ON THE SOLVABILITY OF A VARIATIONAL INEQUALITY PROBLEM AND APPLICATION TO A PROBLEM OF TWO MEMBRANES

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ABSTRACT. The purpose of this work is to give a continuous convex function, for which we can characterize the subdifferential, in order to reformulate a variational inequality problem: find $u = (u_1, u_2) \in K$ such that for all $v = (v_1, v_2) \in K$, $\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + (f, v - u) \geq 0$ as a system of independent equations, where f belongs to $L^2(\Omega) \times L^2(\Omega)$ and $K = \{v \in H_0^1(\Omega) \times H_0^1(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega\}$.

2000 Mathematics Subject Classification. Primary 35J85.

1. Introduction. We are interested in the following variational inequality problem: find $u = (u_1, u_2) \in K$ such that for all $v = (v_1, v_2) \in K$,

$$\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + (f, v - u) \geq 0, \tag{1.1}$$

where f belongs to $L^2(\Omega) \times L^2(\Omega)$ and K is a closed convex set of $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by

$$K = \{v = (v_1, v_2) \in H_0^1(\Omega) \times H_0^1(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega\}. \tag{1.2}$$

Thanks to the orthogonal projection of the space $L^2(\Omega) \times L^2(\Omega)$ onto the cone $\mathcal{K}$ defined by

$$\mathcal{K} = \{v = (v_1, v_2) \in L^2(\Omega) \times L^2(\Omega) : v_1 \geq v_2 \text{ a.e. in } \Omega\}, \tag{1.3}$$

we construct a functional φ for which we can characterize the subdifferential at a point u, in order to reformulate problem (1.1) to a variational inequality without constraints; that is, find $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega) \times H_0^1(\Omega)$,

$$\int_\Omega \nabla u_1 \nabla (v_1 - u_1) + \int_\Omega \nabla u_2 \nabla (v_2 - u_2) + \varphi(v) - \varphi(u) + (h, v - u) \geq 0, \tag{1.4}$$

where φ is a continuous convex function from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $\mathbb{R}$ and h is an element of $L^2(\Omega) \times L^2(\Omega)$ depending only on f.

We prove that the solution $u = (u_1, u_2)$ can be obtained as a solution of a system of independent two Dirichlet problems

$$u_1, u_2 \in H_0^1(\Omega), \quad \Delta u_1 = g_1, \quad \Delta u_2 = g_2 \text{ in } \Omega, \tag{1.5}$$

where g_1 and g_2 are two functions of $L^2(\Omega)$ determined in terms of f_1 and f_2. We will give an algorithm for computing these functions.

This approach can be applied to study a variational inequality arising from a problem of two membranes [2].

2. Formulation of the problem. Let Ω be an open bounded set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. We equip $H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm

$$a(u,v) = \int_\Omega \nabla u_1 \nabla v_1 + \int_\Omega \nabla u_2 \nabla v_2, \tag{2.1}$$

where

$$u = (u_1, u_2), \ v = (v_1, v_2) \in H_0^1(\Omega) \times H_0^1(\Omega). \tag{2.2}$$

For $r \in L^2(\Omega)$, we let

$$r^+ = \max\{r, 0\}, \qquad r^- = \min\{r, 0\}. \tag{2.3}$$

For $f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$, we let

$$f^+ = (f_1^+, f_2^-), \qquad f^- = (f_1^-, f_2^+). \tag{2.4}$$

For $v = (v_1, v_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$, we let

$$v_+ = \left(v_1 + \frac{(v_2 - v_1)^+}{2}, v_2 - \frac{(v_2 - v_1)^+}{2} \right), \qquad v_- = \left(-\frac{(v_2 - v_1)^+}{2}, \frac{(v_2 - v_1)^+}{2} \right) \tag{2.5}$$

the projection of v onto the cone $\mathcal{K}$ given by (1.3) with respect to the scalar product of $L^2(\Omega) \times L^2(\Omega)$ (respectively, the projection with respect to the scalar product of $L^2(\Omega) \times L^2(\Omega)$ on the polar cone of $\mathcal{K}$ defined by $\mathcal{K}^0 = \{v = (-r, r) \in L^2(\Omega) \times L^2(\Omega) : r \geq 0 \text{ a.e. on } \Omega\}$). We easily verify that

