

SUBMANIFOLDS OF F -STRUCTURE MANIFOLD SATISFYING

$$F^K + (-)^{K+1}F = 0$$

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ABSTRACT. The purpose of this paper is to study invariant submanifolds of an n -dimensional manifold M endowed with an F -structure satisfying $F^K + (-)^{K+1}F = 0$ and $F^W + (-)^{W+1}F \neq 0$ for $1 < W < K$, where K is a fixed positive integer greater than 2. The case when K is odd (≥ 3) has been considered in this paper. We show that an invariant submanifold \tilde{M} , embedded in an F -structure manifold M in such a way that the complementary distribution D_m is never tangential to the invariant submanifold $\Psi(\tilde{M})$, is an almost complex manifold with the induced \tilde{F} -structure. Some theorems regarding the integrability conditions of induced \tilde{F} -structure are proved.

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1. Introduction. Invariant submanifolds have been studied by Blair et al. [1], Kubo [4], Yano and Okumura [7, 8], and among others. Yano and Ishihara [6] have studied and shown that any invariant submanifold of codimension 2 in a contact Riemannian manifold is also a contact Riemannian manifold. We consider an F -structure manifold M and study its invariant submanifolds. Let F be a nonzero tensor field of the type $(1, 1)$ and of class C^∞ on an n -dimensional manifold M such that (see [3])

$$F^K + (-)^{K+1}F = 0, \quad F^W + (-)^{W+1}F \neq 0, \quad \text{for } 1 < W < K, \quad (1.1)$$

where K is a fixed positive integer greater than 2. Such a structure on M is called an F -structure of rank r and of degree K . If the rank of F is constant and $r = r(F)$, then M is called an F -structure manifold of degree $K (\geq 3)$.

Let the operator on M be defined as follows (see [3])

$$\ell = (-)^K F^{K-1}, \quad m = I + (-)^{K+1} F^{K-1}, \quad (1.2)$$

where I denotes the identity operator on M . For the operators defined by (1.2), we have

$$\ell + m = I, \quad \ell^2 = \ell; \quad m^2 = m. \quad (1.3)$$

For F satisfying (1.1), there exist complementary distribution D_ℓ and D_m corresponding to the projection operators ℓ and m , respectively. If $\text{rank}(F) = \text{constant}$ on M , then $\dim D_\ell = r$ and $\dim D_m = (n - r)$. We have the following results (see [3]).

$$F\ell = \ell F = F, \quad Fm = mF = 0, \quad (1.4a)$$

$$F^{K-1} = (-)^K \ell, \quad F^{K-1} \ell = -\ell, \quad F^{K-1} m = 0. \quad (1.4b)$$

Thus F^{K-1} acts on D_ℓ as an almost complex structure and on D_m as a null operator.

2. Invariant submanifolds of F -structure manifold. Let \tilde{M} be a differentiable manifold embedded differentially as a submanifold in an n -dimensional C^∞ Riemannian manifold M with an F -structure and we denote its embedding by $\Psi : \tilde{M} \rightarrow M$. Denote by $B : T(\tilde{M}) \rightarrow T(M)$ the differential mapping of Ψ , where $d\Psi = B$ is the Jacobson map of Ψ . $T(\tilde{M})$ and $T(M)$ are tangent bundles of \tilde{M} and M , respectively. We call $T(\tilde{M}, M)$ as the set of all vectors tangent to the submanifold $\Psi(\tilde{M})$. It is known that $B : T(\tilde{M}) \rightarrow T(\tilde{M}, M)$ is an isomorphism (see [5]).

