ON HENSTOCK-DUNFORD AND HENSTOCK-PETTIS INTEGRALS

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ABSTRACT. We give the Riemann-type extensions of Dunford integral and Pettis integral, Henstock-Dunford integral and Henstock-Pettis integral. We discuss the relationships between the Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral. We prove the Harnack extension theorems and the convergence theorems for Henstock-Dunford and Henstock-Pettis integrals.

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1. Introduction. During 1957–1958, R. Henstock and J. Kurzweil, independently, gave a Riemann-type integral called the Henstock-Kurzweil integral (or Henstock integral) (see [7]). It is a kind of nonabsolute integral and contains the Lebesgue integral. It has been proved that this integral is equivalent to the special Denjoy integral [7]. The Dunford, Pettis integrals are generalizations of the Lebesgue integral to Banach-valued functions. In [5], R. A. Gordon gave two Denjoy-type extensions of the Dunford, Pettis integrals, the Denjoy-Dunford and Denjoy-Pettis integrals, and discussed their properties.

In this paper, we give the Riemann-type extensions of Dunford, Pettis integrals, the Henstock-Dunford, Henstock-Pettis integrals, and discuss the relationships between the Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral. We prove the Harnack extension theorems and the convergence theorems for Henstock-Dunford and Henstock-Pettis integrals.

Throughout this paper, X denotes a real Banach space and X^* its dual. $B(X^*) = \{x^* \in X^* : \| x^* \| \leq 1\}$ is the unit ball in X^*. $I_0 = [a,b]$ is a closed interval in $\mathbb{R}$.

We first give some preliminaries. A partition D of $[a,b]$ is a finite collection of interval-point pairs (I,t) with the intervals nonoverlapping and their union $[a,b]$. Here t is the associated point of I. We write $D = \{(I,t)\}$, it is said to be δ-fine partition of $[a,b]$ if for each interval-point pair (I,t), we have $t \in I \subset (t - \delta(t), t + \delta(t))$.

DEFINITION 1.1 (see [7]). A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable if there exists a function $F : [a,b] \to \mathbb{R}$ such that for every $\epsilon > 0$ there is a function $\delta(t) > 0$ such that for any δ-fine partition $D = \{[u,v]; t\}$ of $[a,b]$, we have

$$\left| \sum [f(t)(v-u) - F(u,v)] \right| < \epsilon, \tag{1.1}$$

where the sum $\sum$ is understood to be over $D = \{([u,v],t)\}$ and $F(u,v) = F(v) - F(u)$. We write $(H) \int_{I_0} f = F(I_0)$.

The function f is said to be Henstock integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is Henstock integrable on $[a,b]$. We write $(H)\int_{I_0} f\chi_E = (H)\int_E f$.

DEFINITION 1.2 (see [1, 5, 7]). A function $f : [a,b] \to \mathbb{R}$ is Denjoy (or special Denjoy) integrable if there exists an ACG (or ACG^*) function $F : [a,b] \to \mathbb{R}$ such that $D_{\mathrm{ap}}F(t) = f(t)$ (or $F'(t) = f(t)$) almost everywhere on $[a,b]$. Where $D_{\mathrm{ap}}F(t)$ denotes the approximate derivative of F at t. We write $(D)\int_{I_0} f = F(I_0)$ (or $(D^*)\int_{I_0} f = F(I_0)$).

The function f is said to be Denjoy (or special Denjoy) integrable on the set $E \subset [a,b]$ if the function $f\chi_E$ is Denjoy (or special Denjoy) integrable on $[a,b]$. We write $(D)\int_{I_0} f\chi_E = (D)\int_E f$ (or $(D^*)\int_{I_0} f\chi_E = (D^*)\int_E f$).

If f is special Denjoy integrable, then f is Denjoy integrable.

LEMMA 1.3 (see [7]). *A function $f : [a,b] \to \mathbb{R}$ is Henstock integrable on $[a,b]$ if and only if f is the special Denjoy integrable on $[a,b]$.*

DEFINITION 1.4 (see Gordon [5]). (a) A function $f : [a,b] \to X$ is Denjoy-Dunford integrable on $[a,b]$ if for each x^* in X^* the function x^*f is Denjoy integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*. We write $x_{I_0}^{**} = (DD)\int_{I_0} f = F(I_0)$ and F is called the primitive of f on I_0.

(b) A function $f : [a,b] \to X$ is Denjoy-Pettis integrable on $[a,b]$ if f is Denjoy-Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$. We write $x_{I_0}^{**} = (DP)\int_{I_0} f = F(I_0)$ and F is called the primitive of f on I_0.

