

MIXED PROBLEM WITH NONLOCAL BOUNDARY CONDITIONS FOR A THIRD-ORDER PARTIAL DIFFERENTIAL EQUATION OF MIXED TYPE

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ABSTRACT. We study a mixed problem with integral boundary conditions for a third-order partial differential equation of mixed type. We prove the existence and uniqueness of the solution. The proof is based on two-sided a priori estimates and on the density of the range of the operator generated by the considered problem.

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1. Introduction. In the rectangle $\Omega = (0, \ell) \times (0, T)$, we consider the equation

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left(a(x, t) \frac{\partial^2 u}{\partial x \partial t} \right) = f(x, t), \quad (1.1)$$

where $a(x, t)$ is bounded with $0 < a_0 < a(x, t) \leq a_1$ and has bounded partial derivatives such that $0 < a_2 \leq \partial a(x, t)/\partial t \leq a_3$ and $0 < a_4 \leq \partial a(x, t)/\partial x \leq a_5$ for $(x, t) \in \bar{\Omega}$.

To (1.1) we add the initial conditions

$$l_1 u = u(x, 0) = \varphi(x), \quad l_2 u = \frac{\partial u}{\partial t}(x, 0) = \psi(x), \quad x \in (0, \ell), \quad (1.2)$$

the Dirichlet condition

$$u(0, t) = 0, \quad t \in (0, T), \quad (1.3)$$

and the integral condition

$$\int_0^\ell u(\xi, t) d\xi = 0, \quad t \in (0, T), \quad (1.4)$$

where φ and ψ are known functions which satisfy the compatibility conditions given by (1.3) and (1.4), that is,

$$\varphi(0) = 0, \quad \int_0^\ell \varphi(x) dx = 0, \quad \psi(0) = 0, \quad \int_0^\ell \psi(x) dx = 0. \quad (1.5)$$

Boundary-value problems for parabolic equations with integral boundary conditions are investigated by Batten [1], Bouziani and Benouar [2], Cannon [3, 4], Perez Esteva and van der Hoeck [5], Ionkin [8], Kamynin [9], Kartynnik [10], Shi [11], Yurchuk [13], and many references therein. We remark that integral boundary conditions for evolution problems have various applications in chemical engineering, thermoelasticity, underground water flow and population dynamics; see for example, [6, 7, 11, 12].

The present paper is devoted to the study of a mixed problem with boundary integral conditions for a third-order partial differential equation of mixed type.

We associate to problem (1.1), (1.2), (1.3), and (1.4) the operator $L = (\mathcal{L}, l_1, l_2)$, defined from E into F , where E is the Banach space of functions $u \in L_2(\Omega)$, satisfying (1.3) and (1.4), with the finite norm

$$\begin{aligned} \|u\|_E^2 &= \int_{\Omega} (\ell - x)^2 \left[\left| \frac{\partial^2 u}{\partial t^2} \right|^2 + \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^2 \right] dx dt \\ &\quad + \sup_{0 \leq t \leq T} \int_0^\ell (\ell - x)^2 \left[\left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \right] dx + \sup_{0 \leq t \leq T} \int_0^\ell \left[\left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] dx, \end{aligned} \quad (1.6)$$

and F is the Hilbert space of vector-valued functions $\mathcal{F} = (f, \varphi, \psi)$ obtained by completion of the space $L_2(\Omega) \times W_2^2(0, \ell) \times W_2^2(0, \ell)$ with respect to the norm

$$\begin{aligned} \|\mathcal{F}\|_F^2 &= \|(f, \varphi, \psi)\|_F^2 \\ &= \int_{\Omega} (\ell - x)^2 |f|^2 dx dt + \int_0^\ell (\ell - x)^2 \left[\left| \frac{d\varphi}{dx} \right|^2 + \left| \frac{d\psi}{dx} \right|^2 \right] dx + \int_0^\ell [|\varphi|^2 + |\psi|^2] dx. \end{aligned} \quad (1.7)$$

Using the energy inequalities method proposed in [13], we establish two-sided a priori estimates. Then, we prove that the operator L is a linear homeomorphism between the spaces E and F .

