

ON 3-TOPOLOGICAL VERSION OF Θ -REGULARITY

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(Received 14 July 1998)

ABSTRACT. We modify the concept of θ -regularity for spaces with 2 and 3 topologies. The new, more general property is fully preserved by sums and products. Using some bitopological reductions of this property, Michael's theorem for several variants of bitopological paracompactness is proved.

Keywords and phrases. Spaces with 2 and 3 topologies, θ -regularity, paracompactness.

2000 Mathematics Subject Classification. Primary 54E55, 54A10; Secondary 54D20, 54A20.

1. Preliminaries. The term *space* (X, τ, σ, ρ) is referred as a set X with three, generally nonidentical topologies τ, σ , and ρ . We say that $x \in X$ is a (σ, ρ) - θ -cluster point of a filter base Φ in X if for every $V \in \sigma$ such that $x \in V$ and every $F \in \Phi$ the intersection $F \cap \text{cl}_\rho V$ is nonempty. If, for every $V \in \sigma$ with $x \in V$, there is some $F \in \Phi$ with $F \subseteq \text{cl}_\rho V$, we say that $\Phi(\sigma, \rho)$ - θ -converges to x . Then x is called a (σ, ρ) - θ -limit of Φ . If Φ converges or has a cluster point with respect to the topology τ , we say that Φ τ -converges or has a τ -cluster point.

A family is called σ -locally finite if it consists of countably many locally finite subfamilies. (This notion has nothing common with the topology also denoted by σ .) For a family $\Phi \subseteq 2^X$, we denote by Φ^F the family of all finite unions of members of Φ . A family Φ is called *directed* if Φ^F is a refinement of Φ .

We say that the space (X, τ, σ, ρ) is $(\tau - \sigma)$ (semi-) *paracompact with respect to ρ* if every τ -open cover of X has a σ -open refinement which is $(\sigma -)$ locally finite with respect to the topology ρ .

The bitopological space (X, τ, σ) is called *RR-pairwise (semi-) paracompact* if the space is $(\tau - \tau)$ (semi-) paracompact with respect to σ and $(\sigma - \sigma)$ (semi-) paracompact with respect to τ . We say that (X, τ, σ) is *FHP-pairwise (semi-) paracompact* if the space is $(\tau - \sigma)$ (semi-) paracompact with respect to σ and $(\sigma - \tau)$ (semi-) paracompact with respect to τ . Finally, (X, τ, σ) is said to be δ -pairwise (semi-) *paracompact* if the space is $(\tau - (\tau \vee \sigma))$ (semi-) paracompact with respect to $\tau \vee \sigma$ and $(\sigma - (\tau \vee \sigma))$ (semi-) paracompact with respect to $\tau \vee \sigma$ (see [7]).

Recall that the topological space (X, τ) is called (countably) θ -regular [2] if every (countable) filter base in (X, τ) with a θ -cluster point has a cluster point.

2. Main results

THEOREM 2.1. *Let τ, σ, ρ be topologies on X . The following statements are equivalent:*

- (i) *For every (countable) τ -open cover Ω of X and each $x \in X$ there is a σ -open*

neighborhood U of x such that $\text{cl}_\rho U$ can be covered by a finite subfamily of Ω .

(ii) Every (countable) τ -closed filter base Φ with a (σ, ρ) - θ -cluster point has a τ -cluster point.

(iii) Every (countable) filter base Φ with a (σ, ρ) - θ -cluster point has a τ -cluster point.

(iv) For every (countable) filter base Φ in X with no τ -cluster point and every $x \in X$ there are $U \in \sigma$, $V \in \rho$, and $F \in \Phi$ such that $x \in U$, $F \subseteq V$, and $U \cap V = \emptyset$.

PROOF. Suppose (i). Let Φ be a (countable) filter base in X with no τ -cluster point. Then $\Omega = \{X \setminus \text{cl}_\tau F \mid F \in \Phi\}$ is a (countable) τ -open directed cover of X . Let $x \in X$. By (i) there is $U \in \sigma$ with $x \in U$ and $\text{cl}_\rho U \subseteq X \setminus \text{cl}_\tau F$ for some $F \in \Phi$. Denote $V = X \setminus \text{cl}_\rho U$. Then $x \in U$, $F \subseteq V \in \rho$ and $U \cap V = \emptyset$. It follows (iv).

