REFLEXIVE AND DIHEDRAL (CO)HOMOLOGY OF A PRE-ADDITIVE CATEGORY

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ABSTRACT. The group dihedral homology of an algebra over a field with characteristic zero was introduced by Tsygan (1983). The dihedral homology and cohomology of an algebra with involution over commutative ring with identity, associated with the small category, were studied by Krasauskas et al. (1988), Loday (1987), and Lodder (1993). The aim of this work is concerned with dihedral and reflexive (co)homology of small pre-additive category. We also define the free product of involutive algebras associated with this category and study its dihedral homology group. Finally, following Perelygin (1990), we show that a small pre-additive category is Morita equivalence.

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1. Preliminaries. Suppose that $\Delta\mathcal{D}$ is a small category, with objects, the set $\{[0],[1],\ldots,[n],\ldots\}$, and the following family of morphisms $\delta_n^i : [n] \to [n-1]$, $0 \le i \le n$, $\sigma_n^j : [n] \to [n+1]$, $0 \le i \le n$, $\tau_n : [n] \to [n]$, $\rho_n : [n] \to [n]$ such that

$$\delta_{n+1}^j \delta_n^i = \delta_{n+1}^i \delta_n^{i-1}, \quad i < j,$$

$$\sigma_n^j \sigma_{n+1}^i = \sigma_n^i \sigma_{n+1}^{j-1}, \quad i \le j,$$

$$\sigma_{n+1}^j \delta_n^1 = \begin{cases} \sigma_{n-2}^{i-1} \delta_{n-1}^j, & i \le j, \\ \mathrm{Id}, & i = j, j+1, \\ \sigma_{n-2}^i \delta_{n-1}^{i-1}, & i > j, \end{cases} \tag{1.1}$$

$$\tau_n \delta_n^i = \delta_n^{i-1} \tau_{n-1}, \quad 0 \le i \le n, \qquad (\tau_n)^{n+1} = 1_n, \qquad \rho_n^2 = 1,$$

$$\tau_n \sigma_n^j = \sigma_n^{j-1} \tau_{n+1}, \quad 0 \le j \le n, \qquad \tau_n \rho_n = \rho_n \tau_n^{-1}.$$

DEFINITION 1.1. The category $\Delta\mathcal{D}$ is called a dihedral category. Note that the category generated by only the morphisms δ_n^i and σ_n^j is called a simplicial category and is denoted by Δ, the category generated by δ_n^i, σ_n^j, and τ_n is a cyclic category and is denoted by ΔC (see [6]), and the category generated by the family of morphisms δ_n^i, σ_n^j, and ρ_n is called a reflexive category and is denoted by ΔR.

DEFINITION 1.2 (see [3]). Let k be a commutative ring with identity and involution. An algebra over k associated with the category $\Delta\mathcal{D}(\Delta R)$ is an algebra with identity generated by the morphisms $\delta_n^i, \sigma_n^j, \tau_n$, and ρ_n $(\delta_n^i, \sigma_n^j, \rho_n)$.

DEFINITION 1.3. For an arbitrary category $\mathbb{C}$, following [3], for case of presentation, a simplicial object in the category $\mathbb{C}$ is a functor $\mathscr{F} : \Delta^{\mathrm{op}} \to \mathbb{C}$ (the category Δ^{op} is the inverse of Δ).

DEFINITION 1.4. Following [2] (see also [6]), for an arbitrary category $\mathbb{C}$, the dihedral (reflexive) object in $\mathbb{C}$ is a functor $\mathscr{F} : \Delta D^{\mathrm{op}} \to \mathbb{C}$, $(\mathscr{F} : \Delta \mathscr{R}^{\mathrm{op}} \to \mathbb{C})$. If we drop the morphism ρ_n from the group family of morphisms $(\delta_n^i, \sigma_n^j, \tau_n, \rho_n)$, we get a cyclic object of an arbitrary category $\mathbb{C}$ (see [5]). Suppose that $\mathscr{F}([n]) = X_n$, $\mathscr{F}(\delta_n^i) = d_n^i$, $\mathscr{F}(\sigma_n^j) = \mathfrak{s}_n^j$, $\mathscr{F}(\tau_n) = t_n$, $\mathscr{F}(\rho_n) = \mathfrak{r}_n$. We write the dihedral (reflexive) object by the family $(X_n, d_n^i, \mathfrak{s}_n^j, t_n, \mathfrak{r}_n)$ $(X_n, d_n^i, \mathfrak{s}_n^j, \mathfrak{r}_n)$. We can easily check that the morphisms $d_n^i, \mathfrak{s}_n^j, t_n$, and $\mathfrak{r}_n$ satisfy relations (1.1).

