

BIORTHOGONALITY CONDITION FOR AXISYMMETRIC STOKES FLOW IN SPHERICAL GEOMETRIES

S. A. KHURI

(Received 15 September 1997)

ABSTRACT. We derive the biorthogonality condition for axisymmetric Stokes flow in a region between two concentric spheres. This biorthogonality condition is a property satisfied by the eigenfunctions and adjoint eigenfunctions, which is needed to compute the coefficients of the eigenfunction expansion solution of the corresponding creeping flow problem.

Keywords and phrases. Eigenvalues, eigenfunctions, eigenfunction expansion, biorthogonality conditions, Stokes flow.

2000 Mathematics Subject Classification. Primary 76D07, 76D99, 35P10.

1. Introduction. Recently, the eigenfunction expansion method has been used extensively for solving problems of Stokes flow. The method leads to the development of a set of eigenfunctions, adjoint eigenfunctions, biorthogonality conditions and an algorithm for the computation of the eigenfunction expansions. This technique was first introduced by Smith [10] in his solution of the biharmonic problem governing the bending of a semi-infinite strip clamped at its side and loaded at its top edge.

The biorthogonal series expansion method was also used by Joseph [2] in his study of the free surface on the round edge of a flowing liquid filling a torsion flow viscometer and by Joseph and Sturges [3] in the steady flow induced in a rectangular cavity by the uniform translation of a covering