The implications (iv) \Rightarrow (iii) \Rightarrow (ii) are clear. Suppose (ii). Take any (countable) τ -open cover Ω of X . Then $\Phi = \{X \setminus V \mid V \in \Omega^F\}$ is a τ -closed filter base in X with no τ -cluster point. Let $x \in X$. It follows from (ii) that x is not a (σ, ρ) - θ -cluster point of Φ , so there are some $U \in \sigma$ and $V \in \Omega^F$ such that $x \in U$ and $(X \setminus V) \cap \text{cl}_\rho U = \emptyset$. Then $\text{cl}_\rho U \subseteq V$, which implies (i). \square

DEFINITION 2.2. Let τ, σ, ρ be topologies on X . Then (X, τ, σ, ρ) is said to be (countably) (τ, σ, ρ) - θ -regular, if X satisfies any of the conditions (i)–(iv) of Theorem 2.1.

Note that for $\tau = \sigma = \rho$ we obtain the notion of (countably) θ -regular space. Omitting the condition of countability, we get further criteria of (τ, σ, ρ) - θ -regularity.

THEOREM 2.3. Let τ, σ, ρ be topologies on X . The following statements are equivalent:

- (i) X is (τ, σ, ρ) - θ -regular.
- (ii) Every (σ, ρ) - θ -convergent filter base Φ has a τ -cluster point.
- (iii) Every (σ, ρ) - θ -convergent ultrafilter in X is τ -convergent.

PROOF. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear. Conversely, suppose (iii) and take a filter base Φ in X with a (σ, ρ) - θ -cluster point $x \in X$. Let ζ be a σ -open local base of x . Then the family $\Phi' = \{F \cap \text{cl}_\rho V \mid F \in \Phi, V \in \zeta\}$ is a filter base finer than Φ and (σ, ρ) - θ -converging to x . Denote by Γ an ultrafilter finer than Φ' . Then $\Phi' \subseteq \Gamma$ and hence Γ also (σ, ρ) - θ -converges to x . By (iii), Γ is τ -convergent to some $y \in X$. Since Γ is finer than Φ , y is a τ -cluster point of Φ . \square

Similarly as for θ -regularity, there are numbers of simple examples of (τ, σ, ρ) - θ -regular spaces, including various modifications of regularity, compactness, local compactness, or paracompactness and we leave them to the reader. Note, for example, that a space $(\tau - \sigma)$ paracompact with respect to ρ is (τ, ρ, σ) - θ -regular.

REMARK 2.4. One can easily check that (τ, σ, ρ) - θ -regularity is preserved by τ -closed subspaces if we consider the corresponding induced topologies on the subspace. On the other hand, as it is shown in [3], even F_σ -subspace of a compact (non-Hausdorff) space need not be countably θ -regular.

For a family $\{(X_\iota, \tau_\iota, \sigma_\iota, \rho_\iota) \mid \iota \in I\}$ denote by τ, σ, ρ the corresponding sum (product) topologies on $X = \sum_{\iota \in I} X_\iota$ ($X = \prod_{\iota \in I} X_\iota$). It is an easy exercise to prove that the topological sum X of $(\tau_\iota, \sigma_\iota, \rho_\iota)$ - θ -regular spaces X_ι , where $\iota \in I$, is (τ, σ, ρ) - θ -regular.

THEOREM 2.5. Let $X = \sum_{i \in I} X_i$ be the sum space for the family $\{(X_i, \tau_i, \sigma_i, \rho_i) \mid i \in I\}$ with the corresponding sum topologies τ, σ, ρ . Suppose that every X_i is $(\tau_i, \sigma_i, \rho_i)$ - θ -regular. Then X is (τ, σ, ρ) - θ -regular.

THEOREM 2.6. Let $X = \prod_{i \in I} X_i$ be the product space for the family $\{(X_i, \tau_i, \sigma_i, \rho_i) \mid i \in I\}$ with the corresponding product topologies τ, σ, ρ . Suppose that every X_i is $(\tau_i, \sigma_i, \rho_i)$ - θ -regular. Then X is (τ, σ, ρ) - θ -regular.

