

## ANALOGUES OF SOME TAUBERIAN THEOREMS FOR STRETCHINGS

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**ABSTRACT.** We investigate the effect of four-dimensional matrix transformation on new classes of double sequences. Stretchings of a double sequence is defined, and this definition is used to present a four-dimensional analogue of D. Dawson's copy theorem for stretching of a double sequence. In addition, the multidimensional analogue of D. Dawson's copy theorem is used to characterize convergent double sequences using stretchings.

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**1. Introduction.** In this paper,  $RH$ -regular matrices and the stretching of double sequences are used to characterize  $P$ -convergent sequences. To achieve this goal we begin by defining an  $\epsilon$ -Pringsheim-copy and a stretching of double sequences. In addition, the copy theorem of Dawson in [1] will be extended as follows: if each of  $A$  and  $T$  is an  $RH$ -regular matrix, and  $x$  is any bounded double complex sequence with  $\epsilon$  being any bounded positive term double sequence with  $P\text{-}\lim_{i,j} \epsilon_{i,j} = 0$ , then there exists a stretching  $y$  of  $x$  such that  $T(Ay)$  exists and contains an  $\epsilon$ -Pringsheim-copy of  $x$ . By using this extended copy theorem some natural implications and variations of this extended copy theorem will be presented.

### 2. Definitions, notations, and preliminary results

**DEFINITION 2.1** (see [3]). A double sequence  $x = [x_{k,l}]$  has Pringsheim limit  $L$  (denoted by  $P\text{-}\lim x = L$ ) provided that given  $\epsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $|x_{k,l} - L| < \epsilon$  whenever  $k, l > N$ . We will describe such an  $x$  more briefly as " $P$ -convergent."

**DEFINITION 2.2** (see [3]). A double sequence  $x$  is called definite divergent, if for every (arbitrarily large)  $G > 0$  there exist two natural numbers  $n_1$  and  $n_2$  such that  $|x_{n,k}| > G$  for  $n \geq n_1$ ,  $k \geq n_2$ .

**DEFINITION 2.3.** The double sequence  $[y]$  is a double subsequence of the sequence  $[x]$  provided that there exist two increasing double index sequences  $\{n_j\}$  and  $\{k_j\}$  such that if  $z_j = x_{n_j, k_j}$ , then  $y$  is formed by

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & - \end{array} \tag{2.1}$$

The double sequence  $x$  is bounded if and only if there exists a positive number  $M$  such that  $|x_{k,l}| < M$  for all  $k$  and  $l$ . A two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The Silverman-Toeplitz theorem [5, 6] characterizes the regularity of two-dimensional matrix transformations. In [4], Robison presented a four-dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is  $P$ -convergent is not necessarily bounded. The definition of regularity for four-dimensional matrices will be stated below along with the Robison-Hamilton characterization of the regularity of four-dimensional matrices.

**DEFINITION 2.4.** The four-dimensional matrix  $A$  is said to be *RH*-regular if it maps every bounded  $P$ -convergent sequence into a  $P$ -convergent sequence with the same  $P$ -limit.

**THEOREM 2.5** (see [2, 4]). *The four-dimensional matrix  $A$  is RH-regular if and only if*

- (RH<sub>1</sub>)  $P\text{-lim}_{m,n} a_{m,n,k,l} = 0$  for each  $k$  and  $l$ ;
- (RH<sub>2</sub>)  $P\text{-lim}_{m,n} \sum_{k,l=1,1}^{\infty, \infty} a_{m,n,k,l} = 1$ ;
- (RH<sub>3</sub>)  $P\text{-lim}_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$  for each  $l$ ;
- (RH<sub>4</sub>)  $P\text{-lim}_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$  for each  $k$ ;
- (RH<sub>5</sub>)  $\sum_{k,l=1,1}^{\infty, \infty} |a_{m,n,k,l}|$  is  $P$ -convergent; and
- (RH<sub>6</sub>) there exist finite positive integers  $A$  and  $B$  such that  $\sum_{k,l>B} |a_{m,n,k,l}| < A$ .

**EXAMPLE 2.6.** The sequences  $[\gamma_{n,k}] = 1$  and  $[\gamma_{n,k}] = -1$  for each  $n$  and  $k$  are both subsequences of the double sequence whose  $n, k$ th term is  $x_{n,k} = (-1)^n$ . In addition to the two subsequences given, every double sequence of 1's and -1's is a subsequence of this  $x$ .