The function f is said to be integrable in one of the above senses on the set $E \subset [a,b]$ if the function $f\chi_E$ is integrable in that sense on $[a,b]$.

LEMMA 1.5 (see [3]). *A function $f : [a,b] \to X$ is Denjoy-Dunford integrable on $[a,b]$ if and only if x^*f is Denjoy integrable on $[a,b]$ for all $x^* \in X^*$.*

2. Definition and properties. In the following, we give the Riemann-type extensions of Dunford, Pettis integrals, and discuss the relationships between Henstock-Dunford integral and Dunford integral, Henstock-Pettis integral and Pettis integral.

DEFINITION 2.1. (a) A function $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$ if for each x^* in X^* the function x^*f is Henstock integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*. We write $x_{I_0}^{**} = (HD)\int_{I_0} f = F(I_0)$ and F is called the primitive of f on I_0.

(b) A function $f : [a,b] \to X$ is Henstock-Pettis integrable on $[a,b]$ if f is Henstock-Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$. We write $x_{I_0}^{**} = (HP)\int_{I_0} f = F(I_0)$ and F is called the primitive of f on I_0.

The function f is said to be integrable in one of the above senses on the set $E \subset [a,b]$ if the function $f\chi_E$ is integrable in that sense on $[a,b]$.

By the above definitions and Definition 1.4, it is easy to see that if f is Henstock-Dunford (or Henstock-Pettis) integrable on I_0, then f is Denjoy-Dunford (or Denjoy-Pettis) integrable.

THEOREM 2.2. *A function $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$ if and only if $x^* f$ is Henstock integrable on $[a,b]$ for all $x^* \in X^*$.*

PROOF. If f is Henstock-Dunford integrable on $[a,b]$, for every $x^* \in X^*$, by Definition 2.1, $x^* f$ is Henstock integrable on $[a,b]$. Conversely, if $x^* f$ is Henstock integrable on $[a,b]$. It follows from Lemma 1.3 that $x^* f$ is Denjoy integrable on $[a,b]$ and $(D) \int_a^b x^* f = (H) \int_a^b x^* f$. It follows from Lemma 1.5 that f is Denjoy-Dunford integrable on $[a,b]$, and for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = (D) \int_I x^* f$ for all x^* in X, that is, $x_I^{**}(x^*) = (H) \int_I x^* f$ for all x^* in X^*. So f is Henstock-Dunford integrable on $[a,b]$. $\qquad\square$

THEOREM 2.3. *If the function $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$, then each perfect set in $[a,b]$ contains a portion on which f is Dunford integrable.*

PROOF. Since the function $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$, then for each $x^* \in X^*$, $x^* f$ is Henstock integrable on $[a,b]$. It follows from [8] that each perfect set in $[a,b]$ contains a portion on which $x^* f$ is Lebesgue integrable. So f is Dunford integrable on a portion. $\qquad\square$

THEOREM 2.4. *If the function $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$, then there is a sequence $\{X_k\}$ of closed subsets such that $X_k \subset X_{k+1}$ for all k, $\cup_{k=1}^{\infty} X_K = [a,b]$, f is Dunford integrable on each X_k and*

$$\lim_{k \to \infty} (Dunford) \int_{X_k \cap [a,x]} f(t)\,dt = (HD) \int_a^x f(t)\,dt \text{ weakly} \qquad (2.1)$$

uniformly on $[a,b]$.

PROOF. It follows from Theorem 2.2 that a function $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$ if and only if $x^* f$ is Henstock integrable on $[a,b]$ for all $x^* \in X^*$. From [8], $x^* f$ is Henstock integrable on $[a,b]$, then there is a sequence $\{X_k\}$ of closed subsets such that $X_k \subset X_{k+1}$ for all k, $\cup_{k=1}^{\infty} X_K = [a,b]$, $x^* f$ is Lebesgue integrable on each X_k and

$$\lim_{k \to \infty} (L) \int_{X_k \cap [a,x]} x^* f(t)\,dt = (H) \int_a^x x^* f(t)\,dt \qquad (2.2)$$

uniformly on $[a,b]$ for each $x^* \in X^*$. So we obtain that f is Dunford integrable on each X_k and

$$\lim_{k \to \infty} (Dunford) \int_{X_k \cap [a,x]} f(t)\,dt = (HD) \int_a^x f(t)\,dt \text{ weakly} \qquad (2.3)$$

uniformly on $[a,b]$. $\qquad\square$

THEOREM 2.5. *If the function $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$, then there exists a sequence $\{X_k\}$ of closed sets, $\cup_{k=1}^{\infty} X_k = [a,b]$, f is Dunford integrable on each X_k.*

