

ON A FUNCTIONAL EQUATION RELATED TO A GENERALIZATION OF FLETT'S MEAN VALUE THEOREM

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ABSTRACT. In this paper, we characterize all the functions that attain their Flett mean value at a particular point between the endpoints of the interval under consideration. These functions turn out to be cubic polynomials and thus, we also characterize these.

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1. Introduction. In [5], Sahoo and Riedel gave a generalization of Flett's mean value theorem [2] as follows.

THEOREM 1.1. *Let f be a real valued function which is differentiable in $[a, b]$, then there is a point $c \in (a, b)$ such that*

$$f(c) - f(a) = (c - a)f'(c) - \frac{1}{2} \frac{f'(b) - f'(a)}{b - a}(c - a)^2. \quad (1.1)$$

It is easy to see that if $f'(b) = f'(a)$, then this reduces to Flett's mean value theorem.

Aczél [1] and Haruki [3] used the Lagrange mean value theorem to ask the question of which functions attained their mean value at a prescribed point $c \in (a, b)$, in particular, at the midpoint $c = (a + b)/2$. The answer is that only quadratic polynomials have the property that the mean value on any interval is attained at the midpoint of that interval. A natural question to ask is this same question for the above mean value theorem. It turns out that quadratic polynomials satisfy (1.1) for any c , but, more interestingly, cubic polynomials satisfy it for $c = (a + 3b)/4$. Thus, the main question becomes whether cubic polynomials are the only functions having this property.

Following the approach in [1], we *pedesterize* (1.1) to obtain

$$f(c) - f(a) = (c - a)h(c) - \frac{1}{2} \frac{h(b) - h(a)}{b - a}(c - a)^2, \quad (1.2)$$

and now setting $c = (a + 3b)/4$ yields

$$f\left(\frac{a+3b}{4}\right) - f(a) = \frac{3}{4}(b - a)h\left(\frac{a+3b}{4}\right) - \frac{9}{32}(b - a)(h(b) - h(a)) \quad (1.3)$$

or

$$f\left(\frac{a+3b}{4}\right) - f(a) = \frac{3}{4}(b - a) \left[h\left(\frac{a+3b}{4}\right) - \frac{3}{8}(h(b) - h(a)) \right]. \quad (1.4)$$

More generally, setting $c = sa + tb$ with $s + t = 1$ and $0 < s, t < 1$, we obtain

$$f(sa + tb) - f(a) = (sa + tb - a)h(sa + tb) - \frac{1}{2} \frac{h(b) - h(a)}{b - a} (sa + tb - a)^2. \quad (1.5)$$

The question we answer, in this paper, is: What are the functions f, h that satisfy the functional equations (1.4) and (1.5) for all $a, b \in \mathbb{R}$? In solving this functional equation, we do not assume any regularity conditions on f or h .

2. Solution of the functional equation. The main work in solving this functional equation is to reduce (1.4) and (1.5) to a form where we can apply the following result by Székelyhidi [6, Thm. 9.5] and Wilson [7].

THEOREM 2.1. *Let G, S be commutative groups, n a nonnegative integer, φ_i, ψ_i additive functions from G into G and let $\text{Ran}(\varphi_i) \subseteq \text{Ran}(\psi_i)$ ($i = 1, \dots, n + 1$). Then if $h, h_i, \varphi_i, \psi_i$ ($i = 1, \dots, n + 1$) satisfy*

$$h(x) + \sum_{i=1}^{n+1} h_i(\varphi_i(x) + \psi_i(t)) = 0, \quad (2.1)$$

then h is a generalized polynomial of degree at most n .

Thus, we are able to prove our main result (Theorem 2.2).

THEOREM 2.2. *The real valued functions f and h are solutions of the functional equation (1.5) if and only if*

$$f(x) = \begin{cases} Ax^3 + Bx^2 + Cx + D & \text{if } s = \frac{1}{4}, t = \frac{3}{4}, \\ Bx^2 + Cx + D & \text{if } s \neq \frac{1}{4}, t \neq \frac{3}{4}, \end{cases} \quad (2.2)$$

$$h(x) = \begin{cases} 3Ax^2 + 2Bx + C & \text{if } s = \frac{1}{4}, t = \frac{3}{4}, \\ 2Bx + C & \text{if } s \neq \frac{1}{4}, t \neq \frac{3}{4}. \end{cases} \quad (2.3)$$

PROOF. It is easy to check that the functions f and h , given above, do satisfy the functional equation (1.5).

To show that these are the only solutions, we start by rewriting (1.5) using $s + t = 1$ as follows:

$$f(a + t(b - a)) - f(a) = t(b - a) \left[h(a + t(b - a)) - \frac{t}{2} [h(b) - h(a)] \right]. \quad (2.4)$$

Now, letting $u = (b - a)/3$, we obtain

$$f(a + 3tu) - f(a) = 3tu \left[h(a + 3tu) - \frac{t}{2} [h(3u + a) - h(a)] \right]. \quad (2.5)$$

Now, we replace a by $a - tu$ in (2.5) and get

$$f(a + 2tu) - f(a - tu) = 3tu \left[h(a + 2tu) - \frac{t}{2} [h((3 - t)u + a) - h(a - tu)] \right]. \quad (2.6)$$

Similarly, using $a = a - 2tu$ in (2.5), we get

$$f(a+tu) - f(a-2tu) = 3tu \left[h(a+tu) - \frac{t}{2} [h((3-2t)u+a) - h(a-2tu)] \right]. \quad (2.7)$$

Interchanging u with $-u$ in (2.7) gives

$$f(a-tu) - f(a+2tu) = -3tu \left[h(a-tu) - \frac{t}{2} [h((-3+2t)u+a) - h(a+2tu)] \right]. \quad (2.8)$$

