IJMMS 26:11 (2001) 671–678
PII. S0161171201010730
http://ijmms.hindawi.com

ON MATRIX TRANSFORMATIONS CONCERNING THE NAKANO VECTOR-VALUED SEQUENCE SPACE

SUTHEP SUANTAI

(Received 15 August 2000 and in revised form 20 October 2000)

ABSTRACT. We give the matrix characterizations from Nakano vector-valued sequence space $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces E_r, ℓ_∞, $\underline{\ell}_\infty(q)$, bs, and cs, where $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers such that $p_k>1$ for all $k\in\mathbb{N}$ and $r\geq 0$.

2000 Mathematics Subject Classification. 46A45.

1. Introduction. Let $(X,\|\cdot\|)$ be a Banach space, $r\geq 0$ and $p=(p_k)$ a bounded sequence of positive real numbers. We write $x=(x_k)$ with x_k in X for all $k\in\mathbb{N}$. The X-valued sequence spaces $c_0(X,p)$, $c(X,p)$, $\ell_\infty(X,p)$, $\ell(X,p)$, $E_r(X,p)$, $F_r(X,p)$, and $\underline{\ell}_\infty(X,p)$ are defined as

$$\begin{aligned}
c_0(X,p)&=\Big\{x=(x_k):\lim_{k\to\infty}\|x_k\|^{p_k}=0\Big\},\\
c(X,p)&=\Big\{x=(x_k):\lim_{k\to\infty}\|x_k-a\|^{p_k}=0,\ \text{for some } a\in X\Big\},\\
\ell_\infty(X,p)&=\Big\{x=(x_k):\sup_k\|x_k\|^{p_k}<\infty\Big\},\\
\ell(X,p)&=\Big\{x=(x_k):\sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty\Big\},\\
E_r(X,p)&=\Big\{x=(x_k):\sup_k\frac{\|x_k\|^{p_k}}{k^r}<\infty\Big\},\\
F_r(X,p)&=\Big\{x=(x_k):\sum_{k=1}^{\infty}k^r\|x_k\|^{p_k}<\infty\Big\},\\
\underline{\ell}_\infty(X,p)&=\bigcap_{n=1}^{\infty}\Big\{x=(x_k):\sup_k\|x_k\|n^{1/p_k}\Big\}.
\end{aligned}\tag{1.1}$$

When $X=K$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, $\ell(p)$, $E_r(p)$, $F_r(p)$, and $\underline{\ell}_\infty(p)$, respectively. The spaces $c_0(p)$, $c(p)$, and $\ell_\infty(p)$ are known as the sequence spaces of Maddox. These spaces were first introduced and studied by Simons [7] and Maddox [4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and it is known as the Nakano sequence space and the space $\ell(X,p)$ is known as the Nakano vector-valued sequence space. When $p_k=1$ for all $k\in\mathbb{N}$, the spaces $E_r(p)$ and $F_r(p)$ are written as E_r and F_r, respectively. These two sequence spaces were first introduced by Cooke [1]. The space $\underline{\ell}_\infty(p)$ was first defined by Grosse-Erdmann [2] and he has given in [3] characterizations of infinite matrices mapping between scalar-valued sequence spaces of Maddox. Wu and Liu [10] gave necessary and sufficient conditions for infinite matrices mapping from $c_0(X,p)$ and $\ell_\infty(X,p)$ into $c_0(q)$ and $\ell_\infty(q)$. Suantai [8] has given characterizations of infinite matrices mapping $\ell(X,p)$ into ℓ_∞ and $\underline{\ell}_\infty(q)$ when $p_k \le 1$ for all $k \in \mathbb{N}$ and he has also given in [9] characterizations of those infinite matrices mapping from $\ell(X,p)$ into the sequence space E_r when $p_k \le 1$ for all $k \in \mathbb{N}$.

