

T_Ω -SEQUENCES IN ABELIAN GROUPS

ROBERT LEDET and BRADD CLARK

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ABSTRACT. A sequence in an abelian group is called a T -sequence if there exists a Hausdorff group topology in which the sequence converges to zero. This paper describes the fundamental system for the finest group topology in which this sequence converges to zero. A sequence is a T_Ω -sequence if there exist uncountably many different Hausdorff group topologies in which the sequence converges to zero. The paper develops a condition which insures that a sequence is a T_Ω -sequence and examples of T_Ω -sequences are given.

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1. Introduction. Let G be an abelian group and let $\langle a_n \rangle_{n=1}^\infty$ be a nontrivial sequence in G . If 0 is the identity element in G , we can ask what is the finest group topology on G such that $\langle a_n \rangle_{n=1}^\infty$ converges to zero? In the terminology of [2], we are placing the topology of a nonconstant sequence on the subspace $\{a_n\}_{n=1}^\infty \cup \{0\} \subseteq G$ and finding the associated Graev topology. When this topology is Hausdorff, Zelenyuk, and Protasov [4] say that $\langle a_n \rangle_{n=1}^\infty$ is a T -sequence. The purpose of this paper will be to extend some of the results of Zelenyuk and Protasov concerning T -sequences in specific abelian groups. We will develop a fundamental system approach to defining group topologies and use this approach to consider the cardinality of the set of Hausdorff group topologies in which a specific sequence converges to zero. This extends results found in [1].

We assume as additional hypothesis throughout this paper that G is an abelian group and that each sequence under consideration is a one-to-one function from the natural numbers \mathbb{N} into G . Also the notations \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and S^1 will denote the integers, rationals, real, and the circle group, respectively. The subgroup of S^1 which is the set of solutions of the form k/p^n , where $k \in \mathbb{Z}$, p is prime and $n \in \mathbb{N}$, we will denote it as $\mathbb{Z}(p^\infty)$.

2. Fundamental systems generated by sequences. Since G is abelian it is possible to define various fundamental systems in a subgroup and use them as a fundamental system for the entire group. We shall use the terms of the sequence $\langle a_n \rangle_{n=1}^\infty$ to define such a fundamental system for the subgroup generated by $\{a_n\}_{n=1}^\infty$. Let $T(n) = \{0\} \cup \{a_k\}_{k=n}^\infty \cup \{-a_k\}_{k=n}^\infty$, where $-a_k$ denotes the inverse of a_k in G , and let ζ denote the collection of all increasing sequences in \mathbb{N} . Then for $C, D \in \zeta$ we define $U(C, D) = \{g_1 + g_2 + \cdots + g_k \mid g_i \in c_i T(d_i) \text{ for } i \in \{1, 2, \dots, k\}; k \in \mathbb{N}\}$.

PROPOSITION 2.1. $\mathcal{F} = \{U(C, D) \mid C, D \in \zeta\}$ is a fundamental system for G .

PROOF. Suppose that $U(C, D)$ and $U(C', D')$ are elements of \mathcal{F} . For each $i \in \mathbb{N}$ let $c'' = \min\{c_i, c'_i\}$ and $d'' = \max\{d_i, d'_i\}$. Define $C'' = \langle c''_i \rangle_{i=1}^\infty$ and $D'' = \langle d''_i \rangle_{i=1}^\infty$. Clearly, both $C'', D'' \in \zeta$. Since $c_1 T(n) \subseteq c_2 T(n)$ whenever $c_1 \leq c_2$ and $T(n) \subseteq T(m)$ whenever $m \leq n$, we have that $c'_i T(d'_i) \subseteq c_i T(d_i) \cap c'_i T(d'_i)$. Therefore we have $U(C'', D'') \subseteq U(C, D) \cap U(C', D')$.

Now suppose $x \in U(C, D)$. Then $x = g_1 + g_2 + \cdots + g_k$ for some $k \in \mathbb{N}$ and each $g_i \in c_i T(d_i)$ for $i \in \{1, 2, \dots, k\}$. If $C' = \langle c_{k+1}, c_{k+2}, \dots \rangle$ and $D' = \langle d_{k+1}, d_{k+2}, \dots \rangle$ then $x + U(C', D') \subseteq U(C, D)$.

