $x''(t) = Ax(t)$ IN HILBERT SPACES

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ABSTRACT. We prove almost periodicity of solutions of the equation $x''(t) = Ax(t)$ when the linear operator A satisfies an inequality of the form $\text{Re}(Ax,x) \geq 0$.

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1. Introduction. Let H be a Hilbert space equipped with norm $\|\cdot\|$ and scalar product $(\cdot,\cdot)$. Almost periodic functions (in Bochner's sense) are continuous functions $f : \mathbb{R} \to H$ such that for every $\epsilon > 0$, there exists a positive real number l such that every interval $[a,a+l]$ contains at least a point τ such that

$$\sup_{t\in\mathbb{R}} \left\|f(t+\tau)-f(t)\right\| < \epsilon. \tag{1.1}$$

The Bochner's criterion (cf. [1, 3, 4]) states that a function $f : \mathbb{R} \to H$ is almost periodic if and only if for every sequence of real numbers $(\sigma_n)_{n=1}^{\infty}$ there exists a subsequence $(s_n)_{n=1}^{\infty}$ such that $(f(t+s_n))_{n=1}^{\infty}$ is uniformly convergent in $t \in \mathbb{R}$.

We proved in [2] that if $A = A_+ + A_-$, where A_+ is a symmetric linear operator and A_- is a skew-symmetric linear operator such that $\text{Re}(A_+x, A_-x) \geq -c\|A_+x\|^2$ for every $x \in H$, then every solution of $x'(t) = Ax(t)$, $t \in \mathbb{R}$, with a relatively compact range in H is almost periodic.

In this note, we use the technique described in [2] to prove similar results for some classes of linear differential equation of second order $x''(t) = Ax(t)$.

2. Main results

THEOREM 2.1. *Assume that the linear operator A satisfies the inequality of the form $\text{Re}(Ax,x) \geq 0$, for any $x \in H$. Then solutions of the differential equation*

$$x''(t) = Ax(t), \quad t \in \mathbb{R}, \tag{2.1}$$

(that are functions $x(t) \in C^2(\mathbb{R},H)$) with relatively compact ranges in H, are almost periodic.

PROOF. Consider $x(t)$ a solution of (2.1) with a relatively compact range in H and let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(t) = \|x(t)\|^2$. Then ϕ is a bounded function over $\mathbb{R}$.

Moreover, for every $t \in \mathbb{R}$, we have

$$\begin{aligned} \phi'(t) &= (x'(t), x(t)) + (x(t), x'(t)), \\ \phi''(t) &= 2\left[\|x'(t)\|^2 + \operatorname{Re}(Ax(t), x(t))\right] \\ &\geq 0, \end{aligned} \tag{2.2}$$

which shows that ϕ is a convex function over $\mathbb{R}$, therefore it is constant

$$\phi(t) = \phi(0), \quad \forall t \in \mathbb{R}, \tag{2.3}$$

or

$$\|x(t)\| = \|x(0)\|, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

We fix $s \in \mathbb{R}$ and consider the function $y_s(\cdot) : \mathbb{R} \to H$ defined by

$$y_s(t) = x(t+s). \tag{2.5}$$

Then $y_s(t)$ obviously satisfies (2.1). Now fix s_1 and s_2 in $\mathbb{R}$. Then $y_{s_1}(t) - y_{s_2}(t)$ also satisfies (2.1); therefore we have

$$\|y_{s_1}(t) - y_{s_2}(t)\| = \|y_{s_1}(0) - y_{s_2}(0)\|, \quad \forall t \in \mathbb{R}, \tag{2.6}$$

which gives

$$\|x(t+s_1) - x(t+s_2)\| = \|x(s_1) - x(s_2)\|, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

Let $(\sigma_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then by relative compactness of $x(t)$, there exists a subsequence $(s_n)_{n=1}^{\infty} \subset (\sigma_n)_{n=1}^{\infty}$ such that $(x(s_n))_{n=1}^{\infty}$ is Cauchy. Hence for any given $\epsilon > 0$, there exists N such that if $n, m > N$, then

$$\|x(s_n) - x(s_m)\| < \epsilon. \tag{2.8}$$

Consequently,

$$\sup_{t \in \mathbb{R}} \|x(t+s_n) - x(t+s_m)\| < \epsilon. \tag{2.9}$$

We conclude that $x(t)$ is almost periodic by the Bochner's criterion. □

Remark 2.2. Examples of such problem occur when A is a positive or monotone linear operator.

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