**EXAMPLE 2.7.** As another example of a subsequence of a double sequence, we define  $x$  as follows:

$$x_{n,k} := \begin{cases} 1, & \text{if } n = k, \\ \frac{1}{n}, & \text{if } n < k, \\ n, & \text{if } n > k. \end{cases} \quad (2.2)$$

Then the double sequence

$$\gamma_{n,k} := \begin{bmatrix} \frac{1}{2} & 4 & \frac{1}{10} & 20 & \cdot & \cdot \\ 8 & 6 & \frac{1}{12} & 22 & \cdot & \cdot \\ \frac{1}{18} & \frac{1}{16} & \frac{1}{14} & 24 & \cdot & \cdot \\ 32 & 30 & 28 & 26 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad (2.3)$$

is clearly a subsequence of  $x$ .

**REMARK 2.8.** Note that if the double sequence  $x$  contains at most a finite number of unbounded rows and/or columns, then every subsequence of  $x$  is bounded. In addition, the finite number of unbounded rows and/or columns does not affect the  $P$ -convergence or  $P$ -divergence of  $x$  and its subsequences.

**DEFINITION 2.9.** A number  $\beta$  is called a Pringsheim limit point of the double sequence  $x = [x_{n,k}]$  provided that there exists a subsequence  $y = [y_{n,k}]$  of  $[x_{n,k}]$  that has Pringsheim limit  $\beta : P\text{-lim } y_{n,k} = \beta$ .

**EXAMPLE 2.10.** Define the double sequence  $x$  by

$$x_{n,k} := \begin{cases} (-1)^n, & \text{if } n = k, \\ (-2)^n, & \text{if } n = k + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

This double sequence has five Pringsheim limit points, namely  $-2, -1, 0, 1$ , and  $2$ .

**REMARK 2.11.** The definition of a Pringsheim limit point can also be stated as follows:  $\beta$  is a Pringsheim limit point of  $x$  provided that there exist two increasing index sequences  $\{n_i\}$  and  $\{k_i\}$  such that  $\lim_i x_{n_i, k_i} = \beta$ .

**DEFINITION 2.12.** A double sequence  $x$  is divergent in the Pringsheim sense ( $P$ -divergent) provided that  $x$  does not converge in the Pringsheim sense ( $P$ -convergent).

**REMARK 2.13.** [Definition 2.12](#) can also be stated as follows: a double sequence  $x$  is  $P$ -divergent provided that either  $x$  contains at least two subsequences with distinct finite Pringsheim limit points or  $x$  contains an unbounded subsequence. Also note that, if  $x$  contains an unbounded subsequence then  $x$  also contains a definite divergent subsequence.

**EXAMPLE 2.14.** This is an example of a convergent double sequence whose terms form an unbounded set

$$x_{n,k} := \begin{cases} k, & \text{if } n = 1, \\ n, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases} \quad (2.5)$$

**EXAMPLE 2.15.** This is an example of an unbounded divergent double sequence with three finite Pringsheim limit points, namely  $-1, 0$ , and  $1$ :

$$x_{n,k} := \begin{cases} k + 1, & \text{if } n = 1, \\ (-1)^{n+1}, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases} \quad (2.6)$$

**EXAMPLE 2.16.** This is an example of a double sequence which contains an unbounded subsequence

$$x_{n,k} := \begin{cases} n, & \text{if } n = k, \\ -n, & \text{if } n = k + 1, \\ 0, & \text{otherwise.} \end{cases} \quad (2.7)$$

**EXAMPLE 2.17.** For an example of a definite divergent sequence take  $x_{n,k} = n$  for each  $n$  and  $k$ ; then it is also clear that  $x$  contains an unbounded subsequence.

The following propositions are easily verified.

**PROPOSITION 2.18.** *If  $x = [x_{n,k}]$  is  $P$ -convergent to  $L$  then  $x$  cannot converge to a limit  $M$ , where  $M \neq L$ .*

**PROPOSITION 2.19.** *If  $x = [x_{n,k}]$  is  $P$ -convergent to  $L$ , then any subsequence of  $x$  is also  $P$ -convergent to  $L$ .*

**REMARK 2.20.** For an ordinary single-dimensional sequence, any sequence is a subsequence of itself. This, however, is not the case in the two-dimensional plane, as illustrated by the following example.