2. Two-sided a priori estimates

THEOREM 2.1. *For any function $u \in E$, there is the a priori estimate*

$$\|Lu\|_F \leq c \|u\|_E, \quad (2.1)$$

where the constant c is independent of u .

PROOF. Using (1.1) and the initial conditions (1.2), we obtain

$$\begin{aligned} \int_{\Omega} (\ell - x)^2 |\mathcal{L}u|^2 dx dt &\leq 3 \int_{\Omega} (\ell - x)^2 \left[\left| \frac{\partial^2 u}{\partial t^2} \right|^2 + a_5^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + a_1^2 \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^2 \right] dx dt, \\ \int_0^\ell (\ell - x)^2 \left[\left| \frac{d\psi}{dx} \right|^2 + \left| \frac{d\varphi}{dx} \right|^2 \right] dx &\leq \sup_{0 \leq t \leq T} \int_0^\ell (\ell - x)^2 \left[\left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 + \left| \frac{\partial u}{\partial x} \right|^2 \right] dx, \\ \int_0^\ell [|\psi|^2 + |\varphi|^2] dx &\leq \sup_{0 \leq t \leq T} \int_0^\ell \left[\left| \frac{\partial u}{\partial t} \right|^2 + |u|^2 \right] dx. \end{aligned} \quad (2.2)$$

Combining the inequalities (2.2), we obtain (2.1) for $u \in E$. \square

THEOREM 2.2. *For any function $u \in E$, there is the a priori estimate*

$$\|u\|_E \leq \alpha \|Lu\|_F, \quad (2.3)$$

with the constant

$$\alpha = \frac{\max(167/10, a_1)}{\min(\exp(-cT)/20, \exp(-cT)a_0^2/15)}, \quad (2.4)$$

and c is such that

$$c \geq 1, \quad ca_0 - 1 \geq a_3 + 2a_5^2. \quad (2.5)$$

Before proving this theorem, we first give the following two lemmas.

LEMMA 2.3. *For $u \in E$ satisfying the first condition in (1.2),*

$$\begin{aligned} & \frac{1}{2} \int_0^\tau \int_0^\ell (\ell-x)^2 \exp(-ct) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt + \frac{c-1}{2} \int_0^\tau \int_0^\ell (\ell-x)^2 \exp(-ct) \left| \frac{\partial u}{\partial x} \right|^2 dx dt \\ & \geq \frac{1}{2} \int_0^\ell (\ell-x)^2 \exp(-c\tau) \left| \frac{\partial u}{\partial x}(x, \tau) \right|^2 dx - \frac{1}{2} \int_0^\ell (\ell-x)^2 \left| \frac{d\varphi}{dx} \right|^2 dx. \end{aligned} \quad (2.6)$$

PROOF. Starting from

$$\int_0^\tau \int_0^\ell (\ell-x)^2 \exp(-ct) \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \right) \frac{\partial \bar{u}}{\partial x} dx dt, \quad (2.7)$$

then integrating by parts and using elementary inequalities, we obtain (2.6). \square

LEMMA 2.4. *For $u \in E$ satisfying the initial conditions (1.2),*

$$\int_0^\ell \exp(-c\tau) |u(x, \tau)|^2 dx \leq \int_0^\tau \int_0^\ell \exp(-ct) \left| \frac{\partial u}{\partial t} \right|^2 dx dt + \int_0^\ell |\varphi|^2 dx, \quad (2.8)$$

with $c \geq 1$.

PROOF. Integrating by parts the expression

$$\int_0^\tau \int_0^\ell \exp(-ct) u \frac{\partial \bar{u}}{\partial t} dx dt \quad (2.9)$$

and using elementary inequalities yield (2.8). \square

REMARK 2.5. We note that Lemmas 2.3 and 2.4 hold for weaker conditions on u .