PROOF. Since f Henstock-Dunford integrable on $[a,b]$, by Definition 2.1, for each $x^* \in X^*$, $x^* f$ is Henstock integrable on $[a,b]$, and for every interval $I \subset [a,b]$,

$\int_I x^* f = x^* \int_I f$, and $F(I) = \int_I f \in X$. Since $x^* f$ is Henstock integrable, then $x^* F$ is ACG^*. So there is a sequence $\{X_k\}$ of closed subsets such that $\cup_{k=1}^{\infty} X_k = [a,b]$ and $x^* F$ is VB^* on each X_k. From [7, Lemma 6.18], $x^* f$ is Lebesgue integrable on each X_k. So we obtain that f is Dunford integrable on each X_k.   □

THEOREM 2.6. *Suppose that X contains no copy of c_0 and $f : [a,b] \to X$. If the function f is Henstock-Pettis integrable on $[a,b]$, then each perfect set in $[a,b]$ contains a portion on which f is Pettis integrable.*

PROOF. Since the function $f : [a,b] \to X$ is Henstock-Pettis integrable on $[a,b]$, then f is Denjoy-Pettis integrable on $[a,b]$. It follows from [5, Theorem 38] that each perfect set in $[a,b]$ contains a portion on which f is Pettis integrable.

In the fact, from [3, Theorem 10], we have that if each Henstock-Pettis integrable function defined on $[a,b]$ is Pettis integrable on a portion of every close set, then X does not contain c_0.   □

THEOREM 2.7. *Suppose that X contains no copy of c_0 and $f : [a,b] \to X$ is a measurable. If the function $f : [a,b] \to X$ is Henstock-Pettis integrable on $[a,b]$, then there exists a sequence $\{X_k\}$ of closed sets with $X_k \uparrow [a,b]$ such that for each $x^* \in X^*$, f is Pettis integrable on each X_k, and*

$$\lim_{k \to \infty} (Pettis) \int_{X_k} f = (HP) \int_a^b f \text{ weakly.} \tag{2.4}$$

PROOF. Since f is Henstock-Pettis integrable on $[a,b]$, then f is Henstock-Dunford integrable on $[a,b]$. By Theorem 2.4, there is a sequence $\{X_k\}$ of closed subsets such that $X_k \subset X_{k+1}$ for all k, $\cup_{k=1}^{\infty} X_K = [a,b]$, $x^* f$ is Lebesgue integrable on each X_k and

$$\lim_{k \to \infty} (L) \int_{X_k \cap [a,x]} x^* f(t) \, dt = (H) \int_a^x x^* f(t) \, dt \tag{2.5}$$

uniformly on $[a,b]$ for each $x^* \in X^*$. So we obtain that f is Dunford integrable on each X_k. From [2, Theorem 2.5, page 54], f is Pettis integrable on $[a,b]$ and

$$\lim_{k \to \infty} (Pettis) \int_{X_k \cap [a,x]} f(t) \, dt = (HP) \int_a^x f(t) \, dt \text{ weakly} \tag{2.6}$$

uniformly on each X_k, that is,

$$\lim_{k \to \infty} (Pettis) \int_{X_k} f = (HP) \int_a^b f \text{ weakly.} \tag{2.7}$$
                □

In Theorem 2.7, if we remove the condition that f is a measurable, then we have the following theorem.

THEOREM 2.8. *Suppose that X contains no copy of c_0. If the function $f : [a,b] \to X$ is Henstock-Pettis integrable on $[a,b]$, then there exists a sequence $\{X_k\}$ of closed sets, $\cup_{k=1}^{\infty} X_k = [a,b]$, f is Pettis integrable on each X_k.*

PROOF. Since f is Henstock-Pettis integrable on $[a,b]$, by Definition 2.1, for each $x^* \in X^*$, $x^* f$ is Henstock integrable on $[a,b]$, and for every interval $I \subset [a,b]$,

$\int_I x^* f = x^* \int_I f$, and $F(I) = \int_I f \in X$. Since $x^* f$ is Henstock integrable, then $x^* F$ is ACG^*. So there is a sequence $\{X_k\}$ of closed subsets such that $\cup_{k=1}^{\infty} X_k = [a,b]$ and $x^* F$ is VB^* on each X_k. For each $k \in N$, let $(a,b) - X_k = \cup_{n=1}^{\infty}(c_n^k, d_n^k)$. Then

$$\sum_{n=1}^{\infty} \left| x^* \int_{c_n^k}^{d_n^k} f \right| < \infty. \tag{2.8}$$

