

ON 3-TOPOLOGICAL VERSION OF Θ -REGULARITY

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ABSTRACT. We modify the concept of θ -regularity for spaces with 2 and 3 topologies. The new, more general property is fully preserved by sums and products. Using some bitopological reductions of this property, Michael's theorem for several variants of bitopological paracompactness is proved.

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1. Preliminaries. The term *space* (X, τ, σ, ρ) is referred as a set X with three, generally nonidentical topologies τ, σ , and ρ . We say that $x \in X$ is a (σ, ρ) - θ -cluster point of a filter base Φ in X if for every $V \in \sigma$ such that $x \in V$ and every $F \in \Phi$ the intersection $F \cap \text{cl}_\rho V$ is nonempty. If, for every $V \in \sigma$ with $x \in V$, there is some $F \in \Phi$ with $F \subseteq \text{cl}_\rho V$, we say that $\Phi(\sigma, \rho)$ - θ -converges to x . Then x is called a (σ, ρ) - θ -limit of Φ . If Φ converges or has a cluster point with respect to the topology τ , we say that Φ τ -converges or has a τ -cluster point.

A family is called σ -locally finite if it consists of countably many locally finite subfamilies. (This notion has nothing common with the topology also denoted by σ .) For a family $\Phi \subseteq 2^X$, we denote by Φ^F the family of all finite unions of members of Φ . A family Φ is called *directed* if Φ^F is a refinement of Φ .

We say that the space (X, τ, σ, ρ) is $(\tau - \sigma)$ (semi-) paracompact with respect to ρ if every τ -open cover of X has a σ -open refinement which is (σ -) locally finite with respect to the topology ρ .

The bitopological space (X, τ, σ) is called RR-pairwise (semi-) paracompact if the space is $(\tau - \tau)$ (semi-) paracompact with respect to σ and $(\sigma - \sigma)$ (semi-) paracompact with respect to τ . We say that (X, τ, σ) is FHP-pairwise (semi-) paracompact if the space is $(\tau - \sigma)$ (semi-) paracompact with respect to σ and $(\sigma - \tau)$ (semi-) paracompact with respect to τ . Finally, (X, τ, σ) is said to be δ -pairwise (semi-) paracompact if the space is $(\tau - (\tau \vee \sigma))$ (semi-) paracompact with respect to $\tau \vee \sigma$ and $(\sigma - (\tau \vee \sigma))$ (semi-) paracompact with respect to $\tau \vee \sigma$ (see [7]).

Recall that the topological space (X, τ) is called (countably) θ -regular [2] if every (countable) filter base in (X, τ) with a θ -cluster point has a cluster point.

2. Main results

THEOREM 2.1. *Let τ, σ, ρ be topologies on X . The following statements are equivalent:*

- (i) *For every (countable) τ -open cover Ω of X and each $x \in X$ there is a σ -open*

neighborhood U of x such that $\text{cl}_\rho U$ can be covered by a finite subfamily of Ω .

- (ii) Every (countable) τ -closed filter base Φ with a (σ, ρ) - θ -cluster point has a τ -cluster point.
- (iii) Every (countable) filter base Φ with a (σ, ρ) - θ -cluster point has a τ -cluster point.
- (iv) For every (countable) filter base Φ in X with no τ -cluster point and every $x \in X$ there are $U \in \sigma$, $V \in \rho$, and $F \in \Phi$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$.

PROOF. Suppose (i). Let Φ be a (countable) filter base in X with no τ -cluster point. Then $\Omega = \{X \setminus \text{cl}_\tau F \mid F \in \Phi\}$ is a (countable) τ -open directed cover of X . Let $x \in X$. By (i) there is $U \in \sigma$ with $x \in U$ and $\text{cl}_\rho U \subseteq X \setminus \text{cl}_\tau F$ for some $F \in \Phi$. Denote $V = X \setminus \text{cl}_\rho U$. Then $x \in U$, $F \subseteq V \in \rho$ and $U \cap V = \emptyset$. It follows (iv).