PROOF OF THEOREM 2.2. First, define

$$D(L) = \left\{ u \in E \mid \frac{\partial^5 u}{\partial x^2 \partial t^3} \in L_2(\Omega) \right\}, \quad Mu = (\ell-x)^2 \frac{\partial^2 u}{\partial t^2} + 2(\ell-x) J \frac{\partial^2 u}{\partial t^2}, \quad (2.10)$$

where

$$Ju = \int_0^x u(\xi, t) d\xi. \quad (2.11)$$

We consider for $u \in D(L)$ the quadratic formula

$$\operatorname{Re} \int_0^\tau \int_0^\ell \exp(-ct) \mathcal{L}u \bar{Mu} dx dt, \quad (2.12)$$

with the constant c satisfying (2.5), obtained by multiplying (1.1) by $\exp(-ct)Mu$, by

integrating over Ω^τ , where $\Omega^\tau = (0, \ell) \times (0, \tau)$, with $0 \leq \tau \leq T$, and by taking the real part. Integrating by parts (2.12) with the use of boundary conditions (1.3) and (1.4), we obtain

$$\begin{aligned}
 & \operatorname{Re} \int_0^\tau \int_0^\ell \exp(-ct) \mathcal{L}u M \bar{u} dx dt \\
 &= \int_0^\tau \int_0^\ell (\ell-x)^2 \exp(-ct) \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt + \frac{1}{2} \int_0^\tau \int_0^\ell \exp(-ct) \left| J \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \\
 &+ \operatorname{Re} \int_0^\tau \int_0^\ell (\ell-x)^2 \exp(-ct) a \frac{\partial^2 u}{\partial x \partial t} \frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{u}}{\partial x \partial t} \right) dx dt \\
 &+ 2 \operatorname{Re} \int_0^\tau \int_0^\ell \exp(-ct) \frac{\partial u}{\partial t} a \frac{\partial^2 \bar{u}}{\partial t^2} dx dt \\
 &+ 2 \operatorname{Re} \int_0^\tau \int_0^\ell \exp(-ct) \frac{\partial a}{\partial x} \frac{\partial u}{\partial t} J \frac{\partial^2 \bar{u}}{\partial t^2} dx dt. \tag{2.13}
 \end{aligned}$$

On the other hand, by using the elementary inequalities we get

$$\begin{aligned}
 & \operatorname{Re} \int_0^\tau \int_0^\ell \exp(-ct) \mathcal{L}u M \bar{u} dx dt \\
 & \geq \int_0^\tau \int_0^\ell (\ell-x)^2 \exp(-ct) \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \\
 &+ \operatorname{Re} \int_0^\tau \int_0^\ell (\ell-x)^2 \exp(-ct) a \frac{\partial^2 u}{\partial x \partial t} \frac{\partial}{\partial t} \left(\frac{\partial^2 \bar{u}}{\partial x \partial t} \right) dx dt \\
 &+ 2 \operatorname{Re} \int_0^\tau \int_0^\ell \exp(-ct) \frac{\partial u}{\partial t} a \frac{\partial^2 \bar{u}}{\partial t^2} dx dt \\
 &- 2 \operatorname{Re} \int_0^\tau \int_0^\ell \exp(-ct) \left| \frac{\partial a}{\partial x} \right|^2 \left| \frac{\partial u}{\partial t} \right|^2 dx dt. \tag{2.14}
 \end{aligned}$$

Again, integrating by parts the second and third terms of the right-hand side of the inequality (2.14) and taking into account the initial conditions (1.2) give

$$\begin{aligned}
 & \operatorname{Re} \int_0^\tau \int_0^\ell \exp(-ct) \mathcal{L}u M \bar{u} dx dt + \int_0^\ell a(x, 0) |\psi|^2 dx + \frac{1}{2} \int_0^\ell a(x, 0) (\ell-x)^2 \left| \frac{d\psi}{dx} \right|^2 dx dt \\
 & \geq \int_0^\tau \int_0^\ell \exp(-ct) (\ell-x)^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt - 2 \int_0^\tau \int_0^\ell \exp(-ct) \left| \frac{\partial a}{\partial x} \right|^2 \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\
 &+ \frac{1}{2} \int_0^\ell a(x, \tau) \exp(-c\tau) (\ell-x) \left| \frac{\partial^2 u}{\partial x \partial t} (x, \tau) \right|^2 dx \\
 &- \frac{1}{2} \int_0^\tau \int_0^\ell \exp(-ct) \frac{\partial a}{\partial t} (\ell-x)^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt \\
 &+ \frac{c}{2} \int_0^\tau \int_0^\ell \exp(-ct) (\ell-x)^2 a \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt + \int_0^\ell \exp(-c\tau) a(x, \tau) \left| \frac{\partial u}{\partial t} (x, \tau) \right|^2 dx \\
 &- \int_0^\tau \int_0^\ell \exp(-ct) \frac{\partial a}{\partial t} \left| \frac{\partial u}{\partial t} \right|^2 dx dt + c \int_0^\tau \int_0^\ell \exp(-ct) a \left| \frac{\partial u}{\partial t} \right|^2 dx dt. \tag{2.15}
 \end{aligned}$$

