

## $T_\Omega$ -SEQUENCES IN ABELIAN GROUPS

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**ABSTRACT.** A sequence in an abelian group is called a  $T$ -sequence if there exists a Hausdorff group topology in which the sequence converges to zero. This paper describes the fundamental system for the finest group topology in which this sequence converges to zero. A sequence is a  $T_\Omega$ -sequence if there exist uncountably many different Hausdorff group topologies in which the sequence converges to zero. The paper develops a condition which insures that a sequence is a  $T_\Omega$ -sequence and examples of  $T_\Omega$ -sequences are given.

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**1. Introduction.** Let  $G$  be an abelian group and let  $\langle a_n \rangle_{n=1}^\infty$  be a nontrivial sequence in  $G$ . If 0 is the identity element in  $G$ , we can ask what is the finest group topology on  $G$  such that  $\langle a_n \rangle_{n=1}^\infty$  converges to zero? In the terminology of [2], we are placing the topology of a nonconstant sequence on the subspace  $\{a_n\}_{n=1}^\infty \cup \{0\} \subseteq G$  and finding the associated Graev topology. When this topology is Hausdorff, Zelenyuk, and Protasov [4] say that  $\langle a_n \rangle_{n=1}^\infty$  is a  $T$ -sequence. The purpose of this paper will be to extend some of the results of Zelenyuk and Protasov concerning  $T$ -sequences in specific abelian groups. We will develop a fundamental system approach to defining group topologies and use this approach to consider the cardinality of the set of Hausdorff group topologies in which a specific sequence converges to zero. This extends results found in [1].

We assume as additional hypothesis throughout this paper that  $G$  is an abelian group and that each sequence under consideration is a one-to-one function from the natural numbers  $\mathbb{N}$  into  $G$ . Also the notations  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $S^1$  will denote the integers, rationals, real, and the circle group, respectively. The subgroup of  $S^1$  which is the set of solutions of the form  $k/p^n$ , where  $k \in \mathbb{Z}$ ,  $p$  is prime and  $n \in \mathbb{N}$ , we will denote it as  $\mathbb{Z}(p^\infty)$ .

**2. Fundamental systems generated by sequences.** Since  $G$  is abelian it is possible to define various fundamental systems in a subgroup and use them as a fundamental system for the entire group. We shall use the terms of the sequence  $\langle a_n \rangle_{n=1}^\infty$  to define such a fundamental system for the subgroup generated by  $\{a_n\}_{n=1}^\infty$ . Let  $T(n) = \{0\} \cup \{a_k\}_{k=n}^\infty \cup \{-a_k\}_{k=n}^\infty$ , where  $-a_k$  denotes the inverse of  $a_k$  in  $G$ , and let  $\zeta$  denote the collection of all increasing sequences in  $\mathbb{N}$ . Then for  $C, D \in \zeta$  we define  $U(C, D) = \{g_1 + g_2 + \dots + g_k \mid g_i \in c_i T(d_i) \text{ for } i \in \{1, 2, \dots, k\}; k \in \mathbb{N}\}$ .

**PROPOSITION 2.1.**  $\mathcal{F} = \{U(C, D) \mid C, D \in \zeta\}$  is a fundamental system for  $G$ .

**PROOF.** Suppose that  $U(C, D)$  and  $U(C', D')$  are elements of  $\mathcal{F}$ . For each  $i \in \mathbb{N}$  let  $c'' = \min\{c_i, c'_i\}$  and  $d'' = \max\{d_i, d'_i\}$ . Define  $C'' = \langle c''_i \rangle_{i=1}^{\infty}$  and  $D'' = \langle d''_i \rangle_{i=1}^{\infty}$ . Clearly, both  $C'', D'' \in \zeta$ . Since  $c_1 T(n) \subseteq c_2 T(n)$  whenever  $c_1 \leq c_2$  and  $T(n) \subseteq T(m)$  whenever  $m \leq n$ , we have that  $c''_i T(d''_i) \subseteq c_i T(d_i) \cap c'_i T(d'_i)$ . Therefore we have  $U(C'', D'') \subseteq U(C, D) \cap U(C', D')$ .

Now suppose  $x \in U(C, D)$ . Then  $x = g_1 + g_2 + \dots + g_k$  for some  $k \in \mathbb{N}$  and each  $g_i \in c_i T(d_i)$  for  $i \in \{1, 2, \dots, k\}$ . If  $C' = \langle c_{k+1}, c_{k+2}, \dots \rangle$  and  $D' = \langle d_{k+1}, d_{k+2}, \dots \rangle$  then  $x + U(C', D') \subseteq U(C, D)$ .