DEFINITION 1.5. Let k be a commutative ring with identity and involution and let $\mathbb{C}$ be a category of k-modules. The dihedral k-module in $\mathbb{C}$ is defined to be the dihedral objects $(X_n, d_n^i, \mathfrak{s}_n^j, t_n, \mathfrak{r}_n)$ in the category $\mathbb{C}$.

2. The reflexive and dihedral homology of pre-additive category. In this section, we define the dihedral k-module associated with a pre-additive category and study its (co)homology.

DEFINITION 2.1. Let k be a commutative ring with identity and involution. Following [2], the k-category A with an involution is defined to be a small pre-additive category with objects the k-modules of set morphisms $A(i,j)$, where, i,j are in A, and the bilinear maps $A(i,j) x A(j,k) \to A(i,k)$, as morphisms. Suppose that, for all objects $i,j \in A$, there exists a k-linear map $* : A(i,j) \to A(j,i)$, such that $*^2 : A(i,j) \to A(i,j)$. Define the family $M = \{M_n\}_{n \geq 0}$ of k-modules and k-morphisms as follows:

$$M_0 = \bigoplus_{i_0 \in |A|} A(i_0, i_0), \ldots, M_n = \bigoplus_{i_0, i_1, \ldots, i_n \in |A|} A(i_0, i_0) \underset{k}{\otimes} A(i_1, i_2) \underset{k}{\otimes} \cdots \underset{k}{\otimes} A(i_n, i_0). \quad (2.1)$$

On the family $M = (M_n)$, define the morphisms $d_{n'}^i \mathfrak{s}_{n'}^j t_{n'} \mathfrak{r}_n$ as follows:

$$d_n^i : M_n \longrightarrow M_{n-1}, \qquad \mathfrak{s}_n^j : M_n \longrightarrow M_{n-1}, \qquad \mathfrak{r}_n, t_n : M_n \longrightarrow M_n, \qquad (2.2)$$

such that

$$
\begin{aligned}
d_n^i(a_0 \otimes \cdots \otimes a_n) &= d_n^i(a_0 \otimes \cdots \otimes a_n) + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}, \\
d_n^i(a_0 \otimes \cdots \otimes a_n) &= \sum_{k=0}^{n-1} (-1)^k a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n, \\
\mathfrak{s}_n^j(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n, \\
t_n(a_0 \otimes \cdots \otimes a_n) &= (-1)^n (a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}), \\
\mathfrak{r}_n(a_0 \otimes \cdots \otimes a_n) &= \alpha(-1)^{n(n+1)/2} a_0^* \otimes a_n^* \otimes \cdots \otimes a_1^*, \quad \alpha = \pm 1,
\end{aligned}
\qquad (2.3)
$$

where a_i^*'s are the image of the elements a_i $(0 \leq i \leq n)$ under the involution $*$. Clearly, the module $M = \{M_n\}$ under the last morphisms is a dihedral k-module. Now,

we define the dihedral homology group. Suppose that $M = (M_n) = C^{p,q,r}$, $p, q, r > 0$, and consider the complex $(C^{p,q,r}, \delta^i)$, $i = 1, 2$ (see [3]), where

$$\delta^1 : C^{p,q,r} \longrightarrow C^{p-1,q,r}, \qquad \delta^2 : C^{p,q,r} \longrightarrow C^{p,q-1,r}, \qquad (2.4)$$

are defined by

$$\delta^1 = \begin{cases} 1 - T_n, & n = 1 \ (\mathrm{mod}\,2), \\ N_n = 1 + T_n + \cdots + T_n^n, & n = 0 \ (\mathrm{mod}\,2), \end{cases}$$

$$\delta^2 = \begin{cases} b_{n+1} = (-1)^i d_{n+1}^i, & n = 0 \ (\mathrm{mod}\,2), \\ -b = (-1)^i d^i, & n = 1 \ (\mathrm{mod}\,2). \end{cases} \qquad (2.5)$$

Clearly, by definition, $\delta^i \cdot \delta^i = 0$, $i = 1, 2$. The complex $(C^{p,q,r}, \delta^i)$ can be illustrated by Tsygan's bicomplex $\mathbb{C}(M)$ (see [5]):

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
{\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & \\
C^{0,2,2} & \xleftarrow{1-T} & C^{1,2,2} & \xleftarrow{N} & C^{2,2,2} & \xleftarrow{1-T} & C^{3,2,2} & \xleftarrow{N} & \cdots \\
{\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & \\
C^{0,1,1} & \xleftarrow{1-T} & C^{1,1,1} & \xleftarrow{N} & C^{2,1,1} & \xleftarrow{1-T} & C^{3,1,1} & \xleftarrow{N} & \cdots \\
{\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & \\
C^{0,0,0} & \xleftarrow{1-T} & C^{1,0,0} & \xleftarrow{N} & C^{2,0,0} & \xleftarrow{1-T} & C^{3,0,0} & \xleftarrow{N} & \cdots,
\end{array} \qquad (2.6)
$$