Let $U(C, D) \in \mathcal{F}$. For each $i \in \mathbb{N}$ we define

$$c'_i = \begin{cases} \frac{c_{2i}}{2} & \text{if } c_{2i} \text{ is even,} \\ \frac{c_{2i}-1}{2} & \text{if } c_{2i} \text{ is odd.} \end{cases} \quad (2.1)$$

If $C' = \langle c'_i \rangle$ then $C' \in \zeta$ since $C \in \zeta$. Also we have that $2c' \leq c_{2i}$ for all $i \in \mathbb{N}$. Define $D' = \langle d_{2i} \rangle$. Then for each $i \in \mathbb{N}$ we have that $2c'_i T(d_{2i}) \subseteq c_{2i} T(d_{2i})$ and hence $2U(C', D') \subseteq U(C, D)$.

Finally, we note that since $U(C, D)^{-1} = U(C, D)$, \mathcal{F} is a fundamental system. \square

PROPOSITION 2.2. *The group topology generated by \mathcal{F} is the finest group topology on G for which $\langle a_n \rangle_{n=1}^\infty$ converges to zero.*

PROOF. Let τ be any group topology on G for which the sequence $\langle a_n \rangle_{n=1}^\infty$ converges to zero and let $0 \in W \in \tau$. We inductively define a sequence of open sets in τ , say V_1, V_2, \dots , with $0 \in V_i$ for all i , $2V_1 \subseteq W$, and in general $(n+1)V_n \subseteq V_{n-1}$ for $n \geq 2$. We also may assume that each V_i is symmetric.

For any $k \in \mathbb{N}$ we have that $V_1 + 2V_2 + \cdots + kV_k \subseteq W$. Since $\langle a_n \rangle_{n=1}^\infty$ converges to zero in τ , we can find a tail of the sequence in V_i . We choose $d_i \in \mathbb{N}$ so that $T(d_i) \subseteq V_i$ and $d_i > \max\{d_1, \dots, d_{i-1}\}$. Then we have that $kT(d_i) \subseteq kV_k$ and for $D = \langle d_i \rangle$, we have that $U(\mathbb{N}, D) \subseteq W$.

The technique used in Proposition 2.1 can be used to show that various subcollections of \mathcal{F} are also fundamental systems for G . For example if $D = \langle d_i \rangle \in \zeta$ and for $k \in \mathbb{N}$, $D_k = \langle d_{ki} \rangle$, then $\mathcal{F}' = \{U(C, D_k) \mid C \in \zeta, k \in \mathbb{N}\}$ will also form a fundamental system. \square

T_Ω -SEQUENCES. Shelah [3] constructs an example of a nonabelian group that admits only the discrete and indiscrete topologies as group topologies. Certainly, the constant identity sequence in Shelah's group will be a T -sequence which converges in a unique Hausdorff group topology. On the other hand, the sparse sequences in \mathbb{Q} described in [1] are shown to converge to the identity in uncountably many different Hausdorff group topologies. We will call any such sequence a T_Ω -sequence. As we shall see in this section, many sequences in abelian groups are actually T_Ω -sequences.

Our search for T_Ω -sequences will require that we focus our attention on various subcollections of the fundamental system described in Proposition 2.1. To refine our notation we define for $D = \langle d_n \rangle \in \zeta$, $U(\langle d_n \rangle) = \{\sum_{i=1}^n g_i \mid g_i \in T(d_i) \text{ for } i \in \{1, 2, \dots, n\} \text{ and } n \in \mathbb{N}\}$ and $\mathcal{F}_D = \{U(\langle d_{kn} \rangle) \mid k \in \mathbb{N}\}$. Using techniques similar to those used in Proposition 2.1, it can be shown that \mathcal{F}_D is a fundamental system for G .

We will also focus on a subcollection of ζ . For each $c \in \mathbb{R}$ with $c > 2$ we define $c_n = [n^c]$, the greatest integer in n^c . Clearly $C = \langle c_n \rangle_{n=1}^\infty \in \zeta$.

LEMMA 2.3. *If c, d are real numbers with $2 < c < d$ and if $k \in \mathbb{N}$ then we can find $N_k \in \mathbb{N}$ such that for $m \geq N_k$, $c_{km} + m < d_m$.*

PROOF. We can find $N_k \in \mathbb{N}$ such that for $m \geq N_k$ we have that $k^c + 2 < m^{d-c}$. Hence $[(km)^c] + m < [m^d]$ and thus $c_{km} + m < d_m$ for all $m > N_k$. \square

DEFINITION 2.4. Let $S \subseteq G$. For $n \in \mathbb{N}$ and $g \in G$ we say that g has an n -factorization in S if and only if there exists $\{s_1, \dots, s_n\} \subseteq S - \{0\}$ with $g = s_1 + s_2 + \dots + s_n$. The factorization is favorable if and only if $-s_i \notin \{s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n\}$.