**EXAMPLE 2.21.** The sequence

$$x_{n,k} := \begin{cases} 1, & \text{if } n = k = 0, \\ 1, & \text{if } n = 0, k = 1, \\ 1, & \text{if } n = 1, k = 0, \\ 0, & \text{otherwise} \end{cases} \quad (2.8)$$

contains only two subsequences, namely,  $[y_{n,k}] = 0$  for each  $n$  and  $k$ , and

$$z_{n,k} := \begin{cases} 1, & \text{if } n = k = 0, \\ 0, & \text{otherwise;} \end{cases} \quad (2.9)$$

neither subsequences is  $x$ .

The following propositions are easily verified.

**PROPOSITION 2.22.** *If every subsequence of  $x = [x_{k,l}]$  is  $P$ -convergent, then  $x$  is  $P$ -convergent.*

**PROPOSITION 2.23.** *The double sequence  $x$  is  $P$ -convergent to  $L$  if and only if every subsequence of  $x$  is  $P$ -convergent to  $L$ .*

**DEFINITION 2.24.** The double sequence  $y$  contains an  $\epsilon$ -Pringsheim-copy of  $x$  provided that  $y$  contains a subsequence  $y_{n_i, k_j}$  such that  $|y_{n_i, k_j} - x_{i,j}| < \epsilon_{i,j}$ , for  $i, j = 1, 2, \dots$ .

**EXAMPLE 2.25.** Let

$$x_{n,k} := \begin{cases} (-1)^n, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases} \quad (2.10)$$

and let  $P\text{-lim}_{n,k} \epsilon_{n,k} = 0$  with

$$y_{n,k} := \begin{cases} (-1)^n, & \text{if } k = n, \\ \epsilon_{n,k}, & \text{otherwise.} \end{cases} \quad (2.11)$$

Observe that, not only does  $y$  contain an  $\epsilon$ -Pringsheim-copy of  $x$ , but  $y$  itself is an  $\epsilon$ -Pringsheim-copy of  $x$ .

**DEFINITION 2.26.** The double sequence  $y$  is a *stretching* of  $x$  provided that there exist two increasing index sequences  $\{R_i\}_{i=0}^{\infty}$  and  $\{S_j\}_{j=0}^{\infty}$  of integers such that

$$y_{n,k} := \begin{cases} R_0 = S_0 = 1, \\ x_{n,i}, & \text{if } R_{i-1} \leq k < R_i, \\ x_{j,k}, & \text{if } S_{j-1} \leq n < S_j, \\ i, j = 1, 2, \dots. \end{cases} \quad (2.12)$$

**REMARK 2.27.** This definition demonstrates the procedure which is used to construct a stretching of a double sequence  $x$ . This procedure uses a sequence of stages to construct the stretching of  $x$ . These stages are constructed using a sequence of abutting rows and columns of  $x$ . These rows and columns are constructed as follows.

**STAGE 1.** Begin by repeating the first row of  $x$   $R_1$  times and denote the resulting double sequence by  $y^{1,0}$  then repeat the first column of  $y^{1,0}$   $S_1$  times resulting in  $y^{1,1}$ .

**STAGE 2.** Begin by repeating the  $R_1 + 1$  row of  $y^{1,1}$ ,  $R_2 - R_1$  times which yields  $y^{2,1}$  then repeat the  $S_1 + 1$  column of  $y^{2,1}$ ,  $S_2 - S_1$  times which yields  $y^{2,2}$ .

⋮

**STAGE  $i$ .** Begin by repeating the  $1 + \sum_{p=1}^{i-1} R_p$  row of  $y^{i-1,i-1}$ ,  $R_i - R_{i-1}$  times which yields  $y^{i,i-1}$  then repeat the  $1 + \sum_{q=1}^{i-1} S_q$  column of  $y^{i,i-1}$ ,  $S_i - S_{i-1}$  times which yields  $y^{i,i}$ . Note that in each stage we repeat the number of rows and then repeat the number of columns. However the resulting stretching  $y$  of  $x$  is the same, if we first repeat the number of columns and then repeat the numbers of rows. Also note that every sequence itself is a stretching of itself and the sequences that induce this kind of stretching are  $R_i = i$  and  $S_j = j$ .

**EXAMPLE 2.28.** The sequence

$$\begin{array}{cccccccccc} x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\ x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\ x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\ x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\ x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\ \vdots & \cdots \end{array} \quad (2.13)$$

is a stretching of  $x$  induced by  $R_i = 3i$  and  $S_j = 3j$ .