where the morphisms $b, -b' : C^{p,q,r} \to C^{p,q-1,r}$ are given by $b = \sum_{i=0}^{n}(-1)^i d_n^i$, $-b' = \sum_{i=0}^{n-1}(-1)^i d_n^i$, $T = (-1)^n t_n$, $N = 1 + T_n + \cdots + T_n^n$. Following [5], the homology of the bicomplex (2.6) gives the cyclic homology group: $\mathcal{H}C_n(M) = \mathcal{H}_n(\mathbb{C}(M))$. Following [3], if we act by the group $\mathbb{Z}/2$ on the bicomplex (2.6): on the column 2ℓ ($\ell > 0$) by means of the automorphism

$$(-1)^{n(n+1)/2+\ell} r_n = (-1)^\ell R_n, \quad \text{where } R_n(a_0 \otimes \cdots \otimes a_n) = (-1)^{n(n+1)/2} r_n, \qquad (2.7)$$

and on the column $2\ell + 1$ by means of the automorphism $(-1)^{n(n-1)/2+\ell+1} r_n = (-1)^\ell R_n T_n$, we get the tricomplex $(C^{p,q,r}, \delta^i)$, $i = 1, 2, 3$. The differentials δ^1, δ^2 are defined in (2.5) and $\delta^3 : C^{p,q,r+1} \to C^{p,q,r}$ is defined by

$$\delta^3 = \begin{cases} (-1)^n (1 + (-1)^\ell R_n), & n = 0 \ (\mathrm{mod}\,4), \\ (-1)^{n+1} (1 + (-1)^{\ell+1} R_n T_n), & n = 1 \ (\mathrm{mod}\,4), \\ (-1)^n (1 + (-1)^{\ell+1} R_n), & n = 2 \ (\mathrm{mod}\,4), \\ (-1)^{n+1} (1 + (-1)^{\ell+1} R_n T_n), & n = 3 \ (\mathrm{mod}\,4),\ R_n = (-1)^n r_n. \end{cases} \qquad (2.8)$$

Following [4], the dihedral homology of the module $M(^\alpha \mathcal{H}\mathcal{D}(M))$ is the homology of the complex $(C^{p,q,r}, \delta^i)$, $i = 1, 2, 3$, $\alpha = \pm 1$.

DEFINITION 2.2. The dihedral homology of a k-category A with involution is the dihedral homology of the associated dihedral k-module $M = \{M_n\} : {}^{\alpha}\mathscr{HD}_n(A) = {}^{\alpha}\mathscr{HD}_n(M)$, $\alpha = \pm 1$.

DEFINITION 2.3. The reflexive homology of a k-category A with an involution is the reflexive homology of k-module $\{M_{\text{refl}}\}$:

$$ {}^{\alpha}\mathscr{HR}_n(A) = {}^{\alpha}\mathscr{HR}_n(M_{\text{refl}}), \quad \alpha \pm 1, \tag{2.9} $$

where M_{refl} is the reflexive k-module $M = \{M_n\}$. Similarly, if we take the cyclic k-module M_{cycl}, we obtain the cyclic homology of the k-category A (see [6]): $\mathscr{HC}_n(A) = \mathscr{HR}_n(M_{\text{cycl}})$. Following [3, 4], the dihedral (reflexive) homology of the dihedral (reflexive) module M can be considered as derived functor

$$ {}^{\alpha}\mathscr{HD}_n(M) = \text{Tor}_n^{k[\Delta D]^{\text{op}}}\left(k^{\mathscr{D}} \cdot M \cdot\right)\left({}^{\alpha}\mathscr{HR}_n(M) = \text{Tor}_n^{k[\Delta R]^{\text{op}}}\left(k^{\mathscr{R}} \cdot M \cdot\right)\right), \tag{2.10} $$

where $k^{\mathscr{D}}(k^{\mathscr{R}})$ is a trivial dihedral (reflexive) k-module, $k[\Delta D]^{\text{op}}(k[\Delta R]^{\text{op}})$ is the algebra associated with the dihedral (reflexive) category.

Note that (see [3]) the dihedral (reflexive) homology is considered as the hyperhomology of the group $\mathbb{Z}/2$ with coefficients in Tsygan bicomplex (simplicial (Hochschild) complex). The relations between the cyclic and the dihedral homology and also the reflexive and the dihedral homology of pre-additive category are given by the following assertions.