Let \tilde{X} and \tilde{Y} be two C^∞ vector fields defined along $\Psi(\tilde{M})$ and tangent to $\Psi(\tilde{M})$. Let X and Y be the local extensions of \tilde{X} and \tilde{Y} . The restriction of $[X, Y]_{\tilde{M}}$ is determined independently of the choice of these local extensions X and Y . Therefore, we can define

$$[\tilde{X}, \tilde{Y}] = [X, Y]_{\tilde{M}}. \quad (2.1)$$

Since B is an isomorphism, it is easy to see that $[B\tilde{X}, B\tilde{Y}] = B[\tilde{X}, \tilde{Y}]$ for all $\tilde{X}, \tilde{Y} \in T(\tilde{M})$. We denote by G the Riemannian metric tensor of M and put

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(B\tilde{X}, B\tilde{Y}) \quad \forall \tilde{X}, \tilde{Y} \text{ in } T(\tilde{M}), \quad (2.2)$$

where g is the Riemannian metric in M and \tilde{g} is the induced metric of \tilde{M} .

DEFINITION 2.1. We say that \tilde{M} is an invariant submanifold of M if

- (i) the tangent space $T_p(\Psi(\tilde{M}))$ of the submanifold $\Psi(\tilde{M})$ is invariant by the linear mapping F at each point p of $\Psi(\tilde{M})$,
- (ii) for each $\tilde{X} \in T(\tilde{M})$, we have

$$F^{(K-1)/2}(B\tilde{X}) = B\tilde{X}'. \quad (2.3)$$

DEFINITION 2.2. Let \tilde{F} be a $(1, 1)$ -tensor field defined in \tilde{M} such that $\tilde{F}(\tilde{X}) = \tilde{X}'$ and M is an invariant submanifold, then we have

$$F(B\tilde{X}) = B(\tilde{F}\tilde{X}), \quad (2.4a)$$

$$F^{(K-1)/2}(B\tilde{X}) = B(\tilde{F}^{(K-1)/2}\tilde{X}). \quad (2.4b)$$

We see that there are two cases for any invariant submanifold \tilde{M} . We assume the following cases.

CASE 1. The distribution D_m is never tangential to $\Psi(\tilde{M})$.

CASE 2. The distribution D_m is always tangential to $\Psi(\tilde{M})$.

We will consider [Case 1](#) and assume that no vector field of the type mX , where $X \in T(\Psi(\tilde{M}))$ is tangential to $\Psi(\tilde{M})$.

THEOREM 2.3. An invariant submanifold \tilde{M} is an almost complex manifold if the following two conditions are satisfied:

- (i) the distribution D_m is never tangential to $\Psi(\tilde{M})$, and
- (ii) \tilde{F} in \tilde{M} defines an induced almost complex structure satisfying $\tilde{F}^{K-1} = (-)^K I$.

PROOF. Applying $F^{(K-1)/2}$ in (2.4), we obtain

$$F^{(K-1)/2}(F^{(K-1)/2}(B\tilde{X})) = F^{(K-1)/2}(B(\tilde{F}^{(K-1)/2}\tilde{X})). \quad (2.5)$$

Making use of (2.4a) in (2.5), we get

$$F^{K-1}(B\tilde{X}) = B(\tilde{F}^{K-1}\tilde{X}). \quad (2.6)$$

In order to show that vector fields of the type $B\tilde{X}$ belong to the distribution D_ℓ , we suppose that $m(B\tilde{X}) \neq 0$, then using (1.2) we have

$$m(B\tilde{X}) = (I + (-)^{K+1}F^{K-1})B\tilde{X} = B\tilde{X} + (-)^{K+1}F^{K-1}(B\tilde{X}) \quad (2.7)$$

which in view of (2.6) becomes

$$m(B\tilde{X}) = B\tilde{X} + (-)^{K+1}B(\tilde{F}^{K-1}\tilde{X}) = B[\tilde{X} + (-)^{K+1}\tilde{F}^{K-1}\tilde{X}] \quad (2.8)$$

which, contrary to our assumption, shows that $m(B\tilde{X})$ is tangential to $\Psi(\tilde{M})$. Thus $m(B\tilde{X}) = 0$.