In this paper, we extend the results of [8, 9] in case $p_k > 1$ for all $k \in \mathbb{N}$. Moreover, we also give the matrix characterizations from $\ell(X,p)$ and $F_r(X,p)$ into the sequence spaces bs and cs.

2. Notations and definitions. Let $(X,\|\cdot\|)$ be a Banach space, the space of all sequences in X is denoted by $W(X)$, and $\Phi(X)$ denotes the space of all finite sequences in X. When $X = K$, the scalar field of X, the corresponding spaces are written as w and Φ.

A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in \mathbb{N}$, x_k stands for the kth term of x. For $k \in \mathbb{N}$, we denote by e_k the sequence $(0,0,\dots,0,1,0,\dots)$ with 1 in the kth position and by e the sequence $(1,1,1,\dots)$. For $x \in X$ and $k \in \mathbb{N}$, let $e^k(x)$ be the sequence $(0,0,\dots,0,x,0,\dots)$ with x in the kth position and let $e(x)$ be the sequence $(x,x,x,\dots)$. We call a sequence space E normal if $(t_kx_k) \in E$ for all $x = (x_k) \in E$ and $t_k \in K$ with $|t_k| = 1$ for all $t_k \in \mathbb{N}$. A normed sequence space $(E,\|\cdot\|)$ is said to be *norm monotone* if $x = (x_k)$, $y = (y_k) \in E$ with $\|x_k\| \le \|y_k\|$ for all $k \in \mathbb{N}$ we have $\|x\| \le \|y\|$. For a fixed scalar sequence $\mu = (\mu_k)$, the sequence space E_μ is defined as

$$E_\mu = \{x \in W(X) : (\mu_k x_k) \in E\}. \tag{2.1}$$

Let $A = (f_k^n)$ with f_k^n in X', the topological dual of X. Suppose that E is a space of X-valued sequences and F a space of scalar-valued sequences. Then A is said to *map* E into F, written by $A : E \to F$, if for each $x = (x_k) \in E$, $A_n(x) = \sum_{k=1}^\infty f_k^n(x_k)$ converges for each $n \in \mathbb{N}$, and the sequence $Ax = (A_n(x)) \in F$. Let (E,F) denote the set of all infinite matrices mapping from E into F.

Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a K-space if for each $k \in \mathbb{N}$, the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. If, in addition, (E,τ) is a Fréchet (Banach) space, then E is called an FK- (BK-) space. Now, suppose that E contains $\Phi(X)$. Then E is said to have *property* AB if the set $\{\sum_{k=1}^n e^k(x_k) : n \in \mathbb{N}\}$ is bounded in E for every $x = (x_k) \in E$. It is said to have *property* AK if $\sum_{k=1}^n e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = (x_k) \in E$. It has *property* AD if $\Phi(X)$ is dense in E.

It is known that the Nakano sequence space $\ell(X,p)$ is an FK-space with property AK under the paranorm $g(x) = (\sum_{k=1}^\infty \|x_k\|^{p_k})^{1/M}$, where $M = \max\{1, \sup_k p_k\}$. If $p_k > 1$ for all $k \in \mathbb{N}$, then $\ell(X,p)$ is a BK-space with the Luxemburg norm defined by

$$\|(x_k)\| = \inf\left\{\varepsilon > 0 : \sum_{k=1}^\infty \left\|\frac{x_k}{\varepsilon}\right\|^{p_k} \le 1\right\}. \tag{2.2}$$

3. Main results. We first give a characterization of an infinite matrix mapping from $\ell(X,p)$ into E_r when $p_k > 1$ for all $k \in \mathbb{N}$. To do this, we need the following lemma.