THEOREM 2.4. *Let k be a commutative ring and let A be a k-category with an involution. Then there exist the following exact sequences*

$$ \cdots \longrightarrow {}^{-\alpha}\mathscr{HD}_n(A) \longrightarrow \mathscr{HC}_n(A) \longrightarrow {}^{+\alpha}\mathscr{HD}_n(A) \longrightarrow {}^{-\alpha}\mathscr{HD}_{n-1}(A) \longrightarrow \cdots, $$

$$ \cdots \longrightarrow {}^{+\alpha}\mathscr{HR}_n(A) \longrightarrow {}^{+\alpha}\mathscr{HD}_n(A) \longrightarrow {}^{-\alpha}\mathscr{HD}_{n-2}(A) \longrightarrow {}^{+\alpha}\mathscr{HR}_{n-1}(A) \longrightarrow \cdots. \tag{2.11} $$

PROOF. The proof follows from [3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

COROLLARY 2.5. *Let $1/2 \in k$. Then there exists the natural isomorphism*

$$ \mathscr{HC}_n(A) \cong {}^{-\alpha}\mathscr{HD}_n(A) \oplus {}^{+\alpha}\mathscr{HD}_n(A). \tag{2.12} $$

Note that we can define the reflexive cohomology and the dihedral cohomology of a pre-additive category in the same manner.

3. The dihedral homology of free product algebras. In this section, we study the product of the algebras associated with a pre-additive k-category, where k has characteristic zero. Let A, B, and C be arbitrary involutive algebras. The free product of the algebras A and B with respect to algebra C is denoted by $A \underset{c}{*} B$. Following [1], the algebra $A \underset{c}{*} B$ is C-bimodule. For the algebras A, B, and C,

$$ \text{Tor}_i^C(A, A) = \text{Tor}_i^C(A, B) = \text{Tor}_i^C(B, A) = \text{Tor}_i^C(B, B) = 0. \tag{3.1} $$

Consider also the following diagram of algebras and homomorphisms between them:

$$A \underset{i_1}{\overset{j_1}{\rightleftarrows}} C \underset{j_2}{\overset{i_2}{\rightleftarrows}} B, \quad j_1 \circ i_1 = \mathrm{Id}_C = j_2 \circ i_2. \tag{3.2}$$

Following [3], let R^A and R^B be the free involutive resolution of the algebras A and B over the homomorphisms i_1 and i_2, respectively. Consequently, we get the following diagram:

$$R^A \overset{i_1^A}{\longleftarrow} C \overset{i_2^B}{\longrightarrow} R^B . \tag{3.3}$$

Consider the diagrams

$$
\begin{array}{cc}
R^A \overset{i_1^A}{\longleftarrow} C & \qquad R^B \overset{i_2^B}{\longleftarrow} C \\
\pi^A \downarrow \quad \swarrow & \qquad \pi^B \downarrow \quad \swarrow \\
A & \qquad B
\end{array}
\tag{3.4}
$$

Clearly, they are commutative. If we define the homomorphisms j_1^A and j_2^B as follows:

$$R^A \overset{j_1^A}{\longrightarrow} R^B, \quad R^B \overset{j_2^B}{\longrightarrow} C, \quad \text{where } j_1^A = j_1 \circ \pi^A, \ j_2^B = j_2 \circ \pi^A, \tag{3.5}$$

we get the diagram

$$R^A \underset{i_1^A}{\overset{j_1^A}{\rightleftarrows}} C \underset{j_2^B}{\overset{i_2^B}{\rightleftarrows}} R^B, \tag{3.6}$$

where $j_1^A \circ i_1^A = \mathrm{Id}_C$, $j_2^B \circ i_2^B = \mathrm{Id}_C$, since $j_1^A \circ i_1^A = (j_1 \circ \pi^A) \circ i_1^A = j_1 \circ i_1 = \mathrm{Id}_c$, $j_2^B \circ i_2^B = (j_2 \circ \pi^B) \circ i_2^B = j_2 \circ i_2 = \mathrm{Id}_c$. Suppose that $\bar{R}^A = \ker j_1^A$, $\bar{R}^B = \ker j_2^B$. Then $R^A \underset{c}{*} R^B$ is a C-module given by

$$R^A \underset{c}{*} R^B = C + \bar{R}^A \underset{c}{*} \bar{R}^B + \left(R^A \underset{c}{\otimes} R^B \right) + \left(R^B \underset{c}{\otimes} R^A \right) + \left(R^A \underset{c}{\otimes} R^B \underset{c}{\otimes} R^A \right) + \cdots. \tag{3.7}$$

We define an involution on $R^A \underset{c}{*} R^B$ as follows:

$$
\begin{aligned}
(p_0 q_0 p_1 q_1 \cdots &p_n q_n q_{n+1})^* \\
&= p_{n+1}^* q_n^* p_n^* \cdots q_0^* p_0^*, \quad p_i \in R^A, \ 0 \le i \le n+1, \ q_i \in R^B, \ 0 \le i \le n.
\end{aligned}
\tag{3.8}
$$

REMARKS. (i) The differential on $R^A \underset{c}{*} R^B$ is defined by Leibniz formula for differential graded algebras [5].