By using the elementary inequalities on the first integral in the left-hand side of (2.15), we obtain

$$\begin{aligned}
& \frac{33}{2} \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 |\mathcal{L}u|^2 dx dt + \frac{3}{4} \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \\
& + \int_0^\ell a(x, 0) |\psi|^2 dx + \frac{1}{2} \int_0^\ell a(x, 0) (\ell-x)^2 \left| \frac{d\psi}{dx} \right|^2 dx \\
& \geq \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt - 2 \int_0^\tau \int_0^\ell \exp(-ct) \left| \frac{\partial a}{\partial x} \right|^2 \left| \frac{\partial u}{\partial t} \right|^2 dx dt \\
& + \frac{1}{2} \int_0^\ell \exp(-c\tau)(\ell-x)^2 \left| \frac{\partial^2 u(x, \tau)}{\partial x \partial t} \right|^2 dx - \frac{1}{2} \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 \frac{\partial a}{\partial t} \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt \\
& + \frac{c}{2} \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt + \int_0^\ell \exp(-c\tau) a(x, \tau) \left| \frac{\partial u(x, \tau)}{\partial t} \right|^2 dx \\
& - \int_0^\tau \int_0^\ell \exp(-ct) \frac{\partial a}{\partial t} \left| \frac{\partial u}{\partial t} \right|^2 dx dt + c \int_0^\tau \int_0^\ell \exp(-ct) a \left| \frac{\partial u}{\partial t} \right|^2 dx dt.
\end{aligned} \tag{2.16}$$

Now, from (1.1) we have

$$\begin{aligned}
& \frac{1}{5} \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 |\mathcal{L}u|^2 dx dt \\
& + \frac{1}{5} \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 \left| \frac{\partial a}{\partial x} \right|^2 \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt \\
& + \frac{1}{5} \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \\
& \geq \frac{1}{15} \int_0^\tau \int_0^\ell \exp(-ct)(\ell-x)^2 a^2 \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^2 dx dt.
\end{aligned} \tag{2.17}$$

Combining inequalities (2.16), (2.17), and Lemmas 2.3 and 2.4, we get

$$\begin{aligned}
& \frac{167}{10} \int_\Omega (\ell-x)^2 |\mathcal{L}u|^2 dx dt + \frac{a_1}{2} \int_0^\ell (\ell-x)^2 \left| \frac{d\psi}{dx} \right|^2 dx \\
& + a_1 \int_0^\ell |\psi|^2 dx + \frac{1}{2} \int_0^\ell (\ell-x)^2 \left| \frac{d\varphi}{dx} \right|^2 dx + \int_0^\ell |\varphi|^2 dx \\
& \geq \exp(-cT) \left(\frac{1}{20} \int_0^\tau \int_0^\ell (\ell-x)^2 \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt + \frac{1}{2} \int_0^\ell (\ell-x)^2 \left| \frac{\partial^2 u}{\partial x \partial t}(x, \tau) \right|^2 dx dt \right. \\
& + \int_0^\ell |u(x, \tau)|^2 dx + a_0 \int_0^\ell \left| \frac{\partial u}{\partial t}(x, \tau) \right|^2 dx + \frac{1}{2} \int_0^\ell (\ell-x)^2 \left| \frac{\partial u}{\partial x}(x, \tau) \right|^2 dx \\
& \left. + \frac{a_0^2}{15} \int_0^\tau \int_0^\ell (\ell-x)^2 \left| \frac{\partial^3 u}{\partial x^2 \partial t} \right|^2 dx dt \right).
\end{aligned} \tag{2.18}$$