Lemma 3.1. *Let E be an X-valued BK-space which is normal and norm monotone and let $A = (f_k^n)$ be an infinite matrix. Then $A : E \to E_r$ if and only if $\sup_n \sum_{k=1}^{\infty} |f_k^n(x_k)|/n^r < \infty$ for every $x = (x_k) \in E$.*

Proof. If the condition holds true, it follows that

$$\sup_n \frac{\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|}{n^r} \le \sup_n \sum_{k=1}^{\infty} \frac{\left|f_k^n(x_k)\right|}{n^r} < \infty \tag{3.1}$$

for every $x = (x_k) \in E$, hence $A : E \to E_r$.

Conversely, assume that $A : E \to E_r$. Since E and E_r are BK-spaces, by Zeller's theorem, $A : E \to E_r$ is bounded, so there exists $M > 0$ such that

$$\sup_{\substack{n\in\mathbb{N} \\ \|(x_k)\|\le 1}} \frac{\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right|}{n^r} \le M. \tag{3.2}$$

Let $x = (x_k) \in E$ be such that $\|x\| = 1$. For each $n \in \mathbb{N}$, we can choose a scalar sequence (t_k) with $|t_k| = 1$ and $f_k^n(t_k x_k) = |f_k^n(x_k)|$ for all $k \in \mathbb{N}$. Since E is normal and norm monotone, we have $(t_k x_k) \in E$ and $\|(t_k x_k)\| \le 1$. It follows from (3.2) that

$$\sum_{k=1}^{\infty} \frac{\left|f_k^n(x_k)\right|}{n^r} = \frac{\left|\sum_{k=1}^{\infty} f_k^n(t_k x_k)\right|}{n^r} \le M, \tag{3.3}$$

which implies

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} \frac{\left|f_k^n(x_k)\right|}{n^r} \le M. \tag{3.4}$$

It follows from (3.4) that for every $x = (x_k) \in E$,

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} \frac{\left|f_k^n(x_k)\right|}{n^r} \le M\|x\|. \tag{3.5}$$

This completes the proof. □

Theorem 3.2. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$, and let $r \ge 0$. For an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p), E_r)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < \infty. \tag{3.6}$$

Proof. Let $x = (x_k) \in \ell(X,p)$. By (3.6), there are $m_0 \in \mathbb{N}$ and $K > 1$ such that

$$\sum_{k=1}^{\infty} \|f_k^n\|^{q_k} n^{-rq_k} m_0^{-q_k} < K, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Note that for $a, b \ge 0$, we have

$$ab \le a^{p_k} + b^{q_k}. \tag{3.8}$$

It follows by (3.7) and (3.8) that for $n \in \mathbb{N}$,

$$\begin{aligned} n^{-r}\left|\sum_{k=1}^{\infty} f_k^n(x_k)\right| &= n^{-r}\left|\sum_{k=1}^{\infty} f_k^n(m_0^{-1}\cdot m_0 x_k)\right| \\ &\le \sum_{k=1}^{\infty}\left(n^{-r}m_0^{-1}\|f_k^n\|\right)\left(\|m_0x_k\|\right) \\ &\le \sum_{k=1}^{\infty} n^{-rq_k}m_0^{-q_k}\|f_k^n\|^{q_k}+m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k} \\ &\le K+m_0^{\alpha}\sum_{k=1}^{\infty}\|x_k\|^{p_k}, \quad \text{where } \alpha=\sup_k p_k. \end{aligned} \tag{3.9}$$

Hence $\sup n^{-r}|\sum_{k=1}^{\infty} f_k^n(x_k)| < \infty$, so that $Ax \in E_r$.