PROOF. Let Γ be an ultrafilter in X with (σ, ρ) - θ -limit $x = (x_i)_{i \in I} \in X$. Let $\pi_i : X \rightarrow X_i$ be the canonical projection. Then $\pi_i(\Gamma)$ is an ultrafilter on X_i which (σ_i, ρ_i) - θ -converges to x_i . But X_i is $(\tau_i, \sigma_i, \rho_i)$ - θ -regular. Hence, $\pi_i(\Gamma)$ τ_i -converges to some $y_i \in X_i$, which implies that Γ τ -converges to $y = (y_i)_{i \in I}$. It follows that X is (τ, σ, ρ) - θ -regular. \square

The productivity of θ -regularity proved in [4] by a different technique now follows as a corollary.

DEFINITION 2.7. A bitopological space (X, τ, σ) is said to be α -pairwise (countably) θ -regular if X is (countably) (τ, τ, σ) - θ -regular and (countably) (σ, σ, τ) - θ -regular, β -pairwise (countably) θ -regular if X is (countably) (τ, σ, τ) - θ -regular and (countably) (σ, τ, σ) - θ -regular, γ -pairwise (countably) θ -regular if X is (countably) (τ, σ, σ) - θ -regular and (countably) (σ, τ, τ) - θ -regular and finally, δ -pairwise (countably) θ -regular if X is (countably) $(\tau \vee \sigma, \sigma, \tau \vee \sigma)$ - θ -regular and (countably) $(\tau \vee \sigma, \tau, \tau \vee \sigma)$ - θ -regular.

REMARK 2.8. Using the characterization (i) in Theorem 2.1 and refining the open covers of the space several times, one can easily check that β - and γ -versions of pairwise θ -regularity are equivalent and imply the α -version, but not vice versa. Since every pairwise regular space obviously is α -pairwise θ -regular, the real line topologized by the intervals $(-\infty, p)$, $p \in \mathbb{R}$ for τ and (q, ∞) , $q \in \mathbb{R}$ for σ is a proper counterexample.

REMARK 2.9. Observe that RR-pairwise paracompact and FHP-pairwise paracompact spaces are β -pairwise θ -regular and it can be easily seen that a β -pairwise θ -regular space has both topologies θ -regular.

However, for the following bitopological modifications of well-known Michael's theorem [5], only the β - and δ -versions of pairwise (countable) θ -regularity will be useful. In the proof of the next theorem, we slightly modify the technique used in [3].

THEOREM 2.10. Let $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, be topologies on X . Let X be $(\sigma_1 - \sigma_2)$ semi-paracompact with respect to σ_3 , $(\sigma_4 - \sigma_3)$ semiparacompact with respect to σ_2 and countably $(\sigma_2, \sigma_4, \sigma_2)$ - θ -regular. Then X is $(\sigma_1 - \sigma_2)$ paracompact with respect to σ_3 .

PROOF. Let Ω be a σ_1 -open cover of X . Since X is $(\sigma_1 - \sigma_2)$ semiparacompact with respect to σ_3 , it follows that Ω has a σ_2 -open refinement, say $\Omega' = \bigcup_{i=1}^{\infty} \Omega_i$, where every Ω_i is a locally finite with respect to σ_3 family refining Ω .

Let $U_n = \bigcup \{U \mid U \in \Omega_i, i \leq n\}$ for every $n \in \mathbb{N}$. The family $\{U_n\}_{n \in \mathbb{N}}$ is a countable σ_2 -open increasing cover of X and since X is countably $(\sigma_2, \sigma_4, \sigma_2)$ - θ -regular, there exists a σ_4 -open cover Φ of X whose σ_2 -closures refine $\{U_n\}_{n \in \mathbb{N}}$. Because X is $(\sigma_4 - \sigma_3)$

semiparacompact with respect to σ_2 , Φ has a σ_3 -open refinement, say $\Phi' = \bigcup_{i=1}^{\infty} \Phi_i$, consisting of families Φ_i which are locally finite with respect to σ_2 . For every $n \in \mathbb{N}$, let

$$V_n = \bigcup \{B \mid B \in \Phi_i, \text{cl}_{\sigma_2} B \subseteq U_j, i+j \leq n\}. \quad (2.1)$$