**3. Main results.** The following theorem is given its name because of its similarity to the copy theorem of Dawson in [1].

**THEOREM 3.1** (extended copy theorem). *If each of  $A$  and  $T$  is an RH-regular matrix, and  $x$  is any bounded double complex sequence with  $\epsilon$  being any bounded positive term*

double sequence with  $P\text{-}\lim_{i,j} \epsilon_{i,j} = 0$ , then there exists a stretching  $y$  of  $x$  such that  $T(Ay)$  exists and contains an  $\epsilon$ -Pringsheim-copy of  $x$ .

**PROOF.** We begin by introducing a few notations which are used only in this proof. Let

$$\begin{aligned}
 \|A\| &:= \sup_{m,n > \bar{B}} \left( \sum_{k,l} |a_{m,n,k,l}| \right) < K_A, \quad \|T\| := \sup_{m,n > \bar{B}} \left( \sum_{k,l} |t_{m,n,k,l}| \right) < K_T, \\
 M_{i,j} &:= 1 + \sum_{k,l=1}^{i,j} |x_{k,l}|, \quad \delta_{i,j} := \min_{i,j} \left\{ \frac{\epsilon_{k,l}}{1} \leq k \leq i \cup 1 \leq l \leq j \right\}, \\
 K &:= K_A + K_T + \max_{i,j} \left\{ \frac{\epsilon_{k,l}}{1} \leq k \leq i \cup 1 \leq l \leq j \right\} + 1, \quad Q_{i,j} := KM_{i,j} + 1, \\
 c_{i,j}(r,s) &:= \left\{ \frac{(k,l)}{1} \leq k < r_i \cup 1 \leq l < s_j \right\}, \\
 \bar{c}_{i,j}(r,s) &:= \left\{ \frac{(k,l)}{r_i} \leq k < \infty \cup s_j \leq l < \infty \right\}, \quad \bar{b}_{i,j}(r,s) := c_{i,j}(r,s) \setminus c_{i-1,j-1}(r,s).
 \end{aligned} \tag{3.1}$$

Then by (RH<sub>2</sub>) there exist  $m_{\alpha_1}$  and  $n_{\beta_1}$  such that for  $m > m_{\alpha_1} > \bar{B}$  and  $n > n_{\beta_1} > \bar{B}$ , where  $\bar{B}$  is defined by the sixth RH-condition,

$$\left| \sum_{k,l=1}^{\infty, \infty} a_{m,n,k,l} - 1 \right| < \frac{\delta_{\alpha_1, \beta_1}}{16Q_{\alpha_1, \beta_1}}. \tag{3.2}$$

Also by (RH<sub>1</sub>) and (RH<sub>2</sub>) there exist  $a_{\alpha_1}$  and  $b_{\beta_1}$  such that

$$\sum_{(k,l) \in c_{\alpha_1, \beta_1}(m,n)} |t_{a_{\alpha_1}, b_{\beta_1}, k, l}| < \frac{\delta_{\alpha_1, \beta_1}}{8Q_{\alpha_1, \beta_1}}, \quad \left| \sum_{k,l=1}^{\infty, \infty} t_{a_{\alpha_1}, b_{\beta_1}, k, l} - 1 \right| < \frac{\delta_{\alpha_1, \beta_1}}{8Q_{\alpha_1, \beta_1}}. \tag{3.3}$$

In addition, there exist  $\bar{m}_{\alpha_1}, \bar{n}_{\beta_1}, \alpha_2$ , and  $\beta_2$  such that if  $1 \leq \psi \leq a_{\alpha_1}$  and  $1 \leq \omega \leq b_{\beta_1}$ , then

$$\sum_{(k,l) \in \bar{c}_{\alpha_1, \beta_1}(\bar{m}, \bar{n})} |t_{\psi, \omega, k, l}| < \frac{\delta_{\alpha_1, \beta_1}}{16Q_{\alpha_2, \beta_2}}. \tag{3.4}$$

Also, there exist  $r_{\alpha_1} > 1$  and  $s_{\beta_1} > 1$  such that if  $1 \leq m \leq \bar{m}_{\alpha_1}$  and  $1 \leq n \leq \bar{n}_{\beta_1}$  then

$$\sum_{(k,l) \in \bar{c}_{\alpha_1, \beta_1}(r, s)} |a_{m,n,k,l}| \leq \frac{\delta_{\alpha_1, \beta_1}}{16Q_{\alpha_2, \beta_2}}. \tag{3.5}$$