PROPOSITION 2.5. *Let $\langle a_n \rangle_{n=1}^\infty$ be a sequence in G and $S = \{\sum_{i=1}^n g_i \mid g_i \in T(i) \text{ for } 1 \leq i \leq n \text{ and } n \in \mathbb{N}\}$. If*

- (1) *every element of S has only finitely many favorable factorizations in S ;*
- (2) *if $a = \sum_{i=1}^m a_i$ for some $n, m \in \mathbb{N}$, then a has no other favorable factorizations in S ;*

then the sequence $\langle a_n \rangle_{n=1}^\infty$ is a T_Ω -sequence.

PROOF. For any sequence $D = \langle d_n \rangle_{n=1}^\infty \in \zeta$ we have that $U(\langle d_n \rangle_{n=1}^\infty) \subseteq S$. So by (1) we have that for every $g \in S$ there exists a $k \in \mathbb{N}$ such that no favorable factorization of g in S has a factor in $T(k)$. Hence $g \notin U(\langle d_{kn} \rangle_{n=1}^\infty)$ and thus \mathcal{F}_D generates a Hausdorff group topology.

Now choose $C = \langle c_n \rangle_{n=1}^\infty$ and $D = \langle d_n \rangle_{n=1}^\infty$ in ζ with the property that for each $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that $c_{km} + m < d_m$ for all $m \geq N_k$. Suppose that $U(\langle d_n \rangle_{n=1}^\infty)$ is open in the topology generated by \mathcal{F}_C . Then there exists a k such that $U(\langle c_{kn} \rangle_{n=1}^\infty) \subseteq U(\langle d_n \rangle_{n=1}^\infty)$. We have that $a = \sum_{i=1}^{N_k} a_{c_{kN_k}+i} \in U(\langle c_{kn} \rangle_{n=1}^\infty)$. But by (2) and the fact that $c_{kN_k} + N_k < b_{N_k}$, we must conclude that $a \notin U(\langle d_n \rangle_{n=1}^\infty)$. Hence the group topology generated by \mathcal{F}_C is different from the group topology \mathcal{F}_D . By Lemma 2.3 we can find uncountably many different Hausdorff group topologies on G with the property that $\langle a_n \rangle_{n=1}^\infty$ converges to zero. \square

EXAMPLE 2.6. Let $\langle p^n \rangle_{n=1}^\infty$ be the sequence of powers of the prime p in \mathbb{Z} . $\langle p^n \rangle_{n=1}^\infty$ is a T_Ω -sequence.

EXAMPLE 2.7. Let $k \in \mathbb{N}$ and let $\langle a_n \rangle_{n=1}^\infty$ be an increasing sequence in \mathbb{Z} satisfying the inequality $a_{n+1}/a_n > n/k$ for all n . For $n > 2k$ we have that $\sum_{i=1}^m a_{n+i} < a_{n+m+1}$ for each $m \in \mathbb{N}$. Hence $\langle a_n \rangle_{n=1}^\infty$ is a T_Ω -sequence.

EXAMPLE 2.8. Let $Z \in \mathbb{Z}(p^\infty)$. The order of Z is p^n if Z is a p^n -root of unity but not a p^{n-1} -root of unity. We denote the order of Z by $O(Z)$. Now if $O(Z) = p^m$ and $O(w) = p^n$ and $m < n$ we have that $O(Zw) = p^n$. Let $\langle Z_n \rangle_{n=1}^\infty$ be a sequence in $\mathbb{Z}(p^\infty)$ satisfying

$$O(Z_{p^n+i}) \geq p^{n+1} O(Z_{p^n+i-1}) \quad \text{for all } n \in \mathbb{N} \text{ and for } 0 \leq i < p. \quad (2.2)$$

By Proposition 2.5, $\langle Z_n \rangle_{n=1}^\infty$ is a T_Ω -sequence.

EXAMPLE 2.9. Consider \mathbb{R} as the direct sum of uncountably many copies of \mathbb{Q} . If $\langle r_n \rangle_{n=1}^\infty$ is any sequence of linearly independent real numbers, then $\langle r_n \rangle_{n=1}^\infty$ is a T_Ω -sequence.

We end this paper with a question. Does there exist a nontrivial sequence in a group G which is a T -sequence, but not a T_Ω -sequence?

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BRADD CLARK: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, LAFAYETTE, LA 70504 1010, USA

E-mail address: bec1033@louisiana.edu

ROBERT LEDET: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, LAFAYETTE, LA 70504 1010, USA

E-mail address: rm12250@us1.edu

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