For necessity, assume that $A \in (\ell(X,p), E_r)$. For each $k \in \mathbb{N}$, we have $\sup_n n^{-r}|f_k^n(x)| < \infty$ for all $x \in X$ since $e^{(k)}(x) \in \ell(X,p)$. It follows by the uniform bounded principle that for each $k \in \mathbb{N}$ there is $C_k > 1$ such that

$$\sup_n n^{-r}\|f_k^n\| \le C_k. \tag{3.10}$$

Suppose that (3.6) is not true. Then

$$\sup_n \sum_{k=1}^{\infty}\|f_k^n\|^{q_k} n^{-rq_k} m^{-q_k} = \infty, \quad \forall m \in \mathbb{N}. \tag{3.11}$$

For $n \in \mathbb{N}$, we have by (3.10) that for $k, m \in \mathbb{N}$,

$$\begin{aligned} \sum_{j=1}^{\infty}\left\|f_j^n\right\|^{q_j} n^{-rq_j}m^{-q_j} &= \sum_{j=1}^{k}\left\|f_j^n\right\|^{q_j} n^{-rq_j}m^{-q_j} + \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j}m^{-q_j} \\ &\le \sum_{j=1}^{k} C_j^{q_j}m^{-q_j} + \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j}m^{-q_j}. \end{aligned} \tag{3.12}$$

This together with (3.11) give

$$\sup_n \sum_{j>k}\left\|f_j^n\right\|^{q_j} n^{-rq_j}m^{-q_j} = \infty, \quad \forall k, m \in \mathbb{N}. \tag{3.13}$$

By (3.13) we can choose $0 = k_0 < k_1 < k_2 < \cdots$, $m_1 < m_2 < \cdots$, $m_i > 4^i$ and a subsequence (n_i) of positive integers such that for all $i \ge 1$,

$$\sum_{k_{i-1}<j\le k_i}\left\|f_j^{n_i}\right\|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i. \tag{3.14}$$

For each $i \in \mathbb{N}$, we can choose $x_j \in X$ with $\|x_j\| = 1$, for $k_{i-1} < j \le k_i$ such that

$$\sum_{k_{i-1}<j\le k_i}\left|f_j^{n_i}(x_j)\right|^{q_j} n_i^{-rq_j} m_i^{-q_j} > 2^i. \tag{3.15}$$

For each $i \in \mathbb{N}$, let $F_i : (0,\infty) \to (0,\infty)$ be defined by

$$F_i(M) = \sum_{k_{i-1}<j\le k_i} \left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M^{-q_j}. \tag{3.16}$$

Then F_i is continuous and non-increasing such that $F(M) \to 0$ as $M \to \infty$. Thus there exists $M_i > 0$ such that $M_i > m_i$ and

$$F(M_i) = \sum_{k_{i-1}<j\le k_i} \left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M_i^{-q_j} = 2^i. \tag{3.17}$$

Put

$$y = (y_j), \quad y_j = 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j/p_j} \left| f_j^{n_i}(x_j) \right|^{q_j-1} x_j \text{ for } k_{i-1} < j \le k_i. \tag{3.18}$$

Thus

$$\begin{aligned}\sum_{j=1}^{\infty} \|y_j\|^{p_j} &= \sum_{i=1}^{\infty} \sum_{k_{i-1}<j\le k_i} 4^{-ip_j} M_i^{-p_j(q_j-1)} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{p_j(q_j-1)} \\ &\le \sum_{i=1}^{\infty} 4^{-i} \sum_{k_{i-1}<j\le k_i} M_i^{-q_j} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{q_j} \\ &= \sum_{i=1}^{\infty} 4^{-i} \cdot 2^i \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i} = 1.\end{aligned} \tag{3.19}$$

Thus $y = (y_j) \in \ell(X,p)$. Since $\ell(X,p)$ is a BK-space which is normal and norm monotone under the Luxemburg norm, by Lemma 3.1, we obtain that