Now, without loss of generality, we set  $\alpha_p = p$  and  $\beta_q = q$ . Having chosen

$$\left\{ \begin{array}{l} m_p, \bar{m}_p, a_p, r_p \\ n_q, \bar{n}_q, b_q, s_q \end{array} \right\}_{p=0, q=0}^{i-1, j-1} \tag{3.6}$$

with  $m_0 = n_0 = \bar{m}_0 = \bar{n}_0 = a_0 = b_0 = r_0 = s_0 = 1$ , now choose  $m_i > \bar{m}_{i-1}$  and  $n_j > \bar{n}_{j-1}$  such that if  $m > m_i$  and  $n > n_j$  then

$$\left| \sum_{(k,l) \in \bar{c}_{i-1, j-1}(r, s)} a_{m,n,k,l} - 1 \right| < \frac{\delta_{i,j}}{16Q_{i,j} 2^{i+j}}, \tag{3.7}$$

$$\sum_{(k,l) \in c_{i-1,j-1}(r,s)} |a_{m,n,k,l}| < \frac{\delta_{i,j}}{8Q_{i-1,j-1}2^{i+j}}. \quad (3.8)$$

Also choose  $a_i > a_{i-1}$  and  $b_j > b_{j-1}$  such that

$$\sum_{(k,l) \in c_{i,j}(m,n)} |t_{a_i,b_j,k,l}| < \frac{\delta_{i,j}}{8Q_{i,j}}, \quad \left| \sum_{(k,l) \in \tilde{c}_{i,j}(m,n)} t_{a_i,b_j,k,l} - 1 \right| < \frac{\delta_{i,j}}{8Q_{i,j}}. \quad (3.9)$$

Next choose  $\tilde{m}_i > m_i$  and  $\tilde{n}_j > n_j$  such that if  $1 \leq \psi \leq a_i$  and  $1 \leq \omega \leq b_j$  then

$$\sum_{(k,l) \in \tilde{c}_{i,j}(\tilde{m},\tilde{n})} |t_{\psi,\omega,k,l}| < \frac{\delta_{i,j}}{2^{2+i+j}Q_{i+1,j+1}}. \quad (3.10)$$

Then choose  $r_i > r_{i-1}$  and  $s_j > s_{j-1}$  such that if  $1 \leq m \leq \tilde{m}_i$  and  $1 \leq n \leq \tilde{n}_j$  then

$$\sum_{(k,l) \in \tilde{c}_{i,j}(r,s)} |a_{m,n,k,l}| < \frac{\delta_{i,j}}{2^{4+i+j}Q_{i+1,j+1}}, \quad (3.11)$$

where  $m_i, n_j, \tilde{m}_i, \tilde{n}_j, r_i$ , and  $s_j$  are chosen using  $(RH_1)$ ,  $(RH_2)$ ,  $(RH_3)$ , and  $(RH_4)$  such that if  $1 \leq p \leq j-1$  and  $1 \leq q \leq i-1$  the following is obtained:

$$\left| \sum_{(k,l) \in \tilde{b}_{p,j}(r,s)} a_{m,n,k,l} \right| \leq \frac{\delta_{p,j}}{8Q_{p,j}2^{p+j}}, \quad \left| \sum_{(k,l) \in \tilde{b}_{i,q}(r,s)} a_{m,n,k,l} \right| \leq \frac{\delta_{i,q}}{8Q_{i,q}2^{i+q}}. \quad (3.12)$$

Therefore by (3.9) and (3.10) we have

$$\left| \sum_{(k,l) \in c_{i,j}(\tilde{m},\tilde{n}) \setminus c_{i,j}(m,n)} t_{a_i,b_j,k,l} - 1 \right| \leq \frac{\delta_{i,j}}{4Q_{i,j}}, \quad (3.13)$$

and by (3.7), (3.8), and (3.11) we also obtain

$$\left| \sum_{(k,l) \in \tilde{b}_{i,j}(r,s)} a_{m,n,k,l} - 1 \right| < \frac{\delta_{i,j}}{8Q_{i,j}2^{i+j}}, \quad (3.14)$$

where  $m_i \leq m \leq \tilde{m}_i$  and  $n_j \leq n \leq \tilde{n}_j$ . Let  $\{y_{k,l}\}$  be the stretching of  $x$  induced by  $\{r_i\}$  and  $\{s_j\}$ . Since