Let  $U(C, D) \in \mathcal{F}$ . For each  $i \in \mathbb{N}$  we define

$$c'_i = \begin{cases} \frac{c_{2i}}{2} & \text{if } c_{2i} \text{ is even,} \\ \frac{c_{2i}-1}{2} & \text{if } c_{2i} \text{ is odd.} \end{cases} \quad (2.1)$$

If  $C' = \langle c'_i \rangle$  then  $C' \in \zeta$  since  $C \in \zeta$ . Also we have that  $2c' \leq c_{2i}$  for all  $i \in \mathbb{N}$ . Define  $D' = \langle d_{2i} \rangle$ . Then for each  $i \in \mathbb{N}$  we have that  $2c'_i T(d_{2i}) \subseteq c_{2i} T(d_{2i})$  and hence  $2U(C', D') \subseteq U(C, D)$ .

Finally, we note that since  $U(C, D)^{-1} = U(C, D)$ ,  $\mathcal{F}$  is a fundamental system.  $\square$

**PROPOSITION 2.2.** *The group topology generated by  $\mathcal{F}$  is the finest group topology on  $G$  for which  $\langle a_n \rangle_{n=1}^{\infty}$  converges to zero.*

**PROOF.** Let  $\tau$  be any group topology on  $G$  for which the sequence  $\langle a_n \rangle_{n=1}^{\infty}$  converges to zero and let  $0 \in W \in \tau$ . We inductively define a sequence of open sets in  $\tau$ , say  $V_1, V_2, \dots$ , with  $0 \in V_i$  for all  $i$ ,  $2V_1 \subseteq W$ , and in general  $(n+1)V_n \subseteq V_{n-1}$  for  $n \geq 2$ . We also may assume that each  $V_i$  is symmetric.

For any  $k \in \mathbb{N}$  we have that  $V_1 + 2V_2 + \dots + kV_k \subseteq W$ . Since  $\langle a_n \rangle_{n=1}^{\infty}$  converges to zero in  $\tau$ , we can find a tail of the sequence in  $V_i$ . We choose  $d_i \in \mathbb{N}$  so that  $T(d_i) \subseteq V_i$  and  $d_i > \max\{d_1, \dots, d_{i-1}\}$ . Then we have that  $kT(d_i) \subseteq kV_k$  and for  $D = \langle d_i \rangle$ , we have that  $U(\mathbb{N}, D) \subseteq W$ .

The technique used in Proposition 2.1 can be used to show that various subcollections of  $\mathcal{F}$  are also fundamental systems for  $G$ . For example if  $D = \langle d_i \rangle \in \zeta$  and for  $k \in \mathbb{N}$ ,  $D_k = \langle d_{ki} \rangle$ , then  $\mathcal{F}' = \{U(C, D_k) \mid C \in \zeta, k \in \mathbb{N}\}$  will also form a fundamental system.  $\square$

**$T_{\Omega}$ -SEQUENCES.** Shelah [3] constructs an example of a nonabelian group that admits only the discrete and indiscrete topologies as group topologies. Certainly, the constant identity sequence in Shelah's group will be a  $T$ -sequence which converges in a unique Hausdorff group topology. On the other hand, the sparse sequences in  $\mathbb{Q}$  described in [1] are shown to converge to the identity in uncountably many different Hausdorff group topologies. We will call any such sequence a  $T_{\Omega}$ -sequence. As we shall see in this section, many sequences in abelian groups are actually  $T_{\Omega}$ -sequences.

Our search for  $T_{\Omega}$ -sequences will require that we focus our attention on various subcollections of the fundamental system described in Proposition 2.1. To refine our notation we define for  $D = \langle d_n \rangle \in \zeta$ ,  $U(\langle d_n \rangle) = \{\sum_{i=1}^n g_i \mid g_i \in T(d_i) \text{ for } i \in \{1, 2, \dots, n\} \text{ and } n \in \mathbb{N}\}$  and  $\mathcal{F}_D = \{U(\langle d_{kn} \rangle) \mid k \in \mathbb{N}\}$ . Using techniques similar to those used in Proposition 2.1, it can be shown that  $\mathcal{F}_D$  is a fundamental system for  $G$ .

We will also focus on a subcollection of  $\zeta$ . For each  $c \in \mathbb{R}$  with  $c > 2$  we define  $c_n = [n^c]$ , the greatest integer in  $n^c$ . Clearly  $C = \langle c_n \rangle_{n=1}^\infty \in \zeta$ .

**LEMMA 2.3.** *If  $c, d$  are real numbers with  $2 < c < d$  and if  $k \in \mathbb{N}$  then we can find  $N_k \in \mathbb{N}$  such that for  $m \geq N_k$ ,  $c_{km} + m < d_m$ .*

**PROOF.** We can find  $N_k \in \mathbb{N}$  such that for  $m \geq N_k$  we have that  $k^c + 2 < m^{d-c}$ . Hence  $[(km)^c] + m < [m^d]$  and thus  $c_{km} + m < d_m$  for all  $m > N_k$ .  $\square$

**DEFINITION 2.4.** Let  $S \subseteq G$ . For  $n \in \mathbb{N}$  and  $g \in G$  we say that  $g$  has an  $n$ -factorization in  $S$  if and only if there exists  $\{s_1, \dots, s_n\} \subseteq S - \{0\}$  with  $g = s_1 + s_2 + \dots + s_n$ . The factorization is favorable if and only if  $-s_i \notin \{s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n\}$ .