$$\sup_n \sum_{k=1}^{\infty} \frac{|f_k^n(y_k)|}{n^r} < \infty. \tag{3.20}$$

But we have

$$\begin{aligned}\sup_n \sum_{j=1}^{\infty} \frac{\left| f_j^n(y_j) \right|}{n^r} &\ge \sup_i \sum_{j=1}^{\infty} \frac{\left| f_j^{n_i}(y_j) \right|}{n_i^r} \ge \sup_i \sum_{k_{i-1}<j\le k_i} \frac{\left| f_j^{n_i}(y_j) \right|}{n_i^r} \\ &= \sup_i \sum_{k_{i-1}<j\le k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-r(q_j/p_j+1)} \left| f_j^{n_i}(x_j) \right|^{q_j} \\ &= \sup_i \sum_{k_{i-1}<j\le k_i} 4^{-i} M_i^{-(q_j-1)} n_i^{-rq_j} \left| f_j^{n_i}(x_j) \right|^{q_j} \\ &= \sup_i \sum_{k_{i-1}<j\le k_i} \left(\left| f_j^{n_i}(x_j) \right|^{q_j} n_i^{-rq_j} M_i^{-q_j} \right) 4^{-i} M_i \\ &\ge \sup_i 2^i = \infty, \quad \text{because } M_i > 4^i.\end{aligned} \tag{3.21}$$

This is contradictory with (3.20). Therefore (3.6) is satisfied. □

Theorem 3.3. *Let $p=(p_k)$ be a bounded sequence of positive real numbers such that $p_k>1$ for all $k\in\mathbb{N}$, $1/p_k+1/q_k=1$ for all $k\in\mathbb{N}$, $r\ge 0$ and $s\ge 0$. Then for an infinite matrix $A=(f_k^n)$, $A\in(F_r(X,p),E_s)$ if and only if there is $m_0\in\mathbb{N}$ such that*

$$\sup_n\sum_{k=1}^{\infty}\left(k^{-rq_k/p_k}\|f_k^n\|^{q_k}n^{-sq_k}m_0^{-q_k}\right)<\infty. \tag{3.22}$$

Proof. Since $F_r(X,p)=\ell(X,p)_{(k^{r/p_k})}$, it is easy to see that

$$A\in(F_r(X,p),E_s)\Longleftrightarrow(k^{-r/p_k}f_k^n)_{n,k}\in(\ell(X,p)E_s). \tag{3.23}$$

By Theorem 3.2, we have $(k^{-r/p_k}f_k^n)_{n,k}\in(\ell(X,p)E_s)$ if and only if there is $m_0\in\mathbb{N}$ such that

$$\sup_n\sum_{k=1}^{\infty}\left(k^{-rq_k/p_k}\|f_k^n\|^{q_k}n^{-sq_k}m_0^{-q_k}\right)<\infty. \tag{3.24}$$

Thus the theorem is proved. □

Since $E_0=\ell_\infty$, the following two results are obtained directly from Theorems 3.2 and 3.3, respectively.

Corollary 3.4. *Let $p=(p_k)$ be a bounded sequence of positive real numbers with $p_k>1$ for all $k\in\mathbb{N}$ and let $1/p_k+1/q_k=1$ for all $k\in\mathbb{N}$. Then for an infinite matrix $A=(f_k^n)$, $A\in(\ell(X,p),\ell_\infty)$ if and only if there is $m_0\in\mathbb{N}$ such that*

$$\sup_n\sum_{k=1}^{\infty}\|f_k^n\|^{q_k}m_0^{-q_k}<\infty. \tag{3.25}$$

Corollary 3.5. *Let $p=(p_k)$ be a bounded sequence of positive real numbers with $p_k>1$ for all $k\in\mathbb{N}$ and let $1/p_k+1/q_k=1$ for all $k\in\mathbb{N}$. Then for an infinite matrix $A=(f_k^n)$, $A\in(F_r(X,p),\ell_\infty)$ if and only if there is $m_0\in\mathbb{N}$ such that*

$$\sup_n\sum_{k=1}^{\infty}\left(k^{-rq_k/p_k}\|f_k^n\|^{q_k}m_0^{-q_k}\right)<\infty. \tag{3.26}$$