$$\begin{aligned} (A\mathcal{Y})_{m,n} - x_{i,j} &= \sum_{k,l=1}^{r_{i-1}-1, s_{j-1}-1} a_{m,n,k,l} y_{k,l} + \sum_{(k,l) \in \tilde{b}_{i,j}(r,s)} a_{m,n,k,l} y_{k,l} - x_{i,j} \\ &\quad + \sum_{p,q=i+1, j+1}^{\infty, \infty} \sum_{(k,l) \in \tilde{b}_{p,q}(r,s)} a_{m,n,k,l} y_{k,l}, \end{aligned} \quad (3.15)$$

if  $i, j > 1$ , with  $m_i \leq m \leq \tilde{m}_i$  and  $n_j \leq n \leq \tilde{n}_j$  the following is obtained:

$$\left| \sum_{k,l=1}^{r_{i-1}-1, s_{j-1}-1} a_{m,n,k,l} y_{k,l} \right| \leq \max \left\{ \frac{|x_{k,l}|}{1} \mid k \leq i-1 \cup 1 \leq l \leq j-1 \right\} \sum_{k,l=1}^{r_{i-1}-1, s_{j-1}-1} |a_{m,n,k,l} y_{k,l}|. \quad (3.16)$$

By (3.8),

$$\left| \sum_{k,l=1}^{r_{i-1}-1, s_{j-1}-1} a_{m,n,k,l} y_{k,l} \right| \leq \max \left\{ \frac{|\chi_{k,l}|}{1} \leq k \leq i-1 \cup 1 \leq l \leq j-1 \right\} \frac{\delta_{i,j}}{8Q_{i-1,j-1}}. \quad (3.17)$$

Since

$$Q_{i-1,j-1} = K \left( 1 + \sum_{k,l=1}^{i-1, j-1} |\chi_{k,l}| \right) + 1 \geq K \max \left\{ \frac{|\chi_{k,l}|}{1} \leq k \leq i-1 \cup 1 \leq l \leq j-1 \right\}, \quad (3.18)$$

the following holds:

$$\left| \sum_{k,l=1}^{r_{i-1}-1, s_{j-1}-1} a_{m,n,k,l} y_{k,l} \right| \leq \frac{\delta_{i,j}}{8K}, \quad (3.19)$$

the following also is obtained:

$$\begin{aligned} \left| \sum_{p,q=i+1,j+1}^{\infty, \infty} \sum_{(k,l) \in \tilde{b}_{p,q}(r,s)} a_{m,n,k,l} y_{k,l} \right| &\leq \sum_{p,q=i+1,j+1}^{\infty, \infty} |\chi_{k,l}| \sum_{(k,l) \in \tilde{b}_{p,q}(r,s)} |a_{m,n,k,l}| \\ &\leq \frac{\delta_{i,j}}{2^4 K} \sum_{p,q=i+1,j+1}^{\infty, \infty} \frac{1}{2^{p+q}} \leq \frac{\delta_{i,j}}{8K}, \end{aligned} \quad (3.20)$$

because

$$\sum_{k,l=r_p, s_q}^{\infty, \infty} |a_{m,n,k,l}| \leq \frac{\delta_{p-1, q-1}}{2^{4+p+q} Q_{p,q}}, \quad \frac{|\chi_{p,q}|}{Q_{p,q}} < \frac{1}{K}. \quad (3.21)$$

Therefore by (3.11),

$$\begin{aligned} \left| \sum_{(k,l) \in \tilde{b}_{i,j}(r,s)} a_{m,n,k,l} y_{k,l} - \chi_{i,j} \right| &\leq \sum_{q=1}^{i-1} |\chi_{i,q}| \left| \sum_{(k,l) \in \tilde{b}_{i,q}(r,s)} a_{m,n,k,l} \right| \\ &\quad + \sum_{p=1}^{j-1} |\chi_{p,j}| \left| \sum_{(k,l) \in \tilde{b}_{p,j}(r,s)} a_{m,n,k,l} \right| \\ &\quad + |\chi_{i,j}| \left| \sum_{(k,l) \in \tilde{b}_{i,j}(r,s)} a_{m,n,k,l} - 1 \right| \\ &\leq \sum_{p,q=1,1}^{i,j} \frac{|\chi_{i,j}|}{Q_{i,j}} \frac{\delta_{p,q}}{2^{p+q+3}} \leq \frac{\delta_{i,j}}{K8} \sum_{p,q=1,1}^{i,j} \frac{1}{2^{p+q}} = \frac{\delta_{i,j}}{K2}. \end{aligned} \quad (3.22)$$