Since X contains no copy of c_0, by Bessaga-Pelczynski theorem [2, page 22], $\sum_{n=1}^{\infty} \int_{c_n^k}^{d_n^k} f$ is unconditionally convergent in norm. Also

$$\sum_{n=1}^{\infty} \sup_{[a_n^k, b_n^k] \subset [c_n^k, d_n^k]} \left| x^* \int_{a_n^k}^{b_n^k} f \right| < \infty. \tag{2.9}$$

By Harnack extension theorem [7, page 41], we have

$$\int_{X_k} x^* f = \int_a^b x^* f - \sum_{n=1}^{\infty} \int_{c_n^k}^{d_n^k} x^* f = x^* \left(\int_a^b f - \sum_{n=1}^{\infty} \int_{c_n^k}^{d_n^k} f \right). \tag{2.10}$$

Hence $\int_{X_k} f = \int_a^b f - \sum_{n=1}^{\infty} \int_{c_n^k}^{d_n^k} f \in X$ and $\int_{X_k} x^* f = x^* \int_{X_k} f$.

So, for every closed set $H \subset X_k$, we have $\int_H x^* f = x^* \int_H f$ and $\int_H f \in X$. Since $\int_a^b f \chi_{X_k} = \int_{X_k} f \in X$, $\int_a^b f \chi_H = \int_H f \in X$, then for every closed interval $I \subset [a,b]$, $\int_I f \chi_{X_k} = \int_{I \cap X_k} f \in X$. By [5, Theorem 23, page 79], $f \chi_{X_k}$ is Pettis integrable on $[a,b]$, that is, f is Pettis integrable on each X_k. $\qquad \square$

3. The extension theorems and convergence theorems. Now we consider the extension theorems and convergence theorems of the Henstock-Dunford and Henstock-Pettis integrals.

THEOREM 3.1. *Let E be a closed subset in $[a,b]$ and $(a,b) - E$ the union of $\{(a_k, b_k)\}$, $k = 1, 2, \ldots$. If $f : [a,b] \to X$ is Henstock-Dunford integrable on E and each interval $[a_k, b_k]$ with*

$$\sum_{k=1}^{\infty} \omega \left(\int_{a_k}^t x^* f, [a_k, b_k] \right) < \infty \tag{3.1}$$

for each $x^* \in X^*$, then f is Henstock-Dunford integrable on $[a,b]$ and

$$\left\langle x^*, (HD) \int_a^b f \right\rangle = \left\langle x^*, (HD) \int_a^b f \chi_E \right\rangle + \sum_{k=1}^{\infty} \left\langle x^*, (HD) \int_{a_k}^{b_k} f \right\rangle \tag{3.2}$$

for each $x^* \in X^*$.

PROOF. From the conditions of Theorem 3.1, we have the function $x^* f$ satisfies the hypothesis of [7, Corollary 7.11]. So we have $x^* f$ is Henstock integrable on $[a,b]$ and

$$(H) \int_a^b x^* f = (H) \int_a^b x^* f \chi_E + \sum_{k=1}^{\infty} (H) \int_{a_k}^{b_k} x^* f. \tag{3.3}$$

It follows from Theorem 2.2 that f is Henstock-Dunford integrable on $[a,b]$ and the above equality means that

$$\left\langle x^*, (HD) \int_a^b f \right\rangle = \left\langle x^*, (HD) \int_a^b f \chi_E \right\rangle + \sum_{k=1}^\infty \left\langle x^*, (HD) \int_{a_k}^{b_k} f \right\rangle \tag{3.4}$$

for each $x^* \in X^*$. $\qquad\square$

THEOREM 3.2. *Let E be a closed subset in $[a,b]$ and $\{(a_k, b_k)\}$ be an enumeration of the intervals contiguous to E in (a,b). Suppose that $f : [a,b] \to X$ is Henstock-Pettis integrable on E and each interval $[a_k, b_k]$. If $\sum_{k=1}^\infty \omega(\int_{a_k}^t x^* f, [a_k, b_k]) < \infty$ for each $x^* \in X^*$ and the series $\sum_{k=1}^\infty (HP) \int_{[a_k,b_k] \cap J} f$ is unconditionally convergent for every subinterval J of $[a,b]$, then f is Henstock-Pettis integrable on $[a,b]$ and*

$$(HP) \int_a^b f = (HP) \int_a^b f \chi_E + \sum_{k=1}^\infty (HP) \int_{a_k}^{b_k} f. \tag{3.5}$$

PROOF. From Theorem 3.1, we have the function f is Henstock-Dunford integrable on $[a,b]$ and $(H) \int_a^b x^* f = (H) \int_a^b x^* f \chi_E + \sum_{k=1}^\infty (H) \int_{a_k}^{b_k} x^* f$. To show that f is in fact Henstock-Pettis integrable on $[a,b]$. We need to show that $(HD) \int_J f$ belongs to X for each closed interval J in $[a,b]$.