Also, in view of (1.4b), (1.3), and (2.6) we obtain

$$\begin{aligned} B(\tilde{F}^{K-1}\tilde{X}) &= F^{K-1}(B\tilde{X}) = (-)^K \ell(B\tilde{X}) = (-)^K (I - m)B\tilde{X} \\ &= (-)^K B\tilde{X} - (-)^K mB\tilde{X}, \\ B(\tilde{F}^{K-1}\tilde{X}) &= (-)^K B\tilde{X}. \end{aligned} \quad (2.9)$$

Since B is an isomorphism, we get

$$\tilde{F}^{K-1} = (-)^K I. \quad (2.10)$$

Let $\mathcal{F}(M)$ be the ring of real-valued differentiable functions on M , and let $\mathcal{X}(M)$ be the module of derivatives of $\mathcal{F}(M)$. Then $\mathcal{X}(M)$ is Lie algebra over the real numbers and the elements of $\mathcal{X}(M)$ are called vector fields. Then M is equipped with $(1,1)$ -tensor field F which is a linear map such that

$$F : \mathcal{X}(M) \longrightarrow \mathcal{X}(M). \quad (2.11)$$

Let M be of degree K and let K be a positive odd integer greater than 2. Then we consider a positive definite Riemannian metric with respect to which D_ℓ and D_m are orthogonal so that

$$g(X, Y) = g(HX, HY) + g(mX, Y), \quad (2.12)$$

where $H = F^{(K-1)/2}$ for all $X, Y \in \mathcal{X}(M)$. \square

DEFINITION 2.4. The induced metric \tilde{g} defined by (2.2) is Hermitian if the following is satisfied:

$$\tilde{g}(H\tilde{X}, H\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}), \quad \text{where } H = F^{(K-1)/2}. \quad (2.13)$$

THEOREM 2.5. If F -structure manifold has the following two properties, that is,

- (a) \tilde{M} is an invariant submanifold of F -structure manifold M such that distribution D_m is never tangential to $\Psi(\tilde{M})$,
- (b) the Riemannian metric g on M is defined by (2.12).

Then the induced metric \tilde{g} of \tilde{M} defined by (2.2) is Hermitian.

PROOF. In view of (2.2) and (2.13) we obtain

$$\tilde{g}(\tilde{F}^{(K-1)/2}\tilde{X}, \tilde{F}^{(K-1)/2}\tilde{Y}) = g(B\tilde{F}^{(K-1)/2}\tilde{X}, B\tilde{F}^{(K-1)/2}\tilde{Y}). \quad (2.14)$$

Applying (2.4) and (2.12) in (2.14), we get

$$\begin{aligned} \tilde{g}(\tilde{F}^{(K-1)/2}\tilde{X}, \tilde{F}^{(K-1)/2}\tilde{Y}) &= g(F^{(K-1)/2}B\tilde{X}, F^{(K-1)/2}B\tilde{Y}) \\ &= g(B\tilde{X}, B\tilde{Y}) - g(mB\tilde{X}, B\tilde{Y}). \end{aligned} \quad (2.15)$$

Since the distribution D_m is never tangential to $\Psi(\tilde{M})$, on using (2.2) we get

$$\tilde{g}(\tilde{F}^{(K-1)/2}\tilde{X}, \tilde{F}^{(K-1)/2}\tilde{Y}) = g(B\tilde{X}, B\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}). \quad (2.16)$$

Now, we consider the second case and assume that the distribution D_m is always tangential to $\Psi(\tilde{M})$. In view of Case 2, we have $m(B\tilde{X}) = B\tilde{X}^*$, where $\tilde{X}^* \in T(\tilde{M})$ for some $\tilde{X}^* \in T(\tilde{M})$.

We define (1,1)-tensor fields \tilde{m} and $\tilde{\ell}$ in \tilde{M} as follows:

$$\tilde{\ell} = (-)^K \tilde{F}^{K-1}, \quad \tilde{m} = \tilde{I} + (-)^{K+1} \tilde{F}^{K-1}, \quad (2.17a)$$

$$\tilde{m}\tilde{X} = \tilde{X}^*, \quad m(B\tilde{X}) = B(\tilde{m}\tilde{X}). \quad (2.17b)$$