The implications (iv) \Rightarrow (iii) \Rightarrow (ii) are clear. Suppose (ii). Take any (countable) τ -open cover Ω of X . Then $\Phi = \{X \setminus V \mid V \in \Omega^F\}$ is a τ -closed filter base in X with no τ -cluster point. Let $x \in X$. It follows from (ii) that x is not a (σ, ρ) - θ -cluster point of Φ , so there are some $U \in \sigma$ and $V \in \Omega^F$ such that $x \in U$ and $(X \setminus V) \cap \text{cl}_\rho U = \emptyset$. Then $\text{cl}_\rho U \subseteq V$, which implies (i). \square

DEFINITION 2.2. Let τ, σ, ρ be topologies on X . Then (X, τ, σ, ρ) is said to be (*countably*) (τ, σ, ρ) - θ -regular, if X satisfies any of the conditions (i)–(iv) of Theorem 2.1.

Note that for $\tau = \sigma = \rho$ we obtain the notion of (countably) θ -regular space. Omitting the condition of countability, we get further criteria of (τ, σ, ρ) - θ -regularity.

THEOREM 2.3. Let τ, σ, ρ be topologies on X . The following statements are equivalent:

- (i) X is (τ, σ, ρ) - θ -regular.
- (ii) Every (σ, ρ) - θ -convergent filter base Φ has a τ -cluster point.
- (iii) Every (σ, ρ) - θ -convergent ultrafilter in X is τ -convergent.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear. Conversely, suppose (iii) and take a filter base Φ in X with a (σ, ρ) - θ -cluster point $x \in X$. Let ζ be a σ -open local base of x . Then the family $\Phi' = \{F \cap \text{cl}_\rho V \mid F \in \Phi, V \in \zeta\}$ is a filter base finer than Φ and (σ, ρ) - θ -converging to x . Denote by Γ an ultrafilter finer than Φ' . Then $\Phi' \subseteq \Gamma$ and hence Γ also (σ, ρ) - θ -converges to x . By (iii), Γ is τ -convergent to some $y \in X$. Since Γ is finer than Φ , y is a τ -cluster point of Φ . \square

Similarly as for θ -regularity, there are numbers of simple examples of (τ, σ, ρ) - θ -regular spaces, including various modifications of regularity, compactness, local compactness, or paracompactness and we leave them to the reader. Note, for example, that a space $(\tau - \sigma)$ paracompact with respect to ρ is (τ, ρ, σ) - θ -regular.

REMARK 2.4. One can easily check that (τ, σ, ρ) - θ -regularity is preserved by τ -closed subspaces if we consider the corresponding induced topologies on the subspace. On the other hand, as it is shown in [3], even F_σ -subspace of a compact (non-Hausdorff) space need not be countably θ -regular.

For a family $\{(X_\iota, \tau_\iota, \sigma_\iota, \rho_\iota) \mid \iota \in I\}$ denote by τ, σ, ρ the corresponding sum (product) topologies on $X = \sum_{\iota \in I} X_\iota$ ($X = \prod_{\iota \in I} X_\iota$). It is an easy exercise to prove that the topological sum X of $(\tau_\iota, \sigma_\iota, \rho_\iota)$ - θ -regular spaces X_ι , where $\iota \in I$, is (τ, σ, ρ) - θ -regular.

THEOREM 2.5. *Let $X = \sum_{\iota \in I} X_\iota$ be the sum space for the family $\{(X_\iota, \tau_\iota, \sigma_\iota, \rho_\iota) \mid \iota \in I\}$ with the corresponding sum topologies τ, σ, ρ . Suppose that every X_ι is $(\tau_\iota, \sigma_\iota, \rho_\iota)$ - θ -regular. Then X is (τ, σ, ρ) - θ -regular.*

THEOREM 2.6. *Let $X = \prod_{\iota \in I} X_\iota$ be the product space for the family $\{(X_\iota, \tau_\iota, \sigma_\iota, \rho_\iota) \mid \iota \in I\}$ with the corresponding product topologies τ, σ, ρ . Suppose that every X_ι is $(\tau_\iota, \sigma_\iota, \rho_\iota)$ - θ -regular. Then X is (τ, σ, ρ) - θ -regular.*

PROOF. Let Γ be an ultrafilter in X with (σ, ρ) - θ -limit $x = (x_\iota)_{\iota \in I} \in X$. Let $\pi_\iota : X \rightarrow X_\iota$ be the canonical projection. Then $\pi_\iota(\Gamma)$ is an ultrafilter on X_ι which $(\sigma_\iota, \rho_\iota)$ - θ -converges to x_ι . But X_ι is $(\tau_\iota, \sigma_\iota, \rho_\iota)$ - θ -regular. Hence, $\pi_\iota(\Gamma)$ τ_ι -converges to some $y_\iota \in X_\iota$, which implies that Γ τ -converges to $y = (y_\iota)_{\iota \in I}$. It follows that X is (τ, σ, ρ) - θ -regular. \square

The productivity of θ -regularity proved in [4] by a different technique now follows as a corollary.

