

## ON THE EXTENDED HARDY'S INEQUALITY

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**ABSTRACT.** We generalize a strengthened version of Hardy's inequality and give a new simpler proof.

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In the recent paper [4], Hardy's inequality was generalized. In this note, the results given in [4] are further generalized and a new much simpler proof is given. The following Hardy's inequality is well known [1, Theorem 349].

**THEOREM 1** (Hardy's inequality). *Let  $\lambda_n > 0$ ,  $A_n = \sum_{k=1}^n \lambda_k$ ,  $a_n \geq 0$  ( $n \in \mathbb{N}$ ),  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$ , then*

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \quad (1)$$

Recently, [4] gave an improvement of Theorem 1, and the following result was proved.

**THEOREM 2.** *Let  $0 < \lambda_{n+1} \leq \lambda_n$ ,  $A_n = \sum_{k=1}^n \lambda_k$ ,  $a_n \geq 0$  ( $n \in \mathbb{N}$ ),  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$ , then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \left(1 - \frac{\lambda_n}{2(A_n + \lambda_n)}\right) \lambda_n a_n. \quad (2)$$

In this note, we will prove the following theorem.

**THEOREM 3.** *Let  $0 < \lambda_{n+1} \leq \lambda_n$ ,  $A_n = \sum_{k=1}^n \lambda_k$ ,  $a_n \geq 0$  ( $n \in \mathbb{N}$ ),  $0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty$ , then*

$$\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \left(1 + \frac{5\lambda_n}{5A_n + \lambda_n}\right)^{-1/2} \lambda_n a_n. \quad (3)$$

To prove Theorem 3, we introduce some lemmas.

**LEMMA 4.** *For  $x > 0$ , then*

$$e \left(1 - \frac{1}{2x+1}\right) < \left(1 + \frac{1}{x}\right)^x < e \left(1 + \frac{5}{5x+1}\right)^{-1/2}. \quad (4)$$

**PROOF.** (i) Define  $f(x)$  as

$$f(x) = x \ln \left(1 + \frac{1}{x}\right) + \frac{1}{2} \ln \left(1 + \frac{5}{5x+1}\right), \quad x \in (0, +\infty). \quad (5)$$

It is obvious that when  $x > 0$ , the inequality

$$\left(1 + \frac{1}{x}\right)^x < e \left(1 + \frac{5}{5x+1}\right)^{-1/2} \quad (6)$$

is equivalent to  $f(x) < 1$ . It is easy to see that

$$f'(x) = -\frac{1}{x+1} + \ln \left(1 + \frac{1}{x}\right) - \frac{25}{2(5x+6)(5x+1)} \quad (7)$$

and for  $x \in (0, +\infty)$ , it can be shown that

$$\begin{aligned} f''(x) &= \frac{1}{(x+1)^2} - \frac{1}{x(x+1)} + \frac{25}{2(5x+1)^2} - \frac{25}{2(5x+6)^2} \\ &= \frac{-125x^3 - 50x^2 + 35x - 72}{2x(x+1)^2(5x+1)^2(5x+6)^2} < 0. \end{aligned} \quad (8)$$

Hence  $f'(x)$  is decreasing on  $(0, +\infty)$ . Then for any  $x \in (0, +\infty)$ , we have  $f'(x) > \lim_{x \rightarrow +\infty} f'(x) = 0$ , thus,  $f(x)$  is increasing on  $(0, +\infty)$ , and  $f(x) < \lim_{x \rightarrow +\infty} f(x) = 1$  for  $x \in (0, +\infty)$ . The inequality (6) is valid.

(ii) Define  $g(x)$  as

$$g(x) = x \ln \left(1 + \frac{1}{x}\right) - \ln \left(1 - \frac{1}{2x+1}\right), \quad x \in (0, +\infty). \quad (9)$$

When  $x > 0$ , the inequality

$$e \left(1 - \frac{1}{2x+1}\right) < \left(1 + \frac{1}{x}\right)^x \quad (10)$$

is equivalent to  $g(x) > 1$ . For  $x \in (0, +\infty)$ , it can be shown that

$$\begin{aligned} g'(x) &= -\frac{1}{x+1} + \ln \left(1 + \frac{1}{x}\right) - \frac{1}{x(2x+1)}, \\ g''(x) &= \frac{5x^2 + 5x + 1}{x^2(x+1)^2(2x+1)^2} > 0. \end{aligned} \quad (11)$$

Hence,  $g'(x)$  is increasing on  $(0, +\infty)$ . Then for any  $x \in (0, +\infty)$ , we have  $g'(x) < \lim_{x \rightarrow +\infty} g'(x) = 0$ , therefore,  $g(x)$  is decreasing on  $(0, +\infty)$  and  $g(x) > \lim_{x \rightarrow +\infty} g(x) = 1$  for  $x \in (0, +\infty)$ . Inequality (10) is valid.