$$a(v_+, v_-) = 0 \tag{2.6}$$

for all $v \in H_0^1(\Omega) \times H_0^1(\Omega)$. A function φ defined from $H_0^1(\Omega) \times H_0^1(\Omega)$ to $\mathbb{R}$ is called lower semi-continuous (l.s.c.) if its epigraph defined by

$$\mathrm{epi}(\varphi) = \{v = (v_1, v_2) \in H_0^1(\Omega) \times H_0^1(\Omega), \lambda \in \mathbb{R} : \varphi(v) \leq \lambda\} \tag{2.7}$$

is closed in $H_0^1(\Omega) \times H_0^1(\Omega) \times \mathbb{R}$. Let $u \in H_0^1(\Omega) \times H_0^1(\Omega)$, we denote by $\partial\varphi(u)$ the subdifferential of φ at u, defined by

$$\partial\varphi(u) = \{\mu \in H^{-1}(\Omega) \times H^{-1}(\Omega) : \varphi(u) - \varphi(v) \leq \langle \mu, u - v \rangle \ \forall v \in H_0^1(\Omega) \times H_0^1(\Omega)\}. \tag{2.8}$$

If φ is a convex l.s.c. function, then for all $v \in H_0^1(\Omega) \times H_0^1(\Omega)$, $\partial\varphi(v) \neq \varnothing$.

Let $f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega)$. We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the scalar product and the norm of $L^2(\Omega) \times L^2(\Omega)$, respectively. We consider the following variational inequality problem: find $u = (u_1, u_2) \in K$ such that

$$a(u, v - u) + (f, v - u) \geq 0 \quad \forall v = (v_1, v_2) \in K. \tag{2.9}$$

It admits a unique solution. The functional φ defined from $L^2(\Omega) \times L^2(\Omega)$ to $\mathbb{R}$ by $v \mapsto (f^+, v_+)$ is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$ and convex.

PROPOSITION 2.1. $u = (u_1, u_2)$ *is a solution of the problem (2.9) if and only if u is the solution of the following problem: find $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that*

$$a(u, v - u) + \varphi(v) - \varphi(u) + (f^-, v - u) \geq 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega). \qquad (2.10)$$

PROOF. It is well known in the general theory of variational inequalities that problem (2.10) admits a unique solution. So, it is sufficient to show that the solution u of (2.10) is an element of K. Let $v = u_+$, then the inequality of (2.10) becomes

$$a(u, -u_-) + \varphi(u) - \varphi(u) + (f^-, -u_-) \geq 0. \qquad (2.11)$$

By the relation (2.6) we deduce that $u_- = 0$, hence $u \in K$. $\qquad \square$

PROPOSITION 2.2. *Problem (2.10) is equivalent to the following problem: find $\mu = (\mu_1, \mu_2) \in L^2(\Omega) \times L^2(\Omega)$, $u = (u_1, u_2) \in H_0^1(\Omega) \times H_0^1(\Omega)$,*

$$a(u, v) + (\mu, v) + (f^-, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega), \ \mu \in \partial\varphi(u). \qquad (2.12)$$

PROOF. If $u \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $\mu \in L^2(\Omega) \times L^2(\Omega)$ are the solution of (2.12), then by definition of $\mu \in \partial\varphi(u)$, we have

$$a(u, v - u) + \varphi(v) - \varphi(u) + (f^-, v - u) \geq 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega). \qquad (2.13)$$

Conversely, let u be the solution of problem (2.10). For $v = u \pm w$, with $w \in H_0^1(\Omega) \times H_0^1(\Omega)$, the inequality of (2.10) gives

$$\begin{aligned} a(u, w) + (f^-, w) &\geq -(f^+, w^+) \geq -\|f^+\| \|w\|, \\ a(u, w) + (f^-, w) &\leq (f^+, (-w)^+) \leq \|f^+\| \|w\|. \end{aligned} \qquad (2.14)$$

We deduce that

$$\left| a(u, w) + (f^-, w) \right| \leq \|f^+\| \|w\|. \qquad (2.15)$$

So the linear form

$$w \longmapsto a(u, w) + (f^-, w) \qquad (2.16)$$

is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$ equipped with the norm of $L^2(\Omega) \times L^2(\Omega)$. Where μ is an element of $L^2(\Omega) \times L^2(\Omega)$. $\qquad \square$