(ii) The chain complex $\bar{R}^A$ is a free C-biomodule resolution of the C-biomodule $\bar{A}$.

(iii) $\bar{A} + C = A = H \cdot (R^A) = H \cdot (\bar{R}^A \oplus C) = H \cdot (\bar{R}^A) + C$, that is, $H \cdot (\bar{R}^A) = \bar{A}$, where H is a hyperhomology of $\bar{R}^A$. From (3.1), (i), (ii), and (iii), we have

$$\mathrm{Tor}_i^c (\bar{A}, \bar{A}) = \mathrm{Tor}_i^c (\bar{A}, \bar{B}) = \mathrm{Tor}_i^c (\bar{B}, \bar{A}) = \mathrm{Tor}_i^c (\bar{B}, \bar{B}) = 0. \tag{3.9}$$

From (3.7) and (3.9), we have $A \underset{c}{*} B = H \cdot (\bar{R}^A \underset{c}{\otimes} \bar{R}^B)$ (see [1]). Consider the following diagram:

$$
\begin{array}{ccc}
 & & R^A \underset{c}{*} R^B \\
 & \overset{i_1^A \underset{c}{*} i_2^B}{\nearrow} & \big\downarrow \\
C = C \underset{c}{*} C & \underset{i_1 \underset{c}{*} i_2}{\longrightarrow} & A \underset{c}{*} B
\end{array}
\tag{3.10}
$$

LEMMA 3.1. *The diagram (3.10) is commutative.*

PROOF. The proof follows from the fact that the differential graded algebra $R^A \underset{c}{*} R^B$ is an involutive resolution of the algebra $A \underset{c}{*} B$ over the inclusion $i_1 \underset{c}{*} i_2$. $\qquad\square$

Consider the complex $T^{(n)}(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B) = C \underset{c \otimes c^{\mathrm{op}}}{\otimes} ((\bar{R}^A \underset{c}{\otimes} \bar{R}^B) \otimes \cdots \otimes (\bar{R}^A \otimes \bar{R}^B))$. We act on the complex by means of an automorphism ${}^\epsilon \gamma_n$ as follows:

$$
{}^\epsilon \gamma_n((p_0 \otimes q_0) \otimes \cdots \otimes (p_n \otimes q_n)) = (-1)^{\ell + v} \epsilon (p_0^* \otimes q_n^*) \otimes (p_n^* \otimes q_{n-1}^*) \otimes \cdots \otimes (p_1^* \otimes q_0^*); \tag{3.11}
$$

where $p_i \in \bar{R}^A$, $q_i \in \bar{R}^B$, $0 \le i \le n$, $\ell = \deg p_0 \sum_{i=1}^n \deg p_i + \sum_{j=1}^n \deg q_i$, $v = \alpha(\alpha-1)/2$, $\alpha = \sum_{i=1}^n \deg p_i + \deg q_i$, $\alpha = \pm 1$. Consider the chain complex homomorphism

$$
\mu : \frac{T^{(n)}\left(C; R^A \underset{c}{*} R^B\right)}{\mathrm{Im}\,(1-t_n) + \mathrm{Im}\,(1 - {}^\epsilon \gamma_n)} \longrightarrow \frac{\left(R^A * R^B\right)}{C + R^A + R^B + \left[R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B\right] + \mathrm{Im}\,(1 - {}^\epsilon \gamma_n)},
\tag{3.12}
$$

such that

$$
\begin{aligned}
\mu(((p_0 \otimes q_0) &\otimes \cdots \otimes (p_n \otimes q_n)) \bmod (\mathrm{Im}\,(1-t_n) + \mathrm{Im}\,(1 - {}^\epsilon \gamma_n))) \\
&= (p_0 q_0 \cdots p_n q_n) \bmod \left(C + \bar{R}^A + \bar{R}^B + \left[R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B\right] + \mathrm{Im}\,(1 - {}^\epsilon r)\right).
\end{aligned}
\tag{3.13}
$$

In the following lemma, we explain the existence of the homomorphism μ and prove that it is an isomorphism.

LEMMA 3.2. *A chain complex homomorphism μ is an isomorphism.*

PROOF. Clearly, in $R^A \underset{c}{*} R^B$, there exists a subcomplex $C + \bar{R}^A + \bar{R}^B$ (but in $T^{(n)}(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B)$ there is not), and we can factorize $R^A \underset{c}{*} R^B$ by this subcomplex. The elements in $\bar{R}^A \underset{c}{\otimes} \bar{R}^B$ can be compared by modulo with the commutant of the algebra $R^A \underset{c}{*} R^B$ with elements in $\bar{R}^A \underset{c}{\otimes} \bar{R}^B$ since $\mathcal{L} \otimes a + (-1)^m (a \otimes \mathcal{L} - (-1)^{(\deg a)\cdot(\deg \mathcal{L})} \mathcal{L} \otimes a) = (-1)^m a \otimes \mathcal{L}$, where

$$
m = \begin{cases} 0, & \text{if } (\deg a) \cdot (\deg \mathcal{L}) \text{ is even,} \\ 1, & \text{if } (\deg a) \cdot (\deg \mathcal{L}) \text{ is odd, } a \in \bar{R}^A, \ \mathcal{L} \in \bar{R}^B. \end{cases}
\tag{3.14}
$$