Theorem 3.6. *Let $p=(p_k)$ and $q=(q_k)$ be bounded sequences of positive real numbers with $p_k>1$ for all $k\in\mathbb{N}$ and let $1/p_k+1/t_k=1$ for all $k\in\mathbb{N}$. Then for an infinite matrix $A=(f_k^n)$, $A\in(\ell(X,p),\underline{\ell}_\infty(q))$ if and only if for each $r\in\mathbb{N}$, there is $m_r\in\mathbb{N}$ such that*

$$\sup_{n,k}\sum_{k=1}^{\infty}r^{t_k/q_n}\|f_k^n\|^{t_k}m_r^{-t_k}<\infty. \tag{3.27}$$

Proof. Since $\underline{\ell}_\infty(q)=\cap_{r=1}^{\infty}\ell_{\infty(r^{1/q_k})}$, it follows that

$$A\in(\ell(X,p),\underline{\ell}_\infty(q))\Longleftrightarrow A\in\left(\ell(X,p),\ell_{\infty(r^{1/q_k})}\right),\quad\forall r\in\mathbb{N}. \tag{3.28}$$

It is easy to show that for $r\in\mathbb{N}$,

$$A\in\left(\ell(X,p),\ell_{\infty(r^{1/q_k})}\right)\Longleftrightarrow(r^{1/q_n}f_k^n)_{n,k}\in(\ell(X,p),\ell_\infty). \tag{3.29}$$

We obtain by Corollary 3.4 that for $r \in \mathbb{N}$, $(r^{1/q_n} f_k^n)_{n,k} \in (\ell(X,p),\ell_\infty)$ if and only if there is $m_r \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} r^{t_k/q_n} \|f_k^n\|^{t_k} m_r^{-t_k} < \infty. \tag{3.30}$$

Thus the theorem is proved. □

Theorem 3.7. *Let $p = (p_k)$ and $q = (q_k)$ be bounded sequences of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/t_k = 1$ for all $k \in \mathbb{N}$. For an infinite matrix $A = (f_k^n)$, $A \in (F_r(X,p),\underline{\ell}_\infty(q))$ if and only if for each $i \in \mathbb{N}$, there is $m_i \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty. \tag{3.31}$$

Proof. Since $F_r(X,p) = \ell(X,p)_{(k^{r/p_k})}$, it implies that

$$A \in (F_r(X,p),\underline{\ell}_\infty(q)) \Longleftrightarrow (k^{-r/p_k} f_k^n)_{n,k} \in (\ell(X,p),\underline{\ell}_\infty(q)). \tag{3.32}$$

It follows from Theorem 3.6 that $A \in (F_r(X,p),\underline{\ell}_\infty(q))$ if and only if for each $i \in \mathbb{N}$, there is $m_i \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} i^{t_k/q_n} k^{-rt_k/p_k} \|f_k^n\|^{t_k} m_i^{-t_k} < \infty. \tag{3.33}$$

□

Theorem 3.8. *Let $p = (p_k)$ be bounded sequence of positive real numbers with $p_k > 1$ for all $n \in \mathbb{N}$ and let $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p),bs)$ if and only if there is $m_0 \in \mathbb{N}$ such that*

$$\sup_n \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} f_k^i \right\|^{q_k} m_0^{-q_k} < \infty. \tag{3.34}$$

Proof. For an infinite matrix $A = (f_k^n)$, we can easily show that

$$A \in (\ell(X,p),bs) \Longleftrightarrow \left(\sum_{i=1}^{n} f_k^i \right)_{n,k} \in (\ell(X,p),\ell_\infty). \tag{3.35}$$

This implies by Corollary 3.4 that $A \in (\ell(X,p),bs)$ if and only if there is $m_0 \in \mathbb{N}$ such that

$$\sup_n \sum_{k=1}^{\infty} \left\| \sum_{i=1}^{n} f_k^i \right\|^{q_k} m_0^{-q_k} < \infty. \tag{3.36}$$

□

Theorem 3.9. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$ and let $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$. Then for an infinite matrix $A = (f_k^n)$, $A \in (\ell(X,p),cs)$ if and only if*