Therefore,

$$|(A\mathcal{Y})_{m,n} - \chi_{i,j}| \leq \frac{\delta_{i,j}}{K8} + \frac{\delta_{i,j}}{K4} + \frac{\delta_{i,j}}{K2} < \frac{\delta_{i,j}}{2K}. \quad (3.23)$$

Note that the inequality (3.23) is true for  $m_1 \leq m \leq \bar{m}_1$  and  $n_1 \leq n \leq \bar{n}_1$ , and also this inequality is true for  $i, j \geq 1$  with  $m_i \leq m \leq \bar{m}_i$  and  $n_j \leq n \leq \bar{n}_j$ . Hence

$$(A\mathcal{Y})_{m,n} = x_{i,j} + u_{i,j}, \quad (3.24)$$

where  $|u_{i,j}| \leq \delta_{i,j}/2K$ . Note that if  $\bar{m}_{i-1} \leq m \leq m_i$  and  $\bar{n}_{j-1} \leq n \leq n_j$ , then the following is obtained:

$$\begin{aligned} |(A\mathcal{Y})_{m,n}| &\leq \left| \sum_{k,l=1}^{r_i-1, s_j-1} a_{m,n,k,l} y_{k,l} \right| + \left| \sum_{p,q=i+1, j+1}^{\infty, \infty} \sum_{k,l \in \bar{b}_{p,q}(r,s)} a_{m,n,k,l} y_{k,l} \right| \\ &\leq \max \left\{ \frac{|x_{k,l}|}{1} \leq k \leq i \cup 1 \leq l \leq j \right\} \sum_{k,l=1}^{r_i-1, s_j-1} |a_{m,n,k,l}| \\ &\quad + \sum_{p,q=i+1, j+1}^{\infty, \infty} |x_{k,l}| \sum_{k,l \in \bar{b}_{p,q}(r,s)} |a_{m,n,k,l}| \\ &\leq K m_{i,j} + \sum_{p,q=i+1, j+1}^{\infty, \infty} |x_{k,l}| \frac{\delta_{p,q}}{2^{4+p+q} Q_{p+1,q+1}} \\ &\leq K m_{i,j} + \frac{\delta_{i,j}}{K4} \sum_{p,q=i+1, j+1}^{\infty, \infty} \frac{1}{2^{p+q}} \\ &\leq K m_{i,j} + 1 = Q_{i,j}. \end{aligned} \quad (3.25)$$

Also, if  $m_{i-1} \leq m \leq m_i$  and  $n_{j-1} \leq n \leq n_j$  then

$$\begin{aligned} \left| \sum_{k,l=1}^{\infty, \infty} a_{m,n,k,l} y_{k,l} \right| &\leq |(A\mathcal{Y})_{m,n} - x_{i,j}| + |x_{i,j}| \\ &\leq \frac{\delta_{i,j}}{2K} + K m_{i,j} \leq K m_{i,j} + 1 = Q_{i,j}. \end{aligned} \quad (3.26)$$

By using (3.25) we now show the existence of  $T(A\mathcal{Y})$ . If  $a_{i-1} < m \leq a_i$  and  $b_{j-1} < n \leq b_j$  then

$$\begin{aligned} \left| \sum_{k,l=\bar{m}_i+1, \bar{n}_j+1}^{\infty, \infty} t_{m,n,k,l} (A\mathcal{Y})_{k,l} \right| &\leq \sum_{r,s=i,j}^{\infty, \infty} \sum_{(p,q) \in \bar{b}_{r+1,s+1}(\bar{m}, \bar{n})} |t_{m,n,p,q} (A\mathcal{Y})_{p,q}| \\ &\leq \sum_{r,s=i,j}^{\infty, \infty} Q_{r+1,s+1} \sum_{(p,q) \in \bar{b}_{r+1,s+1}(\bar{m}, \bar{n})} |t_{m,n,p,q}| \\ &\leq \sum_{r,s=i,j}^{\infty, \infty} Q_{r+1,s+1} \frac{\delta_{r,s}}{2^{2+r+s} Q_{r+1,s+1}} \\ &\leq \delta_{i,j} \frac{1}{4} \sum_{r,s=1}^{\infty, \infty} \frac{1}{2^{r+s}} < \frac{\delta_{i,j}}{4}. \end{aligned} \quad (3.27)$$