Note that the elements in $\bar{R}^A \underset{c}{\otimes} \bar{R}^B \underset{c}{\otimes} \bar{R}^A$ or in $\bar{R}^B \underset{c}{\otimes} \bar{R}^A \underset{c}{\otimes} \bar{R}^B$ can also be compared by modulo with the commutant of the algebra $R^A \underset{c}{*} R^B$ with elements in $\bar{R}^A \underset{c}{\otimes} \bar{R}^B$, at the same time;

$$pqp' - (pqp' - (-1)^{\deg(p \cdot q) \cdot \deg p'} p' pq) = (-1)^{\deg(pq) \cdot \deg p'} (p'p)q, \quad p, p' \in \bar{R}^A, q \in \bar{R}^B. \tag{3.15}$$

In the complex $T^{(n)}(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B) / \mathrm{Im}(1 - t_n) + \mathrm{Im}(1 - {}^\epsilon \gamma_n)$, we have the following:

$$(p_0 \otimes q_0) \otimes \cdots \otimes (p_n \otimes q_n) = (-1)^S (p_n \otimes q_n) \otimes (p_0 \otimes q_0) \otimes \cdots \otimes (p_{n-1} \otimes q_{n-1}), \tag{3.16}$$

where $s = \deg(p_n \otimes q_n) \sum_{i=1}^{n-1} \deg(p_i \otimes q_i)$ and $(p_0 \otimes q_0) \otimes \cdots \otimes (p_n \otimes q_n) = (-1)^{\ell + \nu} \epsilon$ $(p_0^* \otimes q_n^*) \otimes (p_n^* \otimes q_{n-1}^*) \otimes \cdots \otimes (p_1^* \otimes q_0^*)$; $\ell = \deg p_0 \cdot \sum_{i=1}^{n} \deg p_i + \sum_{j=1}^{n} \deg q_j$, $\nu = \alpha(\alpha - 1)/2$, $\alpha = \sum_{i=1}^{n} \deg p_i + \deg q_i$. Clearly, the same relation holds in the complex: $(R^A \underset{c}{*} R^B)/C + \bar{R}^A + \bar{R}^B + [R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B] + \mathrm{Im}(1 - {}^\epsilon r)$ since

$$(p_0 q_0 p_1 q_1 \cdots p_{n-1} q_{n-1})(p_n q_n) = (-1)^{S'} (p_n q_n)(p_0 q_0 p_1 q_1 \cdots p_{n-1} q_{n-1}), \tag{3.17}$$

where $s' = \deg(p_n q_n) \sum_{i=1}^{n-1} \deg(p_i q_i)$, and since $\deg(p_n q_n) = \deg(p_n \otimes q_n)$, $s = s'$ and

$$\begin{aligned} p_0 q_0 p_1 q_1 \cdots p_n q_n &= (-1)^\nu q_n^* p_n^* \cdots q_1^* p_1^* q_0^* p_0^* \\ &= (-1)^{\nu + \ell} (p_0^* q_n^*)(p_n^* q_{n-1}^*) \cdots (p_2^* q_1^*)(p_1^* q_0^*). \end{aligned} \tag{3.18}$$

This gives the required isomorphism. It is easily seen that the differentials in the complexes $\otimes_{n=0}^{\infty} T^{(n)}(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B) / \mathrm{Im}(1 - t_n) + \mathrm{Im}(1 - {}^\epsilon \gamma_n)$ and $(R^A \underset{c}{*} R^B)/C + \bar{R}^A + \bar{R}^B + [R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B] + \mathrm{Im}(1 - {}^\epsilon r)$ coincide. Using the condition $\mathrm{Tor}_i^c(\bar{A}, \bar{B}) = 0$, $i > 0$, we find that $\bar{R}^A \underset{c}{\otimes} \bar{R}^B$ is a free C-module resolution of the algebra $\bar{R} \underset{c}{\otimes} \bar{R}$. Then by considering the isomorphism μ, we get

$$\mathcal{H} \cdot \left(\frac{R^A \underset{c}{*} R^B}{C + \bar{R}^A + \bar{R}^B + \left[R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B \right] + \mathrm{Im}\left(1 - {}^\epsilon r\right)} \right) \overset{\Omega}{\underset{\approx}{=}} \frac{\cdot (C, \bar{A} \otimes \bar{B})}{\mathrm{Im}\left(1 - \overset{\infty}{\underset{n=0}{\oplus}} {}^\epsilon \tilde{\gamma}_n\right)}, \tag{3.19}$$