As the left-hand side of (2.18) is independent of τ , by replacing the right-hand side by its upper bound with respect to τ in the interval $[0, T]$, we obtain the desired inequality. \square

3. Solvability of the problem. From estimates (2.1) and (2.3) it follows that the operator $L : E \rightarrow F$ is continuous and its range is closed in F . Therefore, the inverse operator L^{-1} exists and is continuous from the closed subspace $R(L)$ onto E , which means that L is a homomorphism from E onto $R(L)$. To obtain the uniqueness of solution, it remains to show that $R(L) = F$. The proof is based on the following lemma.

LEMMA 3.1. *Suppose that $\partial^3 a / \partial x^2 \partial t$ is also bounded. Let $D_0(L) = \{u \in D(L) : l_1 u = 0, l_2 u = 0\}$. If for $u \in D_0(L)$ and some $\omega \in L_2(\Omega)$,*

$$\int_{\Omega} (\ell - x) \mathcal{L} u \omega \, dx \, dt = 0, \quad (3.1)$$

then $\omega = 0$.

PROOF. From (3.1) we have

$$\int_{\Omega} (\ell - x) \frac{\partial^2 u}{\partial t^2} \omega \, dx \, dt = \int_{\Omega} (\ell - x) \frac{\partial}{\partial x} \left(a \frac{\partial^2 u}{\partial x \partial t} \right) \omega \, dx \, dt. \quad (3.2)$$

If we introduce the smoothing operators with respect to t (see [13]) $J_{\xi}^{-1} = (I + \xi(\partial / \partial t))^{-1}$ and $(J_{\xi}^{-1})^*$, then these operators provide the solutions of the respective problems

$$\xi \frac{d g_{\xi}(t)}{dt} + g_{\xi}(t) = g(t), \quad g_{\xi}(t)|_{t=0} = 0, \quad (3.3)$$

$$-\xi \frac{d g_{\xi}^*(t)}{dt} + g_{\xi}^*(t) = g(t), \quad g_{\xi}^*(t)|_{t=T} = 0, \quad (3.4)$$

and also have the following properties: for any $g \in L_2(0, T)$, the functions $g_{\xi} = (J_{\xi}^{-1})g$ and $g_{\xi}^* = (J_{\xi}^{-1})^*g$ are in $W_2^1(0, T)$ such that $g_{\xi}|_{t=0} = 0$ and $g_{\xi}^*|_{t=T} = 0$. Moreover, J_{ξ}^{-1} commutes with $\partial / \partial t$, so $\int_0^T |g_{\xi} - g|^2 \, dt \rightarrow 0$ and $\int_0^T |g_{\xi}^* - g|^2 \, dt \rightarrow 0$ for $\xi \rightarrow 0$.

Now, for given $\omega(x, t)$, we introduce the function

$$v(x, t) = \omega(x, t) - \int_0^x \frac{\omega(\xi, t)}{\ell - \xi} \, d\xi. \quad (3.5)$$

Integrating by parts with respect to ξ , we obtain

$$\begin{aligned} \int_0^x v(\xi, t) \, d\xi &= \int_0^x \omega(\xi, t) \, d\xi + \int_0^x \frac{\partial}{\partial \xi} (\ell - \xi) \int_0^{\xi} \frac{\omega(\eta, t)}{\ell - \eta} \, d\eta \, d\xi \\ &= (\ell - x)(\omega(x, t) - v(x, t)), \end{aligned} \quad (3.6)$$

which implies that

$$(\ell - x)v + Jv = (\ell - x)\omega, \quad \int_0^{\ell} v(x, t) \, dx = 0. \quad (3.7)$$

Then, from equality (3.2) we obtain

$$-\int_{\Omega} \frac{\partial^2 u}{\partial t^2} Nv \, dx \, dt = \int_{\Omega} A(t) \frac{\partial u}{\partial t} \bar{v} \, dx \, dt, \quad (3.8)$$

where

$$Nv = (\ell - x)v + Jv, \quad A(t)u = -\frac{\partial}{\partial x} \left((\ell - x)a(x, t) \frac{\partial u}{\partial x} \right). \quad (3.9)$$