Therefore  $T(Ay)$  exists. Also, by (3.25) we now show that  $T(Ay)$  contains an  $\epsilon$ -Pringsheim-copy of  $x$ . First note that

$$\begin{aligned} \left| \sum_{k,l=1}^{\infty, \infty} t_{a_i, b_j, k, l} (Ay)_{k, l} - x_{i, j} \right| &\leq \sum_{k, l=1}^{m_i-1, n_j-1} |t_{a_i, b_j, k, l} (Ay)_{k, l}| \\ &+ \left| \sum_{(k, l) \in \bar{b}_{i, j}(r, s)} t_{a_i, b_j, k, l} (Ay)_{k, l} - x_{i, j} \right| \\ &+ \left| \sum_{k, l=m_i+1, \bar{n}_j+1}^{\infty, \infty} t_{m, n, k, l} (Ay)_{k, l} \right|, \end{aligned} \quad (3.28)$$

with

$$\sum_{k, l=1}^{m_i-1, n_j-1} |t_{a_i, b_j, k, l} (Ay)_{k, l}| = \sum_{k, l=1}^{m_i-1, n_j-1} |t_{a_i, b_j, k, l}| Q_{i, j} \leq Q_{i, j} \frac{\delta_{i, j}}{8Q_{i, j}} = \frac{\delta_{i, j}}{8}, \quad (3.29)$$

$$\begin{aligned} \left| \sum_{(k, l) \in \bar{b}_{i, j}(r, s)} t_{a_i, b_j, k, l} (Ay)_{k, l} - x_{i, j} \right| &= \left| \sum_{(k, l) \in \bar{b}_{i, j}(r, s)} t_{a_i, b_j, k, l} (x_{i, j} + u_{i, j}) - x_{i, j} \right| \\ &\leq |x_{i, j}| \sum_{(k, l) \in \bar{b}_{i, j}(r, s)} |t_{a_i, b_j, k, l} - 1| \\ &+ \sum_{(k, l) \in \bar{b}_{i, j}(r, s)} |t_{a_i, b_j, k, l} u_{i, j}| \\ &\leq \frac{|x_{i, j}| \delta_{i, j}}{Q_{i, j}} + \frac{\delta_{i, j}}{4K} \sum_{(k, l) \in \bar{b}_{i, j}(r, s)} |t_{a_i, b_j, k, l}| \\ &\leq \frac{\delta_{i, j}}{2}, \end{aligned} \quad (3.30)$$

$$\begin{aligned} \left| \sum_{k, l=m_i+1, \bar{n}_j+1}^{\infty, \infty} t_{m, n, k, l} (Ay)_{k, l} \right| &\leq \sum_{r, s=i, j}^{\infty, \infty} \sum_{(p, q) \in \bar{b}_{r+1, s+1}(\bar{m}, \bar{n})} |t_{a_i, b_j, p, q} (Ay)_{p, q}| \\ &\leq \sum_{r, s=i, j}^{\infty, \infty} Q_{r+1, s+1} \sum_{(p, q) \in \bar{b}_{r+1, s+1}(\bar{m}, \bar{n})} |t_{a_i, b_j, p, q}| \\ &\leq \sum_{r, s=i, j}^{\infty, \infty} Q_{r+1, s+1} \frac{\delta_{r, s}}{2^{2+r+s} Q_{r+1, s+1}} \leq \frac{\delta_{i, j}}{4}. \end{aligned} \quad (3.31)$$

Hence,

$$\left| \sum_{k, l=1}^{\infty, \infty} t_{m, n, k, l} (Ay)_{k, l} - x_{i, j} \right| \leq \frac{\delta_{i, j}}{4} + \frac{\delta_{i, j}}{2} + \frac{\delta_{i, j}}{8} < \delta_{i, j} \leq \epsilon_{i, j}. \quad (3.32)$$

This completes the proof of the extended copy theorem.  $\square$

The next two results are immediate corollaries of the extended copy theorem.

**COROLLARY 3.2.** *If  $T$  is any RH-regular matrix summability method and  $A$  is an RH-regular matrix such that  $Ay$  is  $T$ -summable for every stretching  $y$  of  $x$ , then  $x$  is  $P$ -convergent.*

**COROLLARY 3.3.** *If  $T$  is any RH-regular matrix summability method and  $A$  is an RH-regular matrix such that  $Ay$  is absolutely  $T$ -summable for every stretching  $y$  of  $x$ , then  $x$  is  $P$ -convergent.*

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