DEFINITION 2.7. A bitopological space (X, τ, σ) is said to be α -pairwise (countably) θ -regular if X is (countably) (τ, τ, σ) - θ -regular and (countably) (σ, σ, τ) - θ -regular, β -pairwise (countably) θ -regular if X is (countably) (τ, σ, τ) - θ -regular and (countably) (σ, τ, σ) - θ -regular, γ -pairwise (countably) θ -regular if X is (countably) (τ, σ, σ) - θ -regular and (countably) (σ, τ, τ) - θ -regular and finally, δ -pairwise (countably) θ -regular if X is (countably) $(\tau \vee \sigma, \sigma, \tau \vee \sigma)$ - θ -regular and (countably) $(\tau \vee \sigma, \tau, \tau \vee \sigma)$ - θ -regular.

REMARK 2.8. Using the characterization (i) in Theorem 2.1 and refining the open covers of the space several times, one can easily check that β - and γ -versions of pairwise θ -regularity are equivalent and imply the α -version, but not vice versa. Since every pairwise regular space obviously is α -pairwise θ -regular, the real line topologized by the intervals $(-\infty, p)$, $p \in \mathbb{R}$ for τ and (q, ∞) , $q \in \mathbb{R}$ for σ is a proper counterexample.

REMARK 2.9. Observe that RR-pairwise paracompact and FHP-pairwise paracompact spaces are β -pairwise θ -regular and it can be easily seen that a β -pairwise θ -regular space has both topologies θ -regular.

However, for the following bitopological modifications of well-known Michael's theorem [5], only the β - and δ -versions of pairwise (countable) θ -regularity will be useful. In the proof of the next theorem, we slightly modify the technique used in [3].

THEOREM 2.10. *Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, be topologies on X . Let X be $(\sigma_1 - \sigma_2)$ semiparacompact with respect to σ_3 , $(\sigma_4 - \sigma_3)$ semiparacompact with respect to σ_2 and countably $(\sigma_2, \sigma_4, \sigma_2)$ - θ -regular. Then X is $(\sigma_1 - \sigma_2)$ paracompact with respect to σ_3 .*

PROOF. Let Ω be a σ_1 -open cover of X . Since X is $(\sigma_1 - \sigma_2)$ semiparacompact with respect to σ_3 , it follows that Ω has a σ_2 -open refinement, say $\Omega' = \bigcup_{i=1}^{\infty} \Omega_i$, where every Ω_i is a locally finite with respect to σ_3 family refining Ω .

Let $U_n = \bigcup\{U \mid U \in \Omega_i, i \leq n\}$ for every $n \in \mathbb{N}$. The family $\{U_n\}_{n \in \mathbb{N}}$ is a countable σ_2 -open increasing cover of X and since X is countably $(\sigma_2, \sigma_4, \sigma_2)$ - θ -regular, there exists a σ_4 -open cover Φ of X whose σ_2 -closures refine $\{U_n\}_{n \in \mathbb{N}}$. Because X is $(\sigma_4 - \sigma_3)$

semiparacompact with respect to σ_2 , Φ has a σ_3 -open refinement, say $\Phi' = \bigcup_{i=1}^{\infty} \Phi_i$, consisting of families Φ_i which are locally finite with respect to σ_2 . For every $n \in \mathbb{N}$, let

$$V_n = \bigcup \{B \mid B \in \Phi_i, \text{ cl}_{\sigma_2} B \subseteq U_j, i + j \leq n\}. \quad (2.1)$$