Replace $\partial u / \partial t$ by the smoothed function $J_\xi^{-1}(\partial u / \partial t)$ in (3.8) and use the relation

$$A(t)J_\xi^{-1} = J_\xi^{-1}A(\tau) + \xi J_\xi^{-1} \frac{\partial A(\tau)}{\partial \tau} J_\xi^{-1}. \quad (3.10)$$

Then, by taking the adjoint of the operator J_ξ^{-1} , and by integrating by parts with respect to t in the left-hand side, we obtain

$$\int_\Omega \frac{\partial u}{\partial t} \overline{N \frac{\partial v_\xi^*}{\partial t}} dx dt = \int_\Omega A(t) \frac{\partial u}{\partial t} \overline{v_\xi^*} dx dt + \xi \int_\Omega \frac{\partial A}{\partial t} \left(\frac{\partial u}{\partial t} \right)_\xi \overline{v_\xi^*} dx dt. \quad (3.11)$$

The operator $A(t)$ has a continuous inverse on $L_2(0, \ell)$ defined by the relation

$$A^{-1}(t)g = - \int_0^x \frac{d\xi}{a(\xi, t)(\ell - \xi)} \int_0^\xi g(\eta) d\eta + c \int_0^x \frac{d\xi}{a(\xi, t)(\ell - \xi)}, \quad (3.12)$$

where

$$c = \frac{\int_0^\ell (dx/a(x, t)) \int_0^x g(\xi) d\xi}{\int_0^\ell (dx/a(x, t))}, \quad \int_0^\ell A^{-1}(t)g dx = 0. \quad (3.13)$$

Hence, the function $(\partial u / \partial t)_\xi$ can be represented in the form

$$\left(\frac{\partial u}{\partial t} \right)_\xi = J_\xi^{-1} A^{-1}(t) A(t) \frac{\partial u}{\partial t}. \quad (3.14)$$

Then, $(\partial A / \partial t)(\partial u / \partial t)_\xi = A_\xi(t)A(t)(\partial u / \partial t)$, where

$$A_\xi(t) = \left(\frac{\partial^2 a}{\partial x \partial t} J_\xi^{-1} - \frac{\partial a}{\partial t} J_\xi^{-1} \frac{\partial a}{\partial x} \frac{1}{a} \right) \frac{1}{a} \left(\int_0^x g(\eta, t) d\eta - c \right) + \frac{\partial a}{\partial t} J_\xi^{-1} \frac{1}{a} g, \quad (3.15)$$

where the constant c is given by (3.13).

Consequently, equation (3.11) becomes

$$\int_\Omega \frac{\partial u}{\partial t} \overline{N \frac{\partial v_\xi^*}{\partial t}} dx dt = \int_\Omega A(t) \frac{\partial u}{\partial t} (v_\xi^* + \xi A_\xi^* v_\xi^*) dx dt, \quad (3.16)$$

in which the conjugate operator $A_\xi^*(t)$ of $A_\xi(t)$ is defined by

$$A_\xi^* v_\xi^* = \frac{1}{a} (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} v_\xi^* + (Bv_\xi^*)(x) - (Bv_\xi^*)(0) \frac{\int_x^\ell (d\xi/a(\xi, t))}{\int_0^\ell (d\xi/a(\xi, t))}, \quad (3.17)$$

where

$$(Bv_\xi^*)(x) = \int_x^\ell \frac{1}{a(\xi, t)} \left[(J_\xi^{-1})^* \frac{\partial^2 a}{\partial \xi \partial \tau} - \frac{1}{a(\xi, t)} \frac{\partial a}{\partial \xi} (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right] v_\xi^*(\xi, \tau) d\xi. \quad (3.18)$$