□

THEOREM 2.6. *We have*

$$B(\tilde{\ell}\tilde{X}) = \ell(B\tilde{X}). \quad (2.18)$$

PROOF. In view of (2.17a), equation (2.18) assumes the following form:

$$B(\tilde{\ell}\tilde{X}) = B((-)^K \tilde{F}^{K-1}\tilde{X}) = (-)^K B(\tilde{F}^{K-1}\tilde{X}). \quad (2.19)$$

Making use of (2.6) and (2.15) in (2.19), we get

$$B(\tilde{\ell}\tilde{X}) = (-)^K \tilde{F}^{K-1}(B\tilde{X}) = \tilde{\ell}(B\tilde{X}). \quad (2.20)$$

□

THEOREM 2.7. *For $\tilde{\ell}$ and \tilde{m} satisfying (2.17a), we have*

$$\tilde{\ell} + \tilde{m} = \tilde{I}, \quad \tilde{\ell}^2 = \tilde{\ell}, \quad \tilde{m}^2 = \tilde{m}. \quad (2.21)$$

PROOF. From (1.3) we have $\ell + m = I$, which can be written as $(\ell + m)B\tilde{X} = B\tilde{X}$, thus we have

$$\ell B\tilde{X} + mB\tilde{X} = B\tilde{X} \quad (2.22)$$

which in view of (2.17b) and (2.18) becomes

$$B(\tilde{\ell}\tilde{X}) + B(\tilde{m}\tilde{X}) = B(\tilde{\ell} + \tilde{m})\tilde{X} = B\tilde{X}. \quad (2.23)$$

Therefore $\tilde{\ell} + \tilde{m} = \tilde{I}$ since B is an isomorphism. Proof of the other relations follows in a similar manner. □

Theorem 2.7 shows that $\tilde{\ell}$ and \tilde{m} defined by (2.17a) are complementary projection operators on \tilde{M} .

THEOREM 2.8. *If F -structure manifold has the following property, that is, \tilde{M} is an invariant submanifold of F -structure manifold M such that distribution D_m is always tangential to $\Psi(\tilde{M})$. Then there exists an induced \tilde{F} -structure manifold which admits a similar Riemannian metric \tilde{g} satisfying*

$$\tilde{g}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{H}\tilde{X}, \tilde{H}\tilde{Y}) + \tilde{g}(\tilde{m}\tilde{X}\tilde{Y}). \quad (2.24)$$

PROOF. From (2.4b) we get

$$B(\tilde{F}^{(K-1)/2}\tilde{X}) = F^{(K-1)/2}(B\tilde{X}). \quad (2.25)$$

Furthermore,

$$B(\tilde{F}^K\tilde{X}) = F^K(B\tilde{X}) \quad (2.26)$$

which in view of (1.1) and (2.4a) yields

$$B(\tilde{F}^K\tilde{X}) = B(-(-)^{K+1}\tilde{F}\tilde{X}) \quad (2.27)$$

which shows that \tilde{F} defines an \tilde{F} -structure manifold which satisfies

$$\tilde{F}^K + (-)^{K+1}\tilde{F} = 0. \quad (2.28)$$

In consequence of (2.2), (2.4b), and (2.12) we obtain

$$\begin{aligned} \tilde{g}(\tilde{H}\tilde{X}, \tilde{H}\tilde{Y}) + \tilde{g}(\tilde{m}\tilde{X}, \tilde{Y}) &= g(B\tilde{H}\tilde{X}, B\tilde{H}\tilde{Y}) + g(B\tilde{m}\tilde{X}, B\tilde{Y}) \\ &= g(HB\tilde{X}, HB\tilde{Y}) + g(mB\tilde{X}, B\tilde{Y}) \\ &= g(B\tilde{X}, B\tilde{Y}), \quad \text{where } \tilde{H} = \tilde{F}^{(K-1)/2} \end{aligned} \quad (2.29)$$

which in view of the fact that B is an isomorphism gives

$$\tilde{g}(\tilde{H}\tilde{X}, \tilde{H}\tilde{Y}) + \tilde{g}(\tilde{m}\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}). \quad (2.30)$$