The family $\{V_n\}_{n \in \mathbb{N}}$ is a σ_3 -open increasing cover of X . Because the family $\bigcup_{i=1}^n \Phi_i$ is locally finite with respect to σ_2 , we have $\text{cl}_{\sigma_2} V_n \subseteq U_{n-1}$. Finally, for every $n \in \mathbb{N}$ and $U \in \Omega_n$, let

$$W_n(U) = U \setminus \text{cl}_{\sigma_2} V_n. \quad (2.2)$$

It can be easily seen that the family $\Gamma = \{W_n(U) \mid n \in \mathbb{N}, U \in \Omega_n\}$ is a σ_2 -open cover of X which is a refinement of Ω locally finite with respect to σ_3 . Indeed, for every $x \in X$ let $k \in \mathbb{N}$ be the least index such that $x \in U$ for some $U \in \Omega_k$. Since $\text{cl}_{\sigma_2} V_k \subseteq U_{k-1}$, it follows that $x \in W_k(U)$. Hence Γ is a σ_2 -open cover which, obviously, refines Ω . To see that Γ is locally finite with respect to σ_3 , let $x \in X$ and let $m \in \mathbb{N}$ be any index such that $x \in V_m$. Because $\{V_n\}_{n \in \mathbb{N}}$ is an increasing family, we have $V_m \cap W_n(U) = \emptyset$ for every $n \geq m$, $U \in \Omega_n$.

But the family $\bigcup_{i=1}^m \Omega_i$ is locally finite with respect to σ_3 . Let S be a σ_3 -neighborhood of x , intersecting at most finitely many elements of $\bigcup_{i=1}^m \Omega_i$. Since for every $i = 1, 2, \dots, m$, $U \in \Omega_i$, we have $W_i(U) \subseteq U$, the set $S \cap V_m$ is a σ_3 -neighborhood of x , meeting only finitely many sets of the cover Γ . Hence Γ is locally finite with respect to σ_3 and therefore X is $(\sigma_1 - \sigma_2)$ paracompact with respect to σ_3 . \square

In order to obtain a theorem for a bitopological space (X, τ_1, τ_2) from Theorem 2.10 it can be easily seen that there are only three meaningful possibilities for identifying the topologies $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.

CASE (i). $\tau_1 = \sigma_1 = \sigma_4$ and $\tau_2 = \sigma_2 = \sigma_3$.

COROLLARY 2.11. *Let X be countably (τ_2, τ_1, τ_2) - θ -regular and $(\tau_1 - \tau_2)$ semiparacompact with respect to τ_2 . Then X is $(\tau_1 - \tau_2)$ paracompact with respect to τ_2 .*

COROLLARY 2.12. *Let X be a bitopological space. Then X is FHP-pairwise paracompact if and only if X is β -pairwise countably θ -regular and FHP-pairwise semiparacompact.*

PROOF. It is sufficient to use the previous corollary twice. \square

Note that Raghavan and Reilly stated [7, Theorem 3.9] from which it would follow that a pairwise regular δ -pairwise semiparacompact space is δ -pairwise paracompact. Unfortunately, (iv) \Rightarrow (i) in the proof of this theorem is not correct. The authors used [1, Theorem 1.5, page 162] in the proof. However, the assumptions of the theorem are not completely satisfied. They tried to expand a locally finite cover \mathcal{V} to the open one using a closed cover such that every its element meets only finitely many members of \mathcal{V} . However, in general the used closed cover is not locally finite or at least closure preserving. That is not sufficient for the expansion, as the following example shows.

EXAMPLE 2.13. Let $C = \mathbb{N} \times (-1, 1)$, $B = \mathbb{N} \times (0, 1)$, and $A = \mathbb{N} \times (-1, 0)$. We consider the Euclidean topology on C induced from the real plane and let $X = C \cup \{y \mid$

γ is a nonconvergent ultra-closed filter in C , $B \in \gamma$. Let $S(U) = U \cup \{y \mid y \in X \setminus C, U \in \gamma\}$ for any $U \subseteq C$ open in C . Of course, X is a subspace of the Wallman compactification ωC and the sets $S(U)$ constitute a topology base for X . Since C is normal, ωC is Hausdorff and hence X is a $T_{3.5}$ space. Denote $A_n = \{n\} \times \langle -1, 0 \rangle$. The family $\Omega = \{S(B), A_1, A_2, A_3, \dots\}$ is a locally finite cover of X , which has no open locally finite extension.