The left-hand side of (3.16) is a continuous linear functional of $\partial u / \partial t$. Hence, the function $h_\xi = v_\xi^* + \xi A_\xi^* v_\xi^*$ has the derivatives $(\ell - x)(\partial h_\xi / \partial x) \in L_2(\Omega)$, $(\partial / \partial x)((\ell - x)(\partial h_\xi / \partial x)) \in L_2(\Omega)$, and the following conditions are satisfied

$$h_\xi|_{x=0} = 0, \quad h_\xi|_{x=\ell} = 0, \quad (\ell - x) \frac{\partial h_\xi}{\partial x} \Big|_{x=\ell} = 0. \quad (3.19)$$

From (3.17) we have

$$(\ell - x) \frac{\partial h_\xi}{\partial x} = \left(I + \xi \frac{1}{a} (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) \frac{\partial v_\xi^*}{\partial x}, \quad (3.20)$$

$$\begin{aligned} \frac{\partial}{\partial x} \left((\ell - x) \frac{\partial h_\xi}{\partial x} \right) &= \left(I + \xi \frac{1}{a} (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) \frac{\partial}{\partial x} \left((\ell - x) \frac{\partial v_\xi^*}{\partial x} \right) \\ &+ \xi \left[-\frac{(\partial a / \partial x) (J_\xi^{-1})^* (\partial a / \partial \tau)}{a^2} + \frac{1}{a} (J_\xi^{-1})^* \frac{\partial^2 a}{\partial x \partial \tau} \right] (\ell - x) \frac{\partial v_\xi^*}{\partial x}, \end{aligned} \quad (3.21)$$

$$\left[\left(I + \xi \frac{1}{a} (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) v_\xi^* \right]_{x=0} = 0, \quad (3.22)$$

$$\left[\left(I + \xi \frac{1}{a} (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) v_\xi^* \right]_{x=\ell} = 0, \quad (3.23)$$

$$\left[\left(I + \xi \frac{1}{a} (J_\xi^{-1})^* \frac{\partial a}{\partial \tau} \right) (\ell - x) \frac{\partial v_\xi^*}{\partial x} \right]_{x=\ell} = 0. \quad (3.24)$$

Since $\|\xi(1/a)(J_\xi^{-1})^*(\partial a / \partial \tau)\|_{L_2(\Omega)} < 1$ for sufficiently small ξ , the operator $I + \xi(1/a)(J_\xi^{-1})^*(\partial a / \partial \tau)$ has a continuous inverse on $L_2(\Omega)$. In addition, the derivative of the above operator with respect to x is a bounded operator in $L_2(\Omega)$. Therefore, from (3.20) and (3.21), the function v_ξ^* has derivatives $(\ell - x)(\partial v_\xi^* / \partial x) \in L_2(\Omega)$ and $(\partial / \partial x)((\ell - x)(\partial v_\xi^* / \partial x)) \in L_2(\Omega)$.

In a similar way, we show that for each fixed $x \in [0, \ell]$ and sufficiently small ξ , the operator $I + \xi(1/a)(J_\xi^{-1})^*(\partial a / \partial \tau)$ has a continuous inverse on $L_2(0, T)$; hence, (3.22), and (3.23), and (3.24) imply that

$$v_\xi^*|_{x=0} = 0, \quad v_\xi^*|_{x=\ell} = 0, \quad (\ell - x) \frac{\partial v_\xi^*}{\partial x} \Big|_{x=\ell} = 0. \quad (3.25)$$

So, for ξ sufficiently small, the function v_ξ^* has the same properties as h_ξ . In addition, v_ξ^* satisfies the integral condition in (3.7).

Putting $u = \int_0^t \int_0^\tau \exp(c\eta) v_\xi^*(\eta, \tau) d\eta d\tau$ in (3.8), where the constant c satisfies $ca_0 - a_3 - a_3^2/a_0 \geq 0$, and using (3.4), we obtain

$$\int_{\Omega} \exp(ct) v_\xi^* \bar{N}v \, dx dt = - \int_{\Omega} A(t) \frac{\partial u}{\partial t} \exp(-ct) \frac{\partial^2 \bar{u}}{\partial t^2} \, dx dt + \xi \int_{\Omega} A(t) \frac{\partial u}{\partial t} \frac{\partial \bar{v}_\xi^*}{\partial t} \, dx dt. \quad (3.26)$$