Let $E_0 = E \cap J$. Then E_0 is a closed set. Since $f \chi_E$ is Henstock-Pettis integrable on J, then $f \chi_{E_0}$ is Henstock-Pettis integrable on J, that is, f is Henstock-Pettis integrable on E_0. And $\{(a_k, b_k) \cap J\}$ is an enumeration of the intervals contiguous to E_0 in J, so f is Henstock-Pettis integrable on them and $\sum_k (HP) \int_{[a_k,b_k] \cap J} f$ is an unconditionally convergent series in X. Now, if we apply Theorem 3.1 to E_0 in J, we get

$$\left\langle x^*, (HD) \int_J f \right\rangle = \left\langle x^*, (HP) \int_J f \chi_{E_0} \right\rangle + \sum_{k=1}^\infty \left\langle x^*, (HP) \int_{[a_k,b_k] \cap J} f \right\rangle \tag{3.6}$$

for each $x^* \in X^*$, that is,

$$\left\langle x^*, (HD) \int_J f \right\rangle = \left\langle x^*, (HP) \int_J f \chi_{E_0} + \sum_{k=1}^\infty (HP) \int_{[a_k,b_k] \cap J} f \right\rangle \tag{3.7}$$

for each $x^* \in X^*$. We conclude that

$$(HD) \int_J f = (HP) \int_J f \chi_{E_0} + \sum_{k=1}^\infty (HP) \int_{[a_k,b_k] \cap J} f. \tag{3.8}$$

Hence, f is Henstock-Pettis integrable on $[a,b]$ and

$$(HP) \int_a^b f = (HP) \int_a^b f \chi_{E_0} + \sum_{k=1}^\infty (HP) \int_{[a_k,b_k] \cap J} f. \tag{3.9}$$

$\qquad\square$

COROLLARY 3.3. *Suppose that X contains no copy of c_0. Let E be a closed subset in $[a,b]$ and $\{(a_k, b_k)\}$ be an enumeration of the intervals contiguous to E in (a,b). Suppose that $f : [a,b] \to X$ is Henstock-Pettis integrable on E and each interval $[a_k, b_k]$.*

If $\sum_{k=1}^{\infty} \omega(\int_{a_k}^{t} x^* f, [a_k, b_k]) < \infty$ for each $x^* \in X^*$, then f is Henstock-Pettis integrable on $[a,b]$ and

$$(HP) \int_a^b f = (HP) \int_a^b f \chi_E + \sum_{k=1}^{\infty} (HP) \int_{a_k}^{b_k} f. \tag{3.10}$$

THEOREM 3.4. *Suppose that X is weakly sequentially complete and $f : [a,b] \to X$ is Henstock-Dunford integrable on $[a,b]$. If f is measurable, then f is Henstock-Pettis integrable on $[a,b]$.*

PROOF. It is similar to the proof of [5, Theorem 40]. □

LEMMA 3.5 (see [1, 5]). *Let Γ be a family of open intervals in (a,b) and suppose that Γ has the following properties:*
   (1) *if (α, β) and (β, γ) belong to Γ, then (α, γ) belongs to Γ;*
   (2) *if (α, β) belong to Γ, then every open interval in (α, β) belongs to Γ;*
   (3) *if (α, β) belong to Γ for every interval in $[\alpha, \beta] \subset (c, d)$, then (c, d) belongs to Γ;*
   (4) *if all of the intervals contiguous to the perfect set $E \subset [a,b]$ belong to Γ, then there exists an interval I in Γ such that $I \cap E \neq \varnothing$.*
   Then Γ contains the interval (a,b).

LEMMA 3.6. *Suppose that $f_n : [a,b] \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$, and*
   (1) *$f_n \to f$ almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock (or D^*) integrable on $[a,b]$;*
   (2) *the primitives F_n of f_n are continuous uniformly in n and ACG^* uniformly in n.*
   Then f is Henstock (or D^*) integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f. \tag{3.11}$$

DEFINITION 3.7. Let $F : [a,b] \to X$ and let E be a subset of $[a,b]$.

(a) F is BV^* on E if $\sup\{\sum_i \omega(F; [c_i, d_i])\}$ is finite, where the supremum is taken over all finite collections $\{[c_i, d_i]\}$ of nonoverlapping intervals that have endpoints in E, ω denotes the oscillation of F over $[c_i, d_i]$, that is,

$$\omega(F; [c_i, d_i]) = \sup\{\|F(x) - F(y)\|; \ x, y \in [c_i, d_i]\}. \tag{3.12}$$

(b) F is AC^* on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $\sum_i \omega(F; [c_i, d_i]) < \epsilon$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\sum_i (d_i - c_i) < \delta$.