(1) *there is $m_0 \in \mathbb{N}$ such that $\sup_n \sum_{k=1}^{\infty} \| \sum_{i=1}^{n} f_k^i \|^{q_k} m_0^{-q_k} < \infty$ and*

(2) *for each $k \in \mathbb{N}$ and $x \in X$, $\sum_{n=1}^{\infty} f_k^n(x)$ converges.*

Proof. The necessity is obtained by Theorem 3.8 and by the fact that $e^{(k)}(x) \in \ell(X,p)$ for every $k \in \mathbb{N}$ and $x \in X$.

Now, suppose that (1) and (2) hold. By Theorem 3.8, we have $A : \ell(X,p) \to bs$. Let $x = (x_k) \in \ell(X,p)$. Since $\ell(X,p)$ has the AK property, we have $x = \lim_{n\to\infty} \sum_{k=1}^{n} e^{(k)}(x_k)$. By Zeller's theorem, $A : \ell(X,p) \to bs$ is continuous. It implies that

$$Ax = \lim_{n\to\infty} \sum_{k=1}^{n} Ae^{(k)}(x_k). \tag{3.37}$$

By (2), $Ae^{(k)}(x_k) \in cs$ for all $k \in \mathbb{N}$. Since cs is a closed subspace of bs, it implies that $Ax \in cs$, that is, $A : \ell(X,p) \to cs$. □

Acknowledgement. The author would like to thank the Thailand Research Fund for the financial support during the preparation of this paper.

References

[1] R. G. Cooke, *Infinite Matrices and Sequence Spaces*, Macmillan, London, 1950. MR 12,694d. Zbl 040.02501.

[2] K.-G. Grosse-Erdmann, *The structure of the sequence spaces of Maddox*, Canad. J. Math. **44** (1992), no. 2, 298–307. MR 93d:46017. Zbl 777.46008.

[3] ———, *Matrix transformations between the sequence spaces of Maddox*, J. Math. Anal. Appl. **180** (1993), no. 1, 223–238. MR 95d:40017. Zbl 791.47029.

[4] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford Ser. (2) **18** (1967), 345–355. MR 36#4195. Zbl 156.06602.

[5] ———, *Paranormed sequence spaces generated by infinite matrices*, Proc. Cambridge Philos. Soc. **64** (1968), 335–340. MR 36#5565. Zbl 157.43503.

[6] H. Nakano, *Modulared sequence spaces*, Proc. Japan Acad. **27** (1951), 508–512. MR 13,954g. Zbl 044.11302.

[7] S. Simons, *The sequence spaces $l(p_\nu)$ and $m(p_\nu)$*, Proc. London Math. Soc. (3) **15** (1965), 422–436. MR 31#600. Zbl 128.33805.

[8] S. Suantai, *On matrix transformations related to Nakano vector-valued sequence spaces*, Bull. Calcutta Math. Soc. **91** (1999), no. 3, 221–226. MR 2000m:46041. Zbl 943.47025.

[9] S. Suantai and C. Sudsukh, *Matrix transformations of Nakano vector-valued sequence space*, Kyungpook Math. J. **40** (2000), no. 1, 93–97. MR 2001g:46041. Zbl 992.06378.

[10] C. X. Wu and L. Liu, *Matrix transformations on some vector-valued sequence spaces*, Southeast Asian Bull. Math. **17** (1993), no. 1, 83–96. MR 94e:46041. Zbl 794.40006.

Suthep Suantai: Department of Mathematics, Faculty of Science, Chaing Mai University, Chiang Mai, 50200, Thailand

E-mail address: malsuthe@science.cmu.ac.th

Journal of Applied Mathematics and Decision Sciences

Special Issue on Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/jamds/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	June 1, 2009
First Round of Reviews	September 1, 2009
Publication Date	December 1, 2009

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be