

# ON $n$ -FOLD IMPLICATIVE FILTERS OF LATTICE IMPLICATION ALGEBRAS

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**ABSTRACT.** We introduce the notion of  $n$ -fold implicative filters and  $n$ -fold implicative lattice implication algebras. We give characterizations of  $n$ -fold implicative filters and  $n$ -fold implicative lattice implication algebras. Finally, we construct an extension property for  $n$ -fold implicative filter.

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**1. Introduction.** In order to research the logical system whose propositional value is given in a lattice, Xu [2] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [3] introduced the notions of filter and implicative filter in a lattice implication algebra, and investigated their properties. The author of this paper [1] gave an equivalent condition of a filter, and provided some equivalent conditions for a filter to be an implicative filter in a lattice implication algebra. In this paper, we discuss the foldness of implicative filters in lattice implication algebras.

## 2. Preliminaries

**DEFINITION 2.1** (see [2]). By a *lattice implication algebra* we mean a bounded lattice  $(L, \vee, \wedge, 0, 1)$  with order-reversing involution “ $\prime$ ” and a binary operation “ $\rightarrow$ ” satisfying the following axioms:

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z), \quad (2.1)$$

$$x \rightarrow x = 1, \quad (2.2)$$

$$x \rightarrow y = y' \rightarrow x', \quad (2.3)$$

$$x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y, \quad (2.4)$$

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x, \quad (2.5)$$

$$(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z), \quad (2.6)$$

$$(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z), \quad (2.7)$$

for all  $x, y, z \in L$ .

**EXAMPLE 2.2** (see [3]). Let  $L := \{0, a, b, c, 1\}$ . Define the partially-ordered relation on  $L$  as  $0 < a < b < c < 1$ , and define

$$x \wedge y := \min\{x, y\}, \quad x \vee y := \max\{x, y\}, \quad (2.8)$$

TABLE 2.1.

$x$	$x'$	$\rightarrow$	0	$a$	$b$	$c$	1
0	1	0	1	1	1	1	1
$a$	$c$	$a$	$c$	1	1	1	1
$b$	$b$	$b$	$b$	$c$	1	1	1
$c$	$a$	$c$	$a$	$a$	$c$	1	1
1	0	1	0	$a$	$b$	$c$	1

for all  $x, y \in L$  and “ $\vee$ ” and “ $\rightarrow$ ” as in Table 2.1. Then  $(L, \vee, \wedge, ', \rightarrow)$  is a lattice implication algebra.

In what follows, the binary operation “ $\rightarrow$ ” will be denoted by juxtaposition. We can define a partial ordering “ $\leq$ ” on a lattice implication algebra  $L$  by  $x \leq y$  if and only if  $xy = 1$ .

In a lattice implication algebra  $L$ , the following hold (see [2]):

$$0x = 1, \quad 1x = x, \quad x1 = 1, \quad (2.9)$$

$$xy \leq (yz)(xz), \quad (2.10)$$

$$x \leq y \text{ implies } yz \leq xz, \quad zx \leq zy. \quad (2.11)$$

In what follows,  $L$  will denote a lattice implication algebra, unless otherwise specified.

**DEFINITION 2.3** (see [3]). A subset  $F$  of  $L$  is called a *filter* of  $L$  if it satisfies for all  $x, y \in L$  the following:

$$1 \in F, \quad (2.12)$$

$$x \in F, \quad xy \in F \text{ imply } y \in F. \quad (2.13)$$

**DEFINITION 2.4** (see [3]). A subset  $F$  of  $L$  is called an *implicative filter* of  $L$  if it satisfies (2.12) and

$$x(yz) \in F, \quad xy \in F \text{ imply } xz \in F, \quad \forall x, y, z \in L. \quad (2.14)$$

**PROPOSITION 2.5** (see [1, Proposition 3.2]). Every filter  $F$  of  $L$  has the property

$$x \leq y, \quad x \in F \text{ imply } y \in F. \quad (2.15)$$

**3.  $n$ -fold implicative filters.** For any elements  $x$  and  $y$  of  $L$  and any positive integer  $n$ , let  $x^n y$  denote  $x(\cdots(x(xy))\cdots)$  in which  $x$  occurs  $n$  times, and  $x^0 y = y$ .