The family $\{V_n\}_{n \in \mathbb{N}}$ is a σ_3 -open increasing cover of X . Because the family $\bigcup_{i=1}^n \Phi_i$ is locally finite with respect to σ_2 , we have $\text{cl}_{\sigma_2} V_n \subseteq U_{n-1}$. Finally, for every $n \in \mathbb{N}$ and $U \in \Omega_n$, let

$$W_n(U) = U \setminus \text{cl}_{\sigma_2} V_n. \quad (2.2)$$

It can be easily seen that the family $\Gamma = \{W_n(U) \mid n \in \mathbb{N}, U \in \Omega_n\}$ is a σ_2 -open cover of X which is a refinement of Ω locally finite with respect to σ_3 . Indeed, for every $x \in X$ let $k \in \mathbb{N}$ be the least index such that $x \in U$ for some $U \in \Omega_k$. Since $\text{cl}_{\sigma_2} V_k \subseteq U_{k-1}$, it follows that $x \in W_k(U)$. Hence Γ is a σ_2 -open cover which, obviously, refines Ω . To see that Γ is locally finite with respect to σ_3 , let $x \in X$ and let $m \in \mathbb{N}$ be any index such that $x \in V_m$. Because $\{V_n\}_{n \in \mathbb{N}}$ is an increasing family, we have $V_m \cap W_n(U) = \emptyset$ for every $n \geq m$, $U \in \Omega_n$.

But the family $\bigcup_{i=1}^m \Omega_i$ is locally finite with respect to σ_3 . Let S be a σ_3 -neighborhood of x , intersecting at most finitely many elements of $\bigcup_{i=1}^m \Omega_i$. Since for every $i = 1, 2, \dots, m$, $U \in \Omega_i$, we have $W_i(U) \subseteq U$, the set $S \cap V_m$ is a σ_3 -neighborhood of x , meeting only finitely many sets of the cover Γ . Hence Γ is locally finite with respect to σ_3 and therefore X is $(\sigma_1 - \sigma_2)$ paracompact with respect to σ_3 . \square

In order to obtain a theorem for a bitopological space (X, τ_1, τ_2) from Theorem 2.10 it can be easily seen that there are only three meaningful possibilities for identifying the topologies $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.

CASE (i). $\tau_1 = \sigma_1 = \sigma_4$ and $\tau_2 = \sigma_2 = \sigma_3$.

COROLLARY 2.11. *Let X be countably (τ_2, τ_1, τ_2) - θ -regular and $(\tau_1 - \tau_2)$ semiparacompact with respect to τ_2 . Then X is $(\tau_1 - \tau_2)$ paracompact with respect to τ_2 .*

COROLLARY 2.12. *Let X be a bitopological space. Then X is FHP-pairwise paracompact if and only if X is β -pairwise countably θ -regular and FHP-pairwise semiparacompact.*

PROOF. It is sufficient to use the previous corollary twice. \square

Note that Raghavan and Reilly stated [7, Theorem 3.9] from which it would follow that a pairwise regular δ -pairwise semiparacompact space is δ -pairwise paracompact. Unfortunately, (iv) \Rightarrow (i) in the proof of this theorem is not correct. The authors used [1, Theorem 1.5, page 162] in the proof. However, the assumptions of the theorem are not completely satisfied. They tried to expand a locally finite cover \mathcal{V} to the open one using a closed cover such that every its element meets only finitely many members of \mathcal{V} . However, in general the used closed cover is not locally finite or at least closure preserving. That is not sufficient for the expansion, as the following example shows.

EXAMPLE 2.13. Let $C = \mathbb{N} \times \langle -1, 1 \rangle$, $B = \mathbb{N} \times (0, 1)$, and $A = \mathbb{N} \times \langle -1, 0 \rangle$. We consider the Euclidean topology on C induced from the real plane and let $X = C \cup \{y \mid$

\mathcal{Y} is a nonconvergent ultra-closed filter in C , $B \in \mathcal{Y}$. Let $S(U) = U \cup \{\mathcal{Y} \mid \mathcal{Y} \in X \setminus C, U \in \mathcal{Y}\}$ for any $U \subseteq C$ open in C . Of course, X is a subspace of the Wallman compactification ωC and the sets $S(U)$ constitute a topology base for X . Since C is normal, ωC is Hausdorff and hence X is a $T_{3.5}$ space. Denote $A_n = \{n\} \times \langle -1, 0 \rangle$. The family $\Omega = \{S(B), A_1, A_2, A_3, \dots\}$ is a locally finite cover of X , which has no open locally finite extension.