**PROPOSITION 2.5.** *Let  $\langle a_n \rangle_{n=1}^\infty$  be a sequence in  $G$  and  $S = \{\sum_{i=1}^n g_i \mid g_i \in T(i)\} \text{ for } 1 \leq i \leq n \text{ and } n \in \mathbb{N}\}$ . If*

- (1) *every element of  $S$  has only finitely many favorable factorizations in  $S$ ;*
- (2) *if  $a = \sum_{i=n}^m a_i$  for some  $n, m \in \mathbb{N}$ , then  $a$  has no other favorable factorizations in  $S$ ;*

*then the sequence  $\langle a_n \rangle_{n=1}^\infty$  is a  $T_\Omega$ -sequence.*

**PROOF.** For any sequence  $D = \langle d_n \rangle_{n=1}^\infty \in \zeta$  we have that  $U(\langle d_n \rangle_{n=1}^\infty) \subseteq S$ . So by (1) we have that for every  $g \in S$  there exists a  $k \in \mathbb{N}$  such that no favorable factorization of  $g$  in  $S$  has a factor in  $T(k)$ . Hence  $g \notin U(\langle d_{kn} \rangle_{n=1}^\infty)$  and thus  $\mathcal{F}_D$  generates a Hausdorff group topology.

Now choose  $C = \langle c_n \rangle_{n=1}^\infty$  and  $D = \langle d_n \rangle_{n=1}^\infty$  in  $\zeta$  with the property that for each  $k \in \mathbb{N}$  there exists  $N_k \in \mathbb{N}$  such that  $c_{km} + m < d_m$  for all  $m \geq N_k$ . Suppose that  $U(\langle d_n \rangle_{n=1}^\infty)$  is open in the topology generated by  $\mathcal{F}_C$ . Then there exists a  $k$  such that  $U(\langle c_{kn} \rangle_{n=1}^\infty) \subseteq U(\langle d_n \rangle_{n=1}^\infty)$ . We have that  $a = \sum_{i=1}^{N_k} a_{c_{kn}+i} \in U(\langle c_{kn} \rangle_{n=1}^\infty)$ . But by (2) and the fact that  $c_{kn} + N_k < b_{N_k}$ , we must conclude that  $a \notin U(\langle d_n \rangle_{n=1}^\infty)$ . Hence the group topology generated by  $\mathcal{F}_C$  is different from the group topology  $\mathcal{F}_D$ . By Lemma 2.3 we can find uncountably many different Hausdorff group topologies on  $G$  with the property that  $\langle a_n \rangle_{n=1}^\infty$  converges to zero.  $\square$

**EXAMPLE 2.6.** Let  $\langle p^n \rangle_{n=1}^\infty$  be the sequence of powers of the prime  $p$  in  $\mathbb{Z}$ .  $\langle p^n \rangle_{n=1}^\infty$  is a  $T_\Omega$ -sequence.

**EXAMPLE 2.7.** Let  $k \in \mathbb{N}$  and let  $\langle a_n \rangle_{n=1}^\infty$  be an increasing sequence in  $\mathbb{Z}$  satisfying the inequality  $a_{n+1}/a_n > n/k$  for all  $n$ . For  $n > 2k$  we have that  $\sum_{i=1}^m a_{n+i} < a_{n+m+1}$  for each  $m \in \mathbb{N}$ . Hence  $\langle a_n \rangle_{n=1}^\infty$  is a  $T_\Omega$ -sequence.

**EXAMPLE 2.8.** Let  $Z \in \mathbb{Z}(p^\infty)$ . The order of  $Z$  is  $p^n$  if  $Z$  is a  $p^n$ -root of unity but not a  $p^{n-1}$ -root of unity. We denote the order of  $Z$  by  $O(Z)$ . Now if  $O(Z) = p^m$  and  $O(w) = p^n$  and  $m < n$  we have that  $O(Zw) = p^n$ . Let  $\langle Z_n \rangle_{n=1}^\infty$  be a sequence in  $\mathbb{Z}(p^\infty)$  satisfying

$$O(Z_{p^n+i}) \geq p^{n+1} O(Z_{p^n+i-1}) \quad \text{for all } n \in \mathbb{N} \text{ and for } 0 \leq i < p. \quad (2.2)$$

By Proposition 2.5,  $\langle Z_n \rangle_{n=1}^\infty$  is a  $T_\Omega$ -sequence.

**EXAMPLE 2.9.** Consider  $\mathbb{R}$  as the direct sum of uncountably many copies of  $\mathbb{Q}$ . If  $\langle r_n \rangle_{n=1}^\infty$  is any sequence of linearly independent real numbers, then  $\langle r_n \rangle_{n=1}^\infty$  is a  $T_\Omega$ -sequence.

We end this paper with a question. Does there exist a nontrivial sequence in a group  $G$  which is a  $T$ -sequence, but not a  $T_\Omega$ -sequence?

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