(c) F is BVG^* on E if E can be expressed as a countable union of sets on each of which F is BV^*.

(d) F is ACG^* on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC^*.

THEOREM 3.8. *Suppose that X is weakly sequentially complete and*
   (1) *$f_n \to f$ weakly almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$;*
   (2) *the primitives F_n of f_n are continuous uniformly in n and ACG^* uniformly in n.*

Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \ \text{weakly}. \tag{3.13}$$

PROOF. Let

$$\Gamma = \left\{ (\alpha, \beta) \subset [a,b] : f \text{ is Henstock-Pettis integrable on } [\alpha, \beta], \int_\alpha^\beta f_n \longrightarrow \int_\alpha^\beta f \text{ weakly} \right\}. \tag{3.14}$$

We must show that Γ contains (a,b) and by Lemma 3.5 it is sufficient to verify that Γ satisfies Romanovski's four conditions.

Conditions (1) and (2) are easily verified.

Suppose that (α, β) belongs to Γ for every interval $[\alpha, \beta]$ in (c,d). For each positive integer $n > 2/(d-c)$, define $I_n = (c + 1/n, d - 1/n)$ and let $x_n = x^{**}_{I_n}$.

Then we have

$$x^{**}_{(c,d)}(x^*) = \int_c^d x^* f = \lim_{n \to \infty} \int_{I_n} x^* f = \lim_{n \to \infty} x^*(x_n) \tag{3.15}$$

for each x^* in X^*. Since X is weakly sequentially complete, the sequence $\{x_n\}$ converges weakly to an element x_0 of X and we must have $x^{**}_{(c,d)} = x_0$. It follows easily that (c,d) belongs to Γ and this verifies condition (3).

Now let E be a perfect set in $[a,b]$ such that each of the intervals in $[a,b]$ contiguous to E belongs to Γ.

Since $\{F_n\}$ is continuous uniformly in n and ACG^* uniformly in n, then for each $x^* \in X^*$, $\{x^* F_n\}$ is continuous uniformly in n and ACG^* uniformly in n, and $x^* f_n \to x^* f$ almost everywhere in $[a,b]$. It follows from [1] that $x^* f$ is special Denjoy integrable on $[a,b]$. So there exists an interval $[u,v]$ with $u, v \in E$ and $E \cap (u,v) \neq \varnothing$ such that $\{F_n\}$ is AC^* uniformly in n on $P = E \cap (u,v)$ and the series $\sum_k \omega(F_n; [u_k, v_k])$ unconditionally converges where $(u,v) - E = \cup_k (u_k, v_k)$. Hence $\sum_k \omega(\int_{u_k}^t x^* f_n; [u_k, v_k])$ $< \infty$ for each $x^* \in X^*$. By Corollary 3.3, we have

$$\int_u^v f_n = \int_P f_n + \sum_k \int_{u_k}^{v_k} f_n. \tag{3.16}$$

$\{F_n\}$ is AC^* uniformly in n on P, $\{x^* F_n : x^* \in B(X^*), \ n \in \mathbb{N}\}$ is AC^* uniformly in n on P. So $\{x^* f_n : x^* \in B(X^*), \ n \in \mathbb{N}\}$ is uniformly integrable on P (see [2]), that is, for $E \subset P$,

$$\lim_{|E| \to 0} \int_E |x^* f_n| = 0 \quad \text{uniformly in } x^* \in B(X^*) \text{ and } n. \tag{3.17}$$

It follows from [4, Theorem 3] that f is Pettis integrable on P and $\int_P f_n \to \int_P f$ weakly. Since $\{F_n\}$ is AC^* uniformly in n on P, so for every $\epsilon > 0$ there exists N such that $\sum_{k=N}^\infty \| \int_{u_k}^{v_k} f_n \| < \epsilon$, $n = 1, 2, \ldots$. For every $x^* \in B(X^*)$, we have $\sum_{k=N}^\infty |\int_{u_k}^{v_k} x^* f_n| < \epsilon$, $n = 1, 2, \ldots$. So $\sum_{k=N}^\infty |\int_{u_k}^{v_k} x^* f| < \epsilon$. Since X is weakly sequentially complete and X does not contain c_0, hence $\sum_k \int_{u_k}^{v_k} f$ unconditionally converges. By (3.16),

$$x^* \int_u^v f_n = x^* \int_P f_n + x^* \sum_k \int_{u_k}^{v_k} f_n. \tag{3.18}$$

Let $n \to \infty$, we have

$$x_{(u,v)}^{**}(x^*) = x^* \int_P f + x^* \sum_k \int_{u_k}^{v_k} f. \tag{3.19}$$