Indeed, suppose that there are some open U_n with $A_n \subseteq U_n$ for $n \in \mathbb{N}$. Then every U_n must meet $B_n = \{n\} \times (0, 1)$. Choose $x_n \in U_n \cap B_n$ for each $n \in \mathbb{N}$. Let $F_n = \{x_n, x_{n+1}, \dots\}$. Since the sequence x_1, x_2, \dots has no cluster point in C , the collection $\Phi = \{F_n \mid n = 1, 2, \dots\}$ is a closed filter base in C with no cluster point in C . It follows that there is a non-convergent ultra-closed filter, say $\mathcal{Y} \in \omega C$, finer than Φ . But $F_1 \subseteq B$ and since $F_1 \in \Phi \subseteq \mathcal{Y}$, $B \in \mathcal{Y}$. Hence $\mathcal{Y} \in X$. Let W be any open neighborhood of \mathcal{Y} in X . There is some V open in C with $\mathcal{Y} \in S(V) \subseteq W$. Then $V \in \mathcal{Y}$ and hence $V \cap F_n \neq \emptyset$ for every $n \in \mathbb{N}$. Thus for any fixed $m \in \mathbb{N}$ there exists $n \geq m$ such that $x_n \in V \subseteq S(V) \subseteq W$ and therefore W intersects infinitely many elements of $\{U_n \mid n = 1, 2, \dots\}$. Hence Ω cannot be expanded to an open locally finite cover.

On the other hand, the previous example does not refute Raghavan-Reilly's theorem, which still remains open as a question. With a different modification of the concept of pairwise regularity the theorem is correct.

COROLLARY 2.14. *Let X be δ -pairwise countably θ -regular. Then X is δ -pairwise paracompact if and only if X is δ -pairwise semiparacompact.*

PROOF. Since X is countably $(\tau_1 \vee \tau_2, \tau_1, \tau_1 \vee \tau_2)$ - θ -regular and $(\tau_1 - (\tau_1 \vee \tau_2))$ semiparacompact with respect to $\tau_1 \vee \tau_2$, it follows that X is $(\tau_1 - (\tau_1 \vee \tau_2))$ paracompact with respect to $\tau_1 \vee \tau_2$ by Corollary 2.11. But X is also countably $(\tau_1 \vee \tau_2, \tau_2, \tau_1 \vee \tau_2)$ - θ -regular and $(\tau_2 - (\tau_1 \vee \tau_2))$ semiparacompact with respect to $\tau_1 \vee \tau_2$ which implies, also by Corollary 2.11, that X is $(\tau_2 - (\tau_1 \vee \tau_2))$ paracompact with respect to $\tau_1 \vee \tau_2$. Hence X is δ -pairwise paracompact in topologies τ_1, τ_2 . \square

REMARK 2.15. Note that the space X constructed in Example 2.13 is $T_{3.5}$ but not normal—the sets $A, X \setminus C$ are closed, pairwise disjoint but they have no disjoint neighborhoods.

CASE (ii). $\tau_1 = \sigma_1 = \sigma_2$ and $\tau_2 = \sigma_3 = \sigma_4$.

COROLLARY 2.16. *Let X be a bitopological space. Then X is RR-pairwise paracompact if and only if X is β -pairwise countably θ -regular and RR-pairwise semiparacompact.*

CASE (iii). $\tau_1 = \sigma_1 = \sigma_3$ and $\tau_2 = \sigma_2 = \sigma_4$.

COROLLARY 2.17. *Let τ_1, τ_2 be countably θ -regular topologies of X . Suppose that X is $(\tau_1 - \tau_2)$ semiparacompact with respect to τ_1 , and $(\tau_2 - \tau_1)$ semiparacompact with respect to τ_2 . Then X is $(\tau_1 - \tau_2)$ paracompact with respect to τ_1 and $(\tau_2 - \tau_1)$ paracompact with respect to τ_2 .*

Finally, remark that modifying properly the concept of Σ -space for bitopological spaces, combining Theorem 2.6 and the corollaries of Theorem 2.10 similar results as

in [4] (see [6, Nagami's theorem]) for the countable product of paracompact Σ -spaces without necessity of Hausdorff-type separation are also possible.

ACKNOWLEDGEMENT. This research is supported by grant GA ČR 201/97/0216.

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