By virtue of (6) and (10), inequalities (4) are valid. This proves Lemma 4.  $\square$

**REMARK 5.** By a direct calculation, we have

$$\left(1 + \frac{5}{5x+1}\right)^{-1/2} < 1 - \frac{1}{2(x+19/20)} \quad (x > 0). \quad (12)$$

Then by (4) and (12), we have

$$e\left(1 - \frac{1}{2x+1}\right) < \left(1 + \frac{1}{x}\right)^x < e\left[1 - \frac{1}{2(x+19/20)}\right] \quad (x > 0). \quad (13)$$

Inequality (13) is equivalent to

$$\frac{e}{2(x+19/20)} < e - \left(1 + \frac{1}{x}\right)^x < \frac{e}{2x+1} \quad (x > 0). \quad (14)$$

Thus, [1, Lemma 2] is contained in Lemma 4. Inequalities (4) and (14) are the new inequalities on the constant  $e$  (cf. [3, Theorem 3.8.26]; and [2, page 358]).

**LEMMA 6** (see [1, Theorem 9]). *Let  $g_m > 0$ ,  $\alpha_m \geq 0$  ( $m = 1, 2, \dots, n$ ),  $\sum_{m=1}^n g_m = 1$ , then*

$$\alpha_1^{g_1} \alpha_2^{g_2} \cdots \alpha_n^{g_n} \leq \sum_{m=1}^n g_m \alpha_m. \quad (15)$$

**PROOF OF THEOREM 3.** Setting  $c_m > 0$ ,  $g_m = \lambda_m/A_n$ ,  $\alpha_m = c_m a_m$  ( $m = 1, 2, \dots, n$ ), by Lemma 6, we have

$$(c_1 a_1)^{\lambda_1/A_1} (c_2 a_2)^{\lambda_2/A_2} \cdots (c_n a_n)^{\lambda_n/A_n} \leq \frac{1}{A_n} \sum_{m=1}^n \lambda_m c_m a_m. \quad (16)$$

Then we find that

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} &= \sum_{n=1}^{\infty} \lambda_{n+1} \frac{(c_1 a_1)^{\lambda_1/A_1} (c_2 a_2)^{\lambda_2/A_2} \cdots (c_n a_n)^{\lambda_n/A_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}} \\ &\leq \sum_{n=1}^{\infty} \left[ \frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}} \right] \frac{1}{A_n} \sum_{m=1}^n c_m \lambda_m a_m \\ &= \sum_{m=1}^{\infty} \lambda_m a_m c_m \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{A_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}}. \end{aligned} \quad (17)$$

Define  $c_m = ((A_{m+1})/A_m)^{A_m/\lambda_m} A_m$  ( $m = 1, 2, \dots$ ) and  $A_0 = 0$ . Because  $0 < \lambda_{n+1} \leq \lambda_n$  ( $n = 1, 2, \dots$ ), we have

$$\begin{aligned} c_m^{\lambda_m} &= \frac{(A_{m+1})^{A_m}}{A_m^{A_{m-1}}}; \quad (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n} = A_{n+1} \quad (n \in \mathbb{N}); \\ c_m \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{A_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}} &= \left( \frac{A_{m+1}}{A_m} \right)^{A_m/\lambda_m} A_m \sum_{n=m}^{\infty} \frac{\lambda_{n+1}}{A_n A_{n+1}} \\ &= \left( 1 + \frac{\lambda_{m+1}}{A_m} \right)^{A_m/\lambda_m} A_m \sum_{n=m}^{\infty} \left( \frac{1}{A_n} - \frac{1}{A_{n+1}} \right) \\ &\leq \left( 1 + \frac{\lambda_m}{A_m} \right)^{A_m/\lambda_m}. \end{aligned} \quad (18)$$

Then by (4) and (17), we obtain that

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} &\leq \sum_{m=1}^{\infty} \left(1 + \frac{\lambda_m}{A_m}\right)^{A_m/\lambda_m} \lambda_m a_m \\ &\leq e \sum_{m=1}^{\infty} \left(1 + \frac{5\lambda_m}{5A_m + \lambda_m}\right)^{-1/2} \lambda_m a_m. \end{aligned} \quad (19)$$

Hence inequality (3) is valid, and Theorem 3 is proved.  $\square$

**REMARK 7.** With inequality (12), Theorem 3 is obviously an improvement and an extension of [4, Theorem 1].

Setting  $\lambda_n \equiv 1$ , (3) changes into

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{5}{5n+1}\right)^{-1/2} a_n. \quad (20)$$

By inequality (12), we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left[1 - \frac{1}{2(n+19/20)}\right] a_n. \quad (21)$$

Thus, inequalities (20) and (21) are obviously an improvement and extension of [5, Theorem 3.1].

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