Comparing (2.8) and (2.6) gives, for $a, u \in \mathbb{R}$,

$$\begin{aligned} & \left[h(a-tu) - \frac{t}{2} [h((-3+2t)u+a) - h(a+2tu)] \right] \\ &= \left[h(a+2tu) - \frac{t}{2} [h((3-t)u+a) - h(a-tu)] \right], \end{aligned} \quad (2.9)$$

which simplifies to

$$\begin{aligned} & t[h((3-t)u+a) - h((-3+2t)u+a) - (h(a-tu) - h(a+2tu))] \\ &= -2[h(a-tu) - h(a+2tu)]. \end{aligned} \quad (2.10)$$

Collecting the terms of h that have the same argument, we obtain

$$(2-t)h(a+2tu) - (2-t)h(a-tu) - th((3-t)u+a) + th((-3+2t)u+a) = 0. \quad (2.11)$$

Writing $x = a+2tu$ and dividing (2.11) by $(2-t)$ yields

$$h(x) - h(x-3tu) - \frac{t}{2-t} h(x+3(1-t)u) + \frac{t}{2-t} h(x-3u) = 0. \quad (2.12)$$

Thus, since $t \neq 0$ is fixed, (2.12) is of the form of equation (2.1) and hence, $h(x)$ is a generalized polynomial of degree at most 2,

$$h(x) = \beta(x, x) + \alpha(x) + C, \quad (2.13)$$

where β is a symmetric, biadditive function and α is an additive function and C is an arbitrary real constant.

Setting $a = 0$ in (2.5), we get

$$f(x) = x \left[h(x) - \frac{t}{2} \left[h\left(\frac{x}{t}\right) - h(0) \right] \right] + D, \quad (2.14)$$

and substituting from (2.13), we obtain

$$f(x) = x\beta(x, x) + x\alpha(x) + Cx - x \frac{t}{2} \beta\left(\frac{x}{t}, \frac{x}{t}\right) - x \frac{t}{2} \alpha\left(\frac{x}{t}\right) + D. \quad (2.15)$$

To prove the continuity of f and h , let us substitute the solutions given in (2.15) into (2.5). We see that both the left- and the right-hand side of (2.5) are polynomial functions in a and u . The equality of the two sides implies, therefore, the equality

of terms which are of the same degree with respect to a and u . First, comparing the terms of degree 1 with respect to each variable, we get

$$3a \left[\alpha(3tu) - \frac{t}{2} \alpha(3u) \right] + 3tu \left[\alpha(a) - \frac{t}{2} \alpha\left(\frac{a}{t}\right) \right] = 3tu\alpha(a), \quad (2.16)$$

whence, substituting ta instead of a and dividing by $t/2$, we get

$$tu\alpha(a) = 2a\alpha(tu) - ta\alpha(u). \quad (2.17)$$

Dividing both sides by tua , we obtain

$$\frac{\alpha(a)}{a} = 2 \frac{\alpha(tu)}{tu} - \frac{\alpha(u)}{u} \quad \forall a \neq 0 \neq u. \quad (2.18)$$

In particular, $\alpha(a)/a$ does not depend on a and, therefore, $\alpha(a) = 2Ba$ for some constant B .

Now, let us compare the terms of degree 2 with respect to a and those of degree 1 with respect to u . We get

$$6a \left[\beta(a, tu) - \frac{t}{2} \beta\left(\frac{a}{t}, u\right) \right] + 3tu \left[\beta(a, a) - \frac{t}{2} \beta\left(\frac{a}{t}, \frac{a}{t}\right) \right] = 3tu\beta(a, a). \quad (2.19)$$

Rearranging and simplifying, we get

$$6a \left[\beta(a, tu) - \frac{t}{2} \beta\left(\frac{a}{t}, u\right) \right] = \frac{3t^2}{2} \beta\left(\frac{a}{t}, \frac{a}{t}\right), \quad (2.20)$$

or, after substituting ta instead of a and dividing by $3t/2$,

$$4a \left[\beta(ta, tu) - \frac{t}{2} \beta(a, u) \right] = tu\beta(a, a). \quad (2.21)$$

Dividing (2.21) by a^2u , we obtain

$$4 \left[\frac{\beta(ta, tu)}{au} - \frac{t}{2} \frac{\beta(a, u)}{au} \right] = \frac{t\beta(a, a)}{a^2} \quad \text{for } a \neq 0 \neq u. \quad (2.22)$$

Using the symmetry of β , we infer that

$$\frac{\beta(a, a)}{a^2} = \frac{\beta(u, u)}{u^2} \quad \forall u \neq 0 \neq a, \quad (2.23)$$

whence, it follows that $\beta(a, a) = 3Aa^2$ for some constant A . Comparing this with formulae for f and h , we see that

$$\begin{aligned} f(x) &= 3A \left(1 - \frac{1}{2t} \right) x^3 + Bx^2 + Cx + D, \\ h(x) &= 3Ax^2 + 2Bx + C. \end{aligned} \quad (2.24)$$

Inserting (2.24) into (1.5), we get, after simplifying,

$$27t \left(1 - \frac{1}{2t} \right) Aa^2u + 81t^2 \left(1 - \frac{1}{2t} \right) Aau^2 = 9tAa^2u + 27t^2Aau^2 \quad \forall a, u \in \mathbb{R}, \quad (2.25)$$

whence, it follows that $A = 0$ provided $t \neq 3/4$. Note that, for $t = 3/4$, we have $3A(1 - (1/2t)) = A$ and the assertion follows from (2.24). \square

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