where ${}^\epsilon \tilde{\gamma}_n$ is an automorphism on the graded K-module $H \cdot (\mathbb{Z}/(n+1); T^{(n)}(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B))$ is induced by the automorphism ${}^\epsilon \gamma_n$ on the complex $T^{(n)}(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B)$. From the isomorphism μ, we get the following isomorphism:

$$\begin{aligned} {}^\epsilon \mathcal{H} \mathcal{D}_i (C \xrightarrow{i_1} A) \oplus {}^\epsilon \mathcal{H} \mathcal{D}_i (C \xrightarrow{i_2} B) &\oplus \left\{ \frac{\overset{\Omega}{\underset{\approx}{}}_i (C, \bar{A} \otimes \bar{B})}{\mathrm{Im}\left(1 - \overset{\infty}{\underset{n=0}{\oplus}} {}^\epsilon \tilde{\gamma}_n\right)} \right\} \\ &= \mathcal{H} \cdot \left(\frac{R^A \underset{c}{*} R^B}{C + \bar{R}^A + \bar{R}^B + \left[R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B \right] + \mathrm{Im}\left(1 - {}^\epsilon r\right)} \right) \\ &\oplus {}^\epsilon \mathcal{H} \mathcal{D}_i (C \xrightarrow{i_1} A) \oplus {}^\epsilon \mathcal{H} \mathcal{D}_i (C \xrightarrow{i_2} B). \qquad \square \end{aligned} \tag{3.20}$$

LEMMA 3.3. *The right-hand side of relation (3.20) is isomorphic to the group* ${}^\epsilon \mathcal{H} \mathcal{D}_i (C \xrightarrow{i_1 * i_2} A \underset{c}{*} B)$.

PROOF. Since

$$\frac{R^A \underset{c}{*} R^B}{\left(C + \left[R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B\right] + \operatorname{Im}\left(1 - {}^\epsilon\mathbf{r}\right)\right)}$$

$$= \frac{\left(R^A \underset{c}{*} R^B\right)}{C + \left[R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B\right] + \operatorname{Im}\left(1 - {}^\epsilon\mathbf{r}\right) + \bar{R}^A + \bar{R}^B}$$

$$+ \left\{\frac{\bar{R}^A}{\left[\bar{R}^A, \bar{R}^A\right] + \operatorname{Im}\left(1 - {}^\epsilon\mathbf{r}_R A\right)}\right\} + \left\{\frac{\bar{R}^B}{\left[\bar{R}^B, \bar{R}^B\right] + \operatorname{Im}\left(1 - {}^\epsilon\mathbf{r}_R B\right)}\right\} \tag{3.21}$$

$$= \left\{\frac{R^A * R^B}{C + \left[R^A \underset{c}{*} R^B, R^A \underset{c}{*} R^B\right] + \operatorname{Im}\left(1 - {}^\epsilon\mathbf{r}\right) + \bar{R}^A + \bar{R}^B}\right\}$$

$$+ \left\{\frac{R^A}{C + \left[R^A, R^A\right] + \operatorname{Im}\left(1 - {}^\epsilon\mathbf{r}_R A\right)}\right\} + \left\{\frac{R^B}{C + \left[R^B, R^B\right] + \operatorname{Im}\left(1 - {}^\epsilon\mathbf{r}_R B\right)}\right\},$$

we have

$${}^\epsilon\mathcal{HD}_i\left(C \xrightarrow{i_1 * i_2} A \underset{c}{*} B\right) = \left\{\frac{\overset{\Omega}{\sim}_i \left(C, \bar{A} \otimes \bar{B}\right)}{\operatorname{Im}\left(1 - \underset{n=0}{\overset{\infty}{\oplus}} {}^\epsilon\tilde{\gamma}_n\right)}\right\} + {}^\epsilon\mathcal{HD}_i\left(C \xrightarrow{i_1} A\right) \oplus {}^\epsilon\mathcal{HD}_i\left(C \xrightarrow{i_2} B\right). \tag{3.22}$$
□

LEMMA 3.4. *The following isomorphism holds:*

$${}^\epsilon\mathcal{HD}_i\left(A \underset{c}{*} B\right) \oplus {}^\epsilon\mathcal{HD}_i(C) = {}^\epsilon\mathcal{HD}_i(A) \oplus {}^\epsilon\mathcal{HD}_i(B) \oplus \frac{\overset{\Omega}{\sim}_i \left(C, \bar{A} \oplus \bar{B}\right)}{\operatorname{Im}\left(1 - \underset{n=0}{\overset{\infty}{\oplus}} {}^\epsilon\tilde{\gamma}_n\right)}. \tag{3.23}$$