Integrating by parts each term in the left-hand side of (3.26) and taking the real parts yield

$$\begin{aligned}
 & \operatorname{Re} \int_{\Omega} A(t) \frac{\partial u}{\partial t} \exp(-ct) \frac{\partial^2 \bar{u}}{\partial t^2} dx dt \\
 & \geq \frac{c}{2} \int_{\Omega} (\ell - x) a(x, t) \exp(-ct) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt \\
 & - \frac{1}{2} \int_{\Omega} (\ell - x) \frac{\partial a}{\partial t} \exp(-ct) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt, \\
 & \operatorname{Re} \left(-\xi \int_{\Omega} A(t) \frac{\partial u}{\partial t} \frac{\partial \bar{v}_\xi^*}{\partial t} dx dt \right) \geq \frac{-\xi a_3^2}{2a_0} \int_{\Omega} (\ell - x) \exp(-ct) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 dx dt.
 \end{aligned} \tag{3.27}$$

Now, using (3.27) in (3.26) with the choice of c indicated above we have

$$2 \operatorname{Re} \int_{\Omega} \exp(ct) v_\xi^* \bar{N} v dx dt \leq 0. \tag{3.28}$$

Then, for $\xi \rightarrow 0$ we obtain $2 \operatorname{Re} \int_{\Omega} \exp(ct) v \bar{N} v dx dt \leq 0$, that is,

$$2 \operatorname{Re} \int_{\Omega} \exp(ct) (\ell - x) |v|^2 dx dt + 2 \operatorname{Re} \int_{\Omega} \exp(ct) v J \bar{v} dx dt \leq 0. \tag{3.29}$$

Since $\operatorname{Re} \int_{\Omega} \exp(ct) v J \bar{v} dx dt = 0$, we conclude that $v = 0$; hence, $\omega = 0$, which ends the proof of the lemma. \square

THEOREM 3.2. *The range $R(L)$ of L coincides with F .*

PROOF. Since F is a Hilbert space, we have $R(L) = F$ if and only if the relation

$$\int_{\Omega} (\ell - x)^2 \mathcal{L} u \bar{f} dx dt + \int_0^\ell \left[(\ell - x)^2 \left(\frac{dl_1 u}{dx} \frac{d\bar{\varphi}}{dx} + \frac{dl_2 u}{dx} \frac{d\bar{\psi}}{dx} \right) \right] dx + \int_0^\ell (l_1 u \bar{\varphi} + l_2 u \bar{\psi}) dx = 0, \tag{3.30}$$

for arbitrary $u \in E$ and $(f, \varphi, \psi) \in F$, implies that $f = 0$, $\varphi = 0$ and $\psi = 0$. Putting $u \in D_0(L)$ in (3.30), we conclude from Lemma 3.1 that $(\ell - x)f = 0$. Hence,

$$\int_0^\ell \left[(\ell - x)^2 \left(\frac{dl_1 u}{dx} \frac{d\bar{\varphi}}{dx} + \frac{dl_2 u}{dx} \frac{d\bar{\psi}}{dx} \right) + l_1 u \bar{\varphi} + l_2 u \bar{\psi} \right] dx = 0 \quad \forall u \in D(L). \tag{3.31}$$

Setting

$$D_{0k}(L) = \left\{ u \in D(L) : u^{(k)}|_{t=0} = 0, k = 0, 1 \right\}, \tag{3.32}$$

and taking $u \in D_{01}(L)$ in (3.31) yield

$$\int_0^\ell \left[(\ell - x)^2 \frac{dl_1 u}{dx} \frac{d\bar{\varphi}}{dx} + l_1 u \bar{\varphi} \right] dx = 0. \tag{3.33}$$

The range of the trace operator l_1 is everywhere dense in Hilbert space with the norm $[\int_0^\ell ((\ell - x)^2 |\frac{d\varphi}{dx}|^2 + |\varphi|^2) dx]^{1/2}$; hence, $\varphi = 0$. Likewise, for $u \in D_{00}(L)$, we get $\psi = 0$. \square

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