**DEFINITION 3.1.** Let  $n$  be a positive integer. A subset  $F$  of  $L$  is called an  *$n$ -fold implicative filter* of  $L$ , if it satisfies (2.12) and

$$x^n(yz) \in F, \quad x^n y \in F \text{ imply } x^n z \in F, \quad \forall x, y, z \in L. \quad (3.1)$$

Note that the 1-fold implicative filter is an implicative filter.

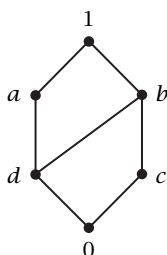


FIGURE 3.1.

TABLE 3.1.

$x$	$x'$
0	1
$a$	$c$
$b$	$d$
$c$	$a$
$d$	$b$
1	0

(a)

	0	$a$	$b$	$c$	$d$	1
0	1	1	1	1	1	1
$a$	$c$	1	$b$	$c$	$b$	1
$b$	$d$	$a$	1	$b$	$a$	1
$c$	$a$	$a$	1	1	$a$	1
$d$	$b$	1	1	$b$	1	1
1	0	$a$	$b$	$c$	$d$	1

(b)

**EXAMPLE 3.2.** Let  $L := \{0, a, b, c, d, 1\}$  be a set with Figure 3.1 as a partial ordering. Define a unary operation “ $\prime$ ” and a binary operation denoted by juxtaposition on  $L$  as in Tables 3.1a and 3.1b, respectively.

Define  $\vee$ - and  $\wedge$ -operations on  $L$  as follows:

$$x \vee y := (xy)y, \quad x \wedge y := ((x'y')y')', \quad \forall x, y \in L. \quad (3.2)$$

Then  $L$  is a lattice implication algebra. It is easy to check that  $F := \{b, c, 1\}$  is an  $n$ -fold implicative filter of  $L$ .

**THEOREM 3.3.** Every  $n$ -fold implicative filter of  $L$  is a filter of  $L$ .

**PROOF.** Let  $F$  be an  $n$ -fold implicative filter of  $L$ . Taking  $x = 1$  in (3.1) and using (2.9), we conclude that  $yz \in F$  and  $y \in F$  imply  $z \in F$ , that is, (2.13) holds. Hence  $F$  is a filter of  $L$ .  $\square$

The converse of Theorem 3.3 is not true. For example, let  $L$  be a lattice implication algebra in Example 3.2. Then  $\{1\}$  is a filter of  $L$ , but  $\{1\}$  is not a 1-fold implicative filter of  $L$  because  $d^1(bc) = db = 1$  and  $d^1b = 1$ , but  $d^1c = b \neq 1$ .

We give conditions for a filter to be an  $n$ -fold implicative filter.

**THEOREM 3.4.** Let  $F$  be a filter of  $L$ . Then the following statements are equivalent:

- (i)  $F$  is an  $n$ -fold implicative filter of  $L$ .
- (ii)  $x^{n+1}y \in F$  implies  $x^n y \in F$ .
- (iii)  $x^n(yz) \in F$  implies  $(x^n y)(x^n z) \in F$ .

**PROOF.** (i) $\Rightarrow$ (ii). Assume that  $F$  is an  $n$ -fold implicative filter of  $L$  and let  $x, y \in L$  be such that  $x^{n+1}y \in F$ . Then  $x^n(xy) \in F$ , and since  $x^n x = 1 \in F$ , it follows from (3.1) that  $x^n y \in F$ .

(ii) $\Rightarrow$ (iii). Suppose (ii) holds and let  $x, y, z \in L$  be such that  $x^n(yz) \in F$ . Since  $x^n(yz) \leq x^n((x^n y)(x^n z))$ , we have

$$x^{n+1}(x^{n-1}((x^n y)z)) = x^n(x^n((x^n y)z)) = x^n((x^n y)(x^n z)) \in F. \quad (3.3)$$

It follows from (ii) that  $x^{n+1}(x^{n-2}((x^n y)z)) = x^n(x^{n-1}((x^n y)z)) \in F$ . Using (ii) again, we get

$$x^{n+1}(x^{n-3}((x^n y)z)) = x^n(x^{n-2}((x^n y)z)) \in F. \quad (3.4)$$

Repeating this process, we conclude that  $(x^n y)(x^n z) = x^n((x^n y)z) \in F$ .