We set

$$C = \{v \in L^2(\Omega) \times L^2(\Omega), \ (v, v) \leq \varphi(v) \ \forall v \in L^2(\Omega) \times L^2(\Omega)\}. \qquad (2.17)$$

LEMMA 2.3. *Let $u \in L^2(\Omega) \times L^2(\Omega)$, then the following properties are equivalent:*
(a) $\mu \in \partial\varphi(u)$.
(b) $\mu \in C$ *and* $(\mu, u) = \varphi(u)$.
(c) $\mu \in C$ *and* $(v - \mu, u) \leq 0$ *for all* $v \in C$.

PROOF. (a)$\Rightarrow$(b). Let $\mu \in \partial\varphi(u)$, we have

$$\varphi(v) - \varphi(u) \geq (\mu, v - u) \quad \forall v \in L^2(\Omega) \times L^2(\Omega). \qquad (2.18)$$

We put $v = 0$, next $v = 2u$ in (2.18). Since φ is positively homogeneous of degree 1, we obtain $\varphi(u) = (\mu, u)$ and consequently

$$\varphi(v) \geq (\mu, v) \quad \forall v \in L^2(\Omega) \times L^2(\Omega). \tag{2.19}$$

(c)$\Rightarrow$(a). For all $v \in V$, we have

$$(\mu, v - u) \leq \varphi(v) - (\mu, u) \leq \varphi(v) - (v, u) \quad \forall v \in C. \tag{2.20}$$

Hence for $v \in \partial \varphi(u)$, we have $(v, u) = \varphi(u)$, consequently $\mu \in \varphi(u)$. $\square$

We deduce from Lemma 2.3 the following relations:

$$\mu_1 + \mu_2 = f_1^+ + f_2^-, \quad f_2^- \leq \mu_2 \leq \mu_1 \leq f_1^+ \text{ a.e. in } \Omega. \tag{2.21}$$

Indeed, the function φ being positively homogeneous of degree 1, $\mu \in \partial \varphi(u)$ implies

$$(\mu, u) = \varphi(u), \tag{2.22}$$
$$(\mu, v) \leq \varphi(v) \quad \forall v \in L^2(\Omega) \times L^2(\Omega). \tag{2.23}$$

Finally, it is sufficient to take in (2.23) elements $v = (v_1, v_2)$ with suitable choices on the components v_1 and v_2.

Let $V = H_0^1(\Omega) \times H_0^1(\Omega)$, and taking into account Lemma 2.3, we can write problem (2.12) as follows: find $u \in H_0^1(\Omega) \times H_0^1(\Omega)$, $\mu \in C$,

$$a(u, v) + (\mu, v) + (f^-, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega),$$
$$(v - \mu, u) \leq 0 \quad \forall v \in C. \tag{2.24}$$

Let A be the Riesz-Fréchet representation of $H^{-1}(\Omega) \times H^{-1}(\Omega)$ in $H_0^1(\Omega) \times H_0^1(\Omega)$. We set $M = A(C)$, this is a closed convex subset in $H_0^1(\Omega) \times H_0^1(\Omega)$ characterized by

$$M = \{w \in H_0^1(\Omega) \times H_0^1(\Omega) : a(w, v) \leq \varphi(v) \ \forall v \in H_0^1(\Omega) \times H_0^1(\Omega)\}. \tag{2.25}$$

Problem (2.24) can be written in the following form: find $u \in H_0^1(\Omega) \times H_0^1(\Omega)$, $z \in M$,

$$a(u + z + t, v) = 0 \quad \forall v \in H_0^1(\Omega) \times H_0^1(\Omega),$$
$$a(w - z, u) \leq 0 \quad \forall w \in M. \tag{2.26}$$

with $z = A(\mu)$ and $t = A(f^-)$. Hence

$$u = -z - t, \quad z = P_M(-t), \tag{2.27}$$

where $P_M(-t)$ is the projection of $-t$ onto the closed convex set M with respect to the scalar product $a(\cdot, \cdot)$ of $H_0^1(\Omega) \times H_0^1(\Omega)$.