Hence

$$x_{(u,v)}^{**} = \int_P f + \sum_k \int_{u_k}^{v_k} f \in X, \tag{3.20}$$

that is, f is Henstock-Pettis integrable on $[u,v]$. So $(u,v) \in \Gamma$. This shows that (u,v) belongs to Γ and Γ satisfies condition (4). This completes the proof. $\square$

THEOREM 3.9. *Suppose that X is weakly sequentially complete and $f_n \to f$ weakly almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$. If there is a scalar function g with $\| f_n(\cdot) \| \le g(\cdot)$ almost everywhere for all n and if $\int g < \infty$, then f is Henstock-Pettis integrable on $[a,b]$ and*

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \ \text{weakly.} \tag{3.21}$$

PROOF. It is similar to the proof of Theorem 3.8. $\square$

DEFINITION 3.10. Let $\{f_\alpha\}$ be a family of Henstock-Pettis integrable functions defined on $[a,b]$. The family $\{x^* f_\alpha : x^* \in B(X^*)\}$ is uniformly integrable in the generalized sense on $[a,b]$, if for each perfect set $E \subset [a,b]$ there exists an interval $[c,d] \subset [a,b]$ with $c,d \in E$ and $E \cap (c,d) \ne \varnothing$ such that $\{x^* f_\alpha : x^* \in B(X^*)\}$ is uniformly integrable on $P = E \cap (c,d)$ and for every α the series $\sum_k \int_{c_k}^{d_k} f_\alpha$ is unconditionally convergent where $(c,d) - E = \cup_k (c_k, d_k)$.

THEOREM 3.11. *Suppose that X is weakly sequentially complete and*
(1) *$f_n \to f$ weakly almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$.*
(2) *The family $\{x^* f_n : x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable in the generalized sense on $[a,b]$.*
(3) *For each $x^* \in X^*$, $\lim_{n \to \infty} \int_c^d x^* f_n = \int_c^d x^* f$ uniformly for every $[c,d] \subset [a,b]$.*
Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \ \text{weakly.} \tag{3.22}$$

PROOF. It is similar to the proof of Theorem 3.8. The only difference is that the family $\{x^* f_n : x^* \in B(X^*), n \in \mathbb{N}\}$ is uniformly integrable in the generalized sense on $[a,b]$, then there is a portion $P = E \cap I$ of E such that the family $|x^* f_n \chi_E|$ is uniformly integrable on P. So f is Pettis integrable on P. $\square$

THEOREM 3.12. *Suppose that X is weakly sequentially complete and*
(1) *$f_n \to f$ weakly almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$ and f is measurable,*
(2) *the primitives F_n of f_n are weakly continuous uniformly in n and weakly ACG^* uniformly in n, that is, for every $x^* \in X^*$, $x^* F_n$ are continuous uniformly in n and ACG^* uniformly in n.*

Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \ \text{weakly.} \tag{3.23}$$

PROOF. For each x^* in X^*, we have

(1) $x^* f_n \to x^* f$ almost everywhere on $[a,b]$ as $n \to \infty$, where each $x^* f_n$ is Henstock integrable on $[a,b]$,

(2) the primitives $x^* F_n$ of $x^* f_n$ are continuous uniformly in n and ACG^* uniformly in n. It follows from Lemma 3.6 that $x^* f$ is Henstock integrable on $[a,b]$ and

$$\int_a^b x^* f_n \longrightarrow \int_a^b x^* f \quad \text{as } n \longrightarrow \infty. \tag{3.24}$$

By Theorem 2.2, f is Henstock-Dunford integrable on $[a,b]$. Since X is weakly sequentially complete and f is measurable, by Theorem 3.4, f is Henstock-Pettis integrable on $[a,b]$. □

THEOREM 3.13. *Suppose that the unit ball $B(X^*)$ of X^* is weak* sequentially compact and*

(1) $f_n \to f$ *weakly almost everywhere in $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$,*

(2) *the primitives F_n of f_n are continuous uniformly in n and ACG^* uniformly in n.*

Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \ \text{weakly.} \tag{3.25}$$

PROOF. Suppose that $I \subset I_0$. Let C be the weak closure of $\{\int_I f_n : n \in \mathbb{N}\}$. For each x^* in X^*, $\{x^* F_n : n \in \mathbb{N}\}$ is continuous uniformly in n and ACG^* uniformly in n in $[a,b]$, and further $\int_a^b x^* f_n = x^* \int_a^b f_n$. A convergence theorem, namely Lemma 3.6, guarantees that $x^* f$ is Henstock integrable on $[a,b]$ and $\lim_{n \to \infty} \int_a^b x^* f_n = \int_a^b x^* f$ for each x^* in X^*. We observe that C is bounded and that $C - \{\int_I f_n : n \in \mathbb{N}\}$ contains at most one point. We will prove that C is weakly compact.