Indeed, suppose that there are some open U_n with $A_n \subseteq U_n$ for $n \in \mathbb{N}$. Then every U_n must meet $B_n = \{n\} \times (0, 1)$. Choose $x_n \in U_n \cap B_n$ for each $n \in \mathbb{N}$. Let $F_n = \{x_n, x_{n+1}, \dots\}$. Since the sequence x_1, x_2, \dots has no cluster point in C , the collection $\Phi = \{F_n \mid n = 1, 2, \dots\}$ is a closed filter base in C with no cluster point in C . It follows that there is a non-convergent ultra-closed filter, say $\gamma \in \omega C$, finer than Φ . But $F_1 \subseteq B$ and since $F_1 \in \Phi \subseteq \gamma$, $B \in \gamma$. Hence $\gamma \in X$. Let W be any open neighborhood of γ in X . There is some V open in C with $\gamma \in S(V) \subseteq W$. Then $V \in \gamma$ and hence $V \cap F_n \neq \emptyset$ for every $n \in \mathbb{N}$. Thus for any fixed $m \in \mathbb{N}$ there exists $n \geq m$ such that $x_n \in V \subseteq S(V) \subseteq W$ and therefore W intersects infinitely many elements of $\{U_n \mid n = 1, 2, \dots\}$. Hence Ω cannot be expanded to an open locally finite cover.

On the other hand, the previous example does not refute Raghavan-Reilly's theorem, which still remains open as a question. With a different modification of the concept of pairwise regularity the theorem is correct.

COROLLARY 2.14. *Let X be δ -pairwise countably θ -regular. Then X is δ -pairwise paracompact if and only if X is δ -pairwise semiparacompact.*

PROOF. Since X is countably $(\tau_1 \vee \tau_2, \tau_1, \tau_1 \vee \tau_2)$ - θ -regular and $(\tau_1 - (\tau_1 \vee \tau_2))$ semiparacompact with respect to $\tau_1 \vee \tau_2$, it follows that X is $(\tau_1 - (\tau_1 \vee \tau_2))$ paracompact with respect to $\tau_1 \vee \tau_2$ by Corollary 2.11. But X is also countably $(\tau_1 \vee \tau_2, \tau_2, \tau_1 \vee \tau_2)$ - θ -regular and $(\tau_2 - (\tau_1 \vee \tau_2))$ semiparacompact with respect to $\tau_1 \vee \tau_2$ which implies, also by Corollary 2.11, that X is $(\tau_2 - (\tau_1 \vee \tau_2))$ paracompact with respect to $\tau_1 \vee \tau_2$. Hence X is δ -pairwise paracompact in topologies τ_1, τ_2 . \square

REMARK 2.15. Note that the space X constructed in Example 2.13 is $T_{3.5}$ but not normal—the sets $A, X \setminus C$ are closed, pairwise disjoint but they have no disjoint neighborhoods.

CASE (ii). $\tau_1 = \sigma_1 = \sigma_2$ and $\tau_2 = \sigma_3 = \sigma_4$.

COROLLARY 2.16. *Let X be a bitopological space. Then X is RR-pairwise paracompact if and only if X is β -pairwise countably θ -regular and RR-pairwise semiparacompact.*

CASE (iii). $\tau_1 = \sigma_1 = \sigma_3$ and $\tau_2 = \sigma_2 = \sigma_4$.

COROLLARY 2.17. *Let τ_1, τ_2 be countably θ -regular topologies of X . Suppose that X is $(\tau_1 - \tau_2)$ semiparacompact with respect to τ_1 , and $(\tau_2 - \tau_1)$ semiparacompact with respect to τ_2 . Then X is $(\tau_1 - \tau_2)$ paracompact with respect to τ_1 and $(\tau_2 - \tau_1)$ paracompact with respect to τ_2 .*

Finally, remark that modifying properly the concept of Σ -space for bitopological spaces, combining Theorem 2.6 and the corollaries of Theorem 2.10 similar results as

in [4] (see [6, Nagami's theorem]) for the countable product of paracompact Σ -spaces without necessity of Hausdorff-type separation are also possible.

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