□

3. Integrability conditions. The Nijenhuis tensor N of the type (1.2) of F satisfying (1.1) in M is given by (see [2])

$$N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y], \quad (3.1)$$

and the Nijenhuis tensor \tilde{N} of \tilde{F} satisfying (2.28) in \tilde{M} is given by

$$N(\tilde{X}, \tilde{Y}) = [\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}] - \tilde{F}[\tilde{F}\tilde{X}, \tilde{Y}] - \tilde{F}[\tilde{X}, \tilde{F}\tilde{Y}] + \tilde{F}^2[\tilde{X}, \tilde{Y}]. \quad (3.2)$$

THEOREM 3.1. *The Nijenhuis tensors N and \tilde{N} of M and \tilde{M} given by (3.1) and (3.2) satisfy the following relation:*

$$N(B\tilde{X}, B\tilde{Y}) = B\tilde{N}(\tilde{X}, \tilde{Y}). \quad (3.3)$$

PROOF. We have

$$N(B\tilde{X}, B\tilde{Y}) = [F(B\tilde{X}), F(B\tilde{Y})] - F[F(B\tilde{X}), B\tilde{Y}] - F[B\tilde{X}, F(B\tilde{Y})] + F^2[B\tilde{X}, B\tilde{Y}] \quad (3.4)$$

which in view of (2.4a) becomes

$$\begin{aligned}
 N(B\tilde{X}, B\tilde{Y}) &= B[\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}] - F[B(\tilde{F}\tilde{X}), B\tilde{Y}] - F[(B\tilde{X}, B\tilde{F}\tilde{Y})] + F^2[B\tilde{X}, B\tilde{Y}] \\
 &= B[\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}] - FB[\tilde{F}\tilde{X}, \tilde{Y}] - FB[\tilde{X}, \tilde{F}\tilde{Y}] + BF^2[\tilde{X}, \tilde{Y}] \\
 &= B[\tilde{F}\tilde{X}, \tilde{F}\tilde{Y}] - B\tilde{F}[\tilde{F}, \tilde{X}, \tilde{Y}] - B\tilde{F}[\tilde{X}, \tilde{F}\tilde{Y}] + B\tilde{F}^2[\tilde{X}, \tilde{Y}] = B\tilde{N}(\tilde{X}, \tilde{Y}). \quad \square
 \end{aligned} \tag{3.5}$$

THEOREM 3.2. *The following identities hold:*

$$\begin{aligned}
 B\tilde{N}(\tilde{\ell}\tilde{X}, \tilde{\ell}\tilde{Y}) &= N(\tilde{\ell}B\tilde{X}, \tilde{\ell}B\tilde{Y}), \quad B\tilde{N}(\tilde{m}\tilde{X}, \tilde{m}\tilde{Y}) = N(\tilde{m}B\tilde{X}, \tilde{m}B\tilde{Y}), \\
 B\{\tilde{m}\tilde{n}(\tilde{X}, \tilde{Y})\} &= mN(B\tilde{X}, B\tilde{Y}). \tag{3.6}
 \end{aligned}$$

PROOF. The proof of (3.6) follows by virtue of Theorem 3.1, equations (1.4a), (2.4a), (2.17a), (2.17b), and (3.3). \square

For \tilde{F} satisfying (2.28), there exists complementary distribution $D_{\tilde{\ell}}$ and $D_{\tilde{m}}$ corresponding to the projection operators $\tilde{\ell}$ and \tilde{m} in \tilde{M} given by (2.17a). Then in view of the integrability conditions of \tilde{F} structure we state the following theorems.

THEOREM 3.3. *If D_{ℓ} is integrable in M , then $D_{\tilde{\ell}}$ is also integrable in \tilde{M} . If D_m is integrable in M , then $D_{\tilde{m}}$ is also integrable in \tilde{M} .*

THEOREM 3.4. *If D_{ℓ} and D_m are both integrable in M , then $D_{\tilde{\ell}}$ and $D_{\tilde{m}}$ are also integrable in \tilde{M} .*

THEOREM 3.5. *If F -structure is integrable in M , then the induced structure \tilde{F} is also integrable in \tilde{M} .*

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