Suppose that C is not weakly compact. An appeal to a theorem of James [6, Theorem 1] produces a bounded sequence (x_k^*) in X^*, a sequence (x_n) in C, and an $\epsilon > 0$ such that $x_k^*(x_n) = 0$ for $k > n$ and $x_k^*(x_n) > \epsilon$ for $n \geq k$. By passing to subsequences and relabelling, we can find a subsequence $(\int_I g_n)$ of $(\int_I f_n)$ and a subsequence (y_k^*) of x_k^* such that

$$
\begin{aligned}
y_k^* \int_I g_n &= \int_I y_k^* g_n = 0 \quad \text{for } k > n, \\
y_k^* \int_I g_n &= \int_I y_k^* g_n > \epsilon \quad \text{for } n \geq k, \\
\lim_{n \to \infty} \int_I x^* g_n &= \int_I x^* f \quad \forall x^* \text{ in } X^*.
\end{aligned}
\tag{3.26}
$$

Since the unit ball $B(X^*)$ of X^* is weak* sequentially compact, the sequence (y_k^*) has a subsequence $(y_{k_j}^*)$ which weak* converges to y_0^*, so $\lim_{j \to \infty} y_{k_j}^* f = y_0^* f$ on I_0,

$\lim_{j \to \infty} y_{k_j}^* F = y_0^* F$ on I_0, that is, $\lim_{j \to \infty} \int_I y_{k_j}^* f = \int_I y_0^* f$. To force a contradiction, note that for each k, $\lim_{n \to \infty} \int_I y_k^* f_n = \int_I y_k^* f$. Hence $\int_I y_k^* f \geq \epsilon$ for each k, and $\int_I y_0^* f \geq \epsilon$. On the other hand, notice that since each g_n is Henstock-Pettis integrable, $(y_{k_j}^*)$ weak* converges to y_0^*, hence

$$\lim_{j \to \infty} \int_I y_{k_j}^* g_n = \lim_{j \to \infty} y_{k_j}^* \int_I g_n = y_0^* \int_I g_n = \int_I y_0^* g_n. \qquad (3.27)$$

Since this holds for each n, and since $\lim_{n \to \infty} \int_I y_0^* g_n = \int_I y_0^* f$, we see that $\int_I y_0^* f = 0$. This contradicts the inequality $\int_I y_0^* f \geq \epsilon$, and proves that C is weakly compact. Since $\lim_{n \to \infty} \int_I x^* f_n = \int_I x^* f$, the sequence $(\int_I f_n)$ of the Henstock-Pettis integrals is weakly Cauchy. It follows from the weak compactness of C that $\lim_{n \to \infty} \int_I f_n$ exists weakly in X. Denote $F(I) = \int_I f = \lim_{n \to \infty} \int_I f_n$ weakly, then $x^* F(I) = x^* \int_I f = \int_I x^* f$ for each x^* in X^*. So f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \qquad (3.28)$$

$\square$

COROLLARY 3.14. *Suppose that X is a reflexive Banach space and*
(1) *$f_n \to f$ weakly almost everywhere on $[a,b]$ as $n \to \infty$, where each f_n is Henstock-Pettis integrable on $[a,b]$,*
(2) *the primitives F_n of f_n are weakly continuous uniformly in n and weakly ACG^* uniformly in n on $[a,b]$.*
Then f is Henstock-Pettis integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \qquad (3.29)$$

THEOREM 3.15. *If the following conditions are satisfied:*
(1) *$\lim_{n \to \infty} f_n = f$ weakly almost everywhere on $[a,b]$, where each f_n is Henstock-Dunford integrable on $[a,b]$,*
(2) *the primitives F_n of f_n are weakly continuous uniformly in n and weakly ACG^* uniformly in n.*
Then f is Henstock-Dunford integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \qquad (3.30)$$

PROOF. Since
(1) $\lim_{n \to \infty} x^* f_n = x^* f$ almost everywhere on $[a,b]$,
(2) the primitives $x^* F_n$ of $x^* f_n$ are continuous uniformly in n and ACG^* uniformly in n.
Then, as in the proof of Theorem 3.12, $x^* f$ is Henstock integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b x^* f_n = \int_a^b x^* f. \qquad (3.31)$$

By Theorem 2.2, f is Henstock-Dunford integrable on $[a,b]$ and

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b f \text{ weakly.} \qquad (3.32)$$

$\square$

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