PROOF. This follows from the fact that

$${}^\epsilon\mathcal{HD}_i(A) = {}^\epsilon\mathcal{HD}_i(C \xrightarrow{i_1} A) \oplus {}^\epsilon\mathcal{HD}(C),$$

$${}^\epsilon\mathcal{HD}_i(B) = {}^\epsilon\mathcal{HD}_i(C \xrightarrow{i_2} B) \oplus {}^\epsilon\mathcal{HD}(C), \tag{3.24}$$

$${}^\epsilon\mathcal{HD}_i\left(A \underset{c}{*} B\right) = {}^\epsilon\mathcal{HD}_i\left(C \xrightarrow{i_1 \underset{c}{*} i_2} A \underset{c}{*} B\right) \oplus {}^\epsilon\mathcal{HD}(C).$$

Note that the last three relations are obtained from the long exact sequence of relative dihedral homology of algebras [5]. Following [3] (also, see [4]), the automorphisms t_n and ${}^\epsilon\gamma_n$ give the representation of the dihedral group $\mathcal{D}_{n+1}$ on the complex $T^{(n)}(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B)$, where $(t_n)^{n+1} = ({}^\epsilon\gamma_n)^2 = \operatorname{Id}$, ${}^\epsilon\gamma_n t_n^n = t_n^{-1} {}^\epsilon\gamma_n$. Then if $\operatorname{char}(k) = 0$, we get the following isomorphism:

$$\frac{\overset{\Omega}{\sim}_i \left(C, \bar{A} \otimes \bar{B}\right)}{\operatorname{Im}\left(1 - \underset{n=0}{\overset{\infty}{\oplus}} {}^\epsilon\tilde{\gamma}_n\right)} \cong \underset{n=0}{\overset{\infty}{\oplus}} \mathbb{H} \cdot \left(\mathcal{D}_{n+1}; T^{(n)}\left(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B\right)\right). \tag{3.25}$$

From Lemma 3.3 and relation (3.20), we get

$${}^\epsilon\mathcal{HD}_i\left(A \underset{c}{*} B\right) \oplus {}^\epsilon\mathcal{HD}_i(C)$$
$$= {}^\epsilon\mathcal{HD}_i(A) \oplus {}^\epsilon\mathcal{HD}_i(B) \oplus \left(\underset{n=0}{\overset{\infty}{\oplus}} \mathbb{H} \cdot \left(\mathcal{D}_{n+1}; T^{(n)}\left(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B\right)\right)\right). \tag{3.26}$$
□

So, we have proved the following theorem.

Theorem 3.5. *Consider the following diagram of involutive algebras associated with a pre-additive K-category:*

$$A \underset{i_1}{\overset{j_1}{\rightrightarrows}} C \underset{j_2}{\overset{i_2}{\rightrightarrows}} B, \quad \text{where } j_1 \circ i_1 = \text{Id}_C = j_2 \circ i_2. \tag{3.27}$$

If

$$\text{Tor}_i^C(A,A) = \text{Tor}_i^C(A,B) = \text{Tor}_i^C(B,A) = \text{Tor}_i^c(B,B) = 0, \quad i > 0, \tag{3.28}$$

then we have

$$
\begin{aligned}
{}^{\epsilon}\mathcal{HD}_i\left(A \underset{c}{*} B\right) &\oplus {}^{\epsilon}\mathcal{HD}_i(C) \\
&= {}^{\epsilon}\mathcal{HD}_i(A) \oplus {}^{\epsilon}\mathcal{HD}_i(B) \oplus \left(\overset{\alpha}{\underset{n=0}{\oplus}} \mathbb{H} \cdot \left(\mathcal{D}_{n+1}; T^{(n)}\left(C; \bar{R}^A \underset{c}{\otimes} \bar{R}^B\right)\right)\right).
\end{aligned}
\tag{3.29}
$$

Let A be a k-category with an involution, and let $\text{Mod}\,A$ be the category of right A-modules and $P(A)$ be full subcategory in $\text{Mod}\,A$, consisting of the finite projective modules. Consider the category $M_{\text{rt}}(k)$ with objects, the k-categories with involution and morphisms $f : A \to B$ are k-factors $F : \text{Mod}\,A \to \text{Mod}\,B$, such that for every $X \in P(A)$, $f(X) \in P(B)$, f commutes with an involution. We call these morphisms, Morita-morphisms. Evidently, if f is an equivalence, then f is a Morita-morphism. Following [7] (also, see [8]), the cyclic (co)homology of k-category Morita equivalence. Using this fact and considering the deep results of [3], the following fact follows.

Theorem 3.6. *The reflexive cohomolgy and the dihedral (co)homology of the k-category A with involution are invariant under Morita equivalence.*

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