From the equality of Proposition 2.2, we deduce that the solution u of problem (2.9) verifies the following equations:

$$\Delta u_1 = \mu_1 + f_1^-, \quad \Delta u_2 = \mu_2 + f_2^+ \quad \text{in } \Omega. \tag{2.28}$$

We notice that the prior knowledge of $\mu = (\mu_1, \mu_2)$ in terms of data of problem (2.9) yields the solutions u_1 and u_2 as solutions of two independent Dirichlet problems given by the system (2.28). We recall that for each element f of $L^p(\Omega)$, the solution of the problem

$$u \in H_0^1(\Omega), \quad -\Delta u = f \quad \text{in } \Omega, \tag{2.29}$$

verifies the following properties (see [2]):

$$u \in H^{2,p}(\Omega), \quad \|u\|_{H^{2,p}} \le C\|f\|_{L^p}, \tag{2.30}$$

where C is a constant depending only on p and Ω. We deduce from (2.28) that u_1, u_2 are in $H^2(\Omega)$ and

$$\begin{aligned}
&\|u_1\|_{H^2(\Omega)} \le c_1 \|\mu_1 + f_1^-\|_{L^2(\Omega)}, \\
&\|u_2\|_{H^2(\Omega)} \le c_2 \|\mu_2 + f_2^-\|_{L^2(\Omega)}, \\
&\|u_1 + u_2\|_{H^2(\Omega)} \le c \|f_1 + f_2\|_{L^2(\Omega)},
\end{aligned} \tag{2.31}$$

where c, c_1, and c_2 are constants depending only on Ω. We define the domain of non-coincidence [2] by

$$\Omega^+ = \{x \in \Omega : u_1(x) > u_2(x)\}. \tag{2.32}$$

From relations (2.21), (2.22), and (2.23) we deduce that

$$\mu_1 = f_1^+, \quad \mu_2 = f_2^- \quad \text{a.e. in } \Omega^+. \tag{2.33}$$

When u_1 and u_2 are continuous on Ω, the following relations are verified:

$$\Delta u_1 = f_1, \quad \Delta u_2 = f_2 \quad \text{in } \Omega^+. \tag{2.34}$$

2.1. Algorithm for computing z. We consider the following projection problem:

$$z \in H_0^1(\Omega) \times H_0^1(\Omega), \quad z = P_M(t'), \quad \text{where } t' = -t. \tag{2.35}$$

Let z_0 belong to M, we compute the element w_0 of M which verifies the following inequality:

$$a(w - w_0, z_0 - t') \ge 0 \quad \forall w \in M. \tag{2.36}$$

Next we compute

$$z_1 = P_{[z_0, w_0]}(t'). \tag{2.37}$$

So, the algorithm is: z_n being given in M, we construct w_n verifying

$$a(w - w_n, z_n - t') \ge 0 \quad \forall w \in M. \tag{2.38}$$

Next $z_{n+1} = P_{[z_n, w_n]}(t')$. The sequence $\{z_n\}$ converges in $H_0^1(\Omega) \times H_0^1(\Omega)$ strongly to the solution of problem (2.35) [1]. Since $M = A(C)$, then the inequality (2.38) implies that there exists $\{v_n\}$ in C which verifies

$$(v - v_n, t' - z_n) \le 0 \quad \forall v \in C \tag{2.39}$$

and Lemma 2.3 shows that v_n is an element of $\partial \varphi(t' - z_n)$.

2.2. Application. This method of solvability can be applied to the study of a variational inequality arising from a problem of two membranes [2],

$$\Delta u_1 + \lambda u_1 = f_1, \quad \Delta u_2 = f_2 \text{ in } \Omega^+, \quad u_1 = u_2,$$

$$\frac{\partial u_1}{\partial x_i} = \frac{\partial u_2}{\partial x_i}, \quad 1 \le i \le n, \tag{2.40}$$

$$\Delta u_1 + \left(\frac{\lambda}{2}\right) u_1 = \frac{1}{2}(f_1 + f_2) \quad \text{in } \Omega^-,$$

where Ω^+ and Ω^-, are two parts of Ω (unknown) separated by a hypersurface Γ of $\mathbb{R}^n$ such that $\Omega = \Omega^+ \cup \Gamma \cup \Omega^-$; f_1, f_2 are two regular functions and $\lambda \in \mathbb{R}$. Formally, Ω^+ is the non-coincidence domain given by (2.32).

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