(iii) $\Rightarrow$ (i). Let  $x, y, z \in L$  be such that  $x^n(yz) \in F$  and  $x^n y \in F$ . It follows from (iii) that  $(x^n y)(x^n z) \in F$  and  $x^n y \in F$ , so from (2.13), we have  $x^n z \in F$ . Hence  $F$  is an  $n$ -fold implicative filter of  $L$ .  $\square$

**DEFINITION 3.5.** Let  $n$  be a positive integer. A lattice implication algebra  $L$  is said to be  $n$ -fold implicative if it satisfies the equality  $x^{n+1}y = x^n y$  for all  $x, y \in L$ .

**COROLLARY 3.6.** In an  $n$ -fold implicative lattice implication algebra, the notion of filters and  $n$ -fold implicative filters coincide.

We give a characterization of an  $n$ -fold implicative lattice implication algebra.

**THEOREM 3.7.** A lattice implication algebra  $L$  is  $n$ -fold implicative if and only if the filter  $\{1\}$  of  $L$  is  $n$ -fold implicative.

**PROOF.** Necessity is by Corollary 3.6. Assume that the filter  $\{1\}$  of  $L$  is  $n$ -fold implicative. Noticing that  $x^n((xy)y) = 1$ , and applying Theorem 3.4, we have

$$(x^{n+1}y)(x^n y) = (x^n(xy)) (x^n y) = 1. \quad (3.5)$$

On the other hand, it is clear that  $(x^n y)(x^{n+1}y) = 1$ . Hence  $x^{n+1}y = x^n y$ , as desired.  $\square$

The following is a characterization of an  $n$ -fold implicative filter.

**THEOREM 3.8.** A nonempty subset  $F$  of  $L$  is an  $n$ -fold implicative filter of  $L$  if and only if it satisfies (2.12) and

$$x(y^{n+1}z) \in F, x \in F \text{ imply } y^n z \in F, \quad \forall x, y, z \in L. \quad (3.6)$$

**PROOF.** Suppose that  $F$  is an  $n$ -fold implicative filter of  $L$  and let  $x, y, z \in L$  be such that  $x(y^{n+1}z) \in F$  and  $x \in F$ . Since  $F$  is a filter of  $L$  (see Theorem 3.3), it follows that  $y^{n+1}z \in F$ . Using Theorem 3.4, we know that  $y^n z \in F$ .

Conversely, assume that  $F$  satisfies (2.12) and (3.6). Let  $x, y \in L$  be such that  $xy \in F$  and  $x \in F$ . Then  $x(1^{n+1}y) = xy \in F$  and  $x \in F$ . Thus, by (3.6), we have  $y = 1^n y \in F$ . Hence  $F$  is a filter of  $L$ . Now, if  $x^{n+1}y \in F$  for all  $x, y \in L$ , then  $1(x^{n+1}y) = x^{n+1}y \in F$  and  $1 \in F$ . It follows from (3.6) that  $x^n y \in F$ . Hence  $F$  is an  $n$ -fold implicative filter of  $L$  by Theorem 3.4. This completes the proof.  $\square$

**THEOREM 3.9** (extension property for  $n$ -fold implicative filters). *Let  $F$  and  $G$  be filters of  $L$  such that  $F \subseteq G$ . If  $F$  is  $n$ -fold implicative, then so is  $G$ .*

**PROOF.** Let  $x, y \in L$  be such that  $x^{n+1}y \in G$ . Since  $x^{n+1}((x^{n+1}y)y) = 1 \in F$ , it follows from (2.1) and Theorem 3.4(ii) that

$$(x^{n+1}y)(x^n y) = x^n((x^{n+1}y)y) \in F \subseteq G, \quad (3.7)$$

so that  $x^n y \in G$  since  $G$  is a filter. Using Theorem 3.4, we conclude that  $G$  is an  $n$ -fold implicative filter of  $L$ .  $\square$

Using Theorems 3.7 and 3.9, we have the following theorem.

**THEOREM 3.10.** *A lattice implication algebra is  $n$ -fold implicative if and only if every filter is  $n$ -fold implicative.*

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