

ON n -FOLD IMPLICATIVE FILTERS OF LATTICE IMPLICATION ALGEBRAS

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ABSTRACT. We introduce the notion of n -fold implicative filters and n -fold implicative lattice implication algebras. We give characterizations of n -fold implicative filters and n -fold implicative lattice implication algebras. Finally, we construct an extension property for n -fold implicative filter.

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1. Introduction. In order to research the logical system whose propositional value is given in a lattice, Xu [2] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [3] introduced the notions of filter and implicative filter in a lattice implication algebra, and investigated their properties. The author of this paper [1] gave an equivalent condition of a filter, and provided some equivalent conditions for a filter to be an implicative filter in a lattice implication algebra. In this paper, we discuss the foldness of implicative filters in lattice implication algebras.

2. Preliminaries

DEFINITION 2.1 (see [2]). By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution “ \prime ” and a binary operation “ \rightarrow ” satisfying the following axioms:

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z), \quad (2.1)$$

$$x \rightarrow x = 1, \quad (2.2)$$

$$x \rightarrow y = y' \rightarrow x', \quad (2.3)$$

$$x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y, \quad (2.4)$$

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x, \quad (2.5)$$

$$(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z), \quad (2.6)$$

$$(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z), \quad (2.7)$$

for all $x, y, z \in L$.

EXAMPLE 2.2 (see [3]). Let $L := \{0, a, b, c, 1\}$. Define the partially-ordered relation on L as $0 < a < b < c < 1$, and define

$$x \wedge y := \min\{x, y\}, \quad x \vee y := \max\{x, y\}, \quad (2.8)$$

TABLE 2.1.

x	x'	\rightarrow	0	a	b	c	1
0	1	0	1	1	1	1	1
a	c	a	c	1	1	1	1
b	b	b	b	c	1	1	1
c	a	c	a	a	c	1	1
1	0	1	0	a	b	c	1

for all $x, y \in L$ and “ \prime ” and “ \rightarrow ” as in Table 2.1. Then $(L, \vee, \wedge, \prime, \rightarrow)$ is a lattice implication algebra.

In what follows, the binary operation “ \rightarrow ” will be denoted by juxtaposition. We can define a partial ordering “ \leq ” on a lattice implication algebra L by $x \leq y$ if and only if $xy = 1$.

In a lattice implication algebra L , the following hold (see [2]):

$$0x = 1, \quad 1x = x, \quad x1 = 1, \quad (2.9)$$

$$xy \leq (yz)(xz), \quad (2.10)$$

$$x \leq y \text{ implies } yz \leq xz, \quad zx \leq zy. \quad (2.11)$$

In what follows, L will denote a lattice implication algebra, unless otherwise specified.

DEFINITION 2.3 (see [3]). A subset F of L is called a *filter* of L if it satisfies for all $x, y \in L$ the following:

$$1 \in F, \quad (2.12)$$

$$x \in F, \quad xy \in F \text{ imply } y \in F. \quad (2.13)$$

DEFINITION 2.4 (see [3]). A subset F of L is called an *implicative filter* of L if it satisfies (2.12) and

$$x(yz) \in F, \quad xy \in F \text{ imply } xz \in F, \quad \forall x, y, z \in L. \quad (2.14)$$

PROPOSITION 2.5 (see [1, Proposition 3.2]). *Every filter F of L has the property*

$$x \leq y, \quad x \in F \text{ imply } y \in F. \quad (2.15)$$

3. n -fold implicative filters. For any elements x and y of L and any positive integer n , let $x^n y$ denote $x(\dots(x(xy))\dots)$ in which x occurs n times, and $x^0 y = y$.

DEFINITION 3.1. Let n be a positive integer. A subset F of L is called an n -fold implicative filter of L , if it satisfies (2.12) and

$$x^n(yz) \in F, \quad x^n y \in F \text{ imply } x^n z \in F, \quad \forall x, y, z \in L. \quad (3.1)$$

Note that the 1-fold implicative filter is an implicative filter.

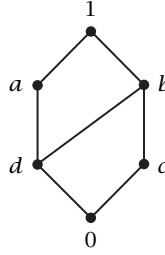


FIGURE 3.1.

TABLE 3.1.

x	x'
0	1
a	c
b	d
c	a
d	b
1	0

(a)

	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

(b)

EXAMPLE 3.2. Let $L := \{0, a, b, c, d, 1\}$ be a set with Figure 3.1 as a partial ordering. Define a unary operation “ \prime ” and a binary operation denoted by juxtaposition on L as in Tables 3.1a and 3.1b, respectively.

Define \vee - and \wedge -operations on L as follows:

$$x \vee y := (xy)y, \quad x \wedge y := ((x'y')y')', \quad \forall x, y \in L. \quad (3.2)$$

Then L is a lattice implication algebra. It is easy to check that $F := \{b, c, 1\}$ is an n -fold implicative filter of L .

THEOREM 3.3. *Every n -fold implicative filter of L is a filter of L .*

PROOF. Let F be an n -fold implicative filter of L . Taking $x = 1$ in (3.1) and using (2.9), we conclude that $yz \in F$ and $y \in F$ imply $z \in F$, that is, (2.13) holds. Hence F is a filter of L . \square

The converse of Theorem 3.3 is not true. For example, let L be a lattice implication algebra in Example 3.2. Then $\{1\}$ is a filter of L , but $\{1\}$ is not a 1-fold implicative filter of L because $d^1(bc) = db = 1$ and $d^1b = 1$, but $d^1c = b \neq 1$.

We give conditions for a filter to be an n -fold implicative filter.

THEOREM 3.4. *Let F be a filter of L . Then the following statements are equivalent:*

- (i) F is an n -fold implicative filter of L .
- (ii) $x^{n+1}y \in F$ implies $x^n y \in F$.
- (iii) $x^n(yz) \in F$ implies $(x^n y)(x^n z) \in F$.

PROOF. (i) \Rightarrow (ii). Assume that F is an n -fold implicative filter of L and let $x, y \in L$ be such that $x^{n+1}y \in F$. Then $x^n(xy) \in F$, and since $x^n x = 1 \in F$, it follows from (3.1) that $x^n y \in F$.

(ii) \Rightarrow (iii). Suppose (ii) holds and let $x, y, z \in L$ be such that $x^n(yz) \in F$. Since $x^n(yz) \leq x^n((x^n y)(x^n z))$, we have

$$x^{n+1}(x^{n-1}((x^n y)z)) = x^n(x^n((x^n y)z)) = x^n((x^n y)(x^n z)) \in F. \quad (3.3)$$

It follows from (ii) that $x^{n+1}(x^{n-2}((x^n y)z)) = x^n(x^{n-1}((x^n y)z)) \in F$. Using (ii) again, we get

$$x^{n+1}(x^{n-3}((x^n y)z)) = x^n(x^{n-2}((x^n y)z)) \in F. \quad (3.4)$$

Repeating this process, we conclude that $(x^n y)(x^n z) = x^n((x^n y)z) \in F$.

(iii) \Rightarrow (i). Let $x, y, z \in L$ be such that $x^n(yz) \in F$ and $x^n y \in F$. It follows from (iii) that $(x^n y)(x^n z) \in F$ and $x^n y \in F$, so from (2.13), we have $x^n z \in F$. Hence F is an n -fold implicative filter of L . \square

DEFINITION 3.5. Let n be a positive integer. A lattice implication algebra L is said to be n -fold implicative if it satisfies the equality $x^{n+1}y = x^n y$ for all $x, y \in L$.

COROLLARY 3.6. In an n -fold implicative lattice implication algebra, the notion of filters and n -fold implicative filters coincide.

We give a characterization of an n -fold implicative lattice implication algebra.

THEOREM 3.7. A lattice implication algebra L is n -fold implicative if and only if the filter $\{1\}$ of L is n -fold implicative.

PROOF. Necessity is by Corollary 3.6. Assume that the filter $\{1\}$ of L is n -fold implicative. Noticing that $x^n((xy)y) = 1$, and applying Theorem 3.4, we have

$$(x^{n+1}y)(x^n y) = (x^n(xy))(x^n y) = 1. \quad (3.5)$$

On the other hand, it is clear that $(x^n y)(x^{n+1}y) = 1$. Hence $x^{n+1}y = x^n y$, as desired. \square

The following is a characterization of an n -fold implicative filter.

THEOREM 3.8. A nonempty subset F of L is an n -fold implicative filter of L if and only if it satisfies (2.12) and

$$x(y^{n+1}z) \in F, \quad x \in F \text{ imply } y^n z \in F, \quad \forall x, y, z \in L. \quad (3.6)$$

PROOF. Suppose that F is an n -fold implicative filter of L and let $x, y, z \in L$ be such that $x(y^{n+1}z) \in F$ and $x \in F$. Since F is a filter of L (see Theorem 3.3), it follows that $y^{n+1}z \in F$. Using Theorem 3.4, we know that $y^n z \in F$.

Conversely, assume that F satisfies (2.12) and (3.6). Let $x, y \in L$ be such that $xy \in F$ and $x \in F$. Then $x(1^{n+1}y) = xy \in F$ and $x \in F$. Thus, by (3.6), we have $y = 1^n y \in F$. Hence F is a filter of L . Now, if $x^{n+1}y \in F$ for all $x, y \in L$, then $1(x^{n+1}y) = x^{n+1}y \in F$ and $1 \in F$. It follows from (3.6) that $x^n y \in F$. Hence F is an n -fold implicative filter of L by Theorem 3.4. This completes the proof. \square

THEOREM 3.9 (extension property for n -fold implicative filters). *Let F and G be filters of L such that $F \subseteq G$. If F is n -fold implicative, then so is G .*

PROOF. Let $x, y \in L$ be such that $x^{n+1}y \in G$. Since $x^{n+1}((x^{n+1}y)y) = 1 \in F$, it follows from (2.1) and Theorem 3.4(ii) that

$$(x^{n+1}y)(x^n y) = x^n((x^{n+1}y)y) \in F \subseteq G, \quad (3.7)$$

so that $x^n y \in G$ since G is a filter. Using Theorem 3.4, we conclude that G is an n -fold implicative filter of L . \square

Using Theorems 3.7 and 3.9, we have the following theorem.

THEOREM 3.10. *A lattice implication algebra is n -fold implicative if and only if every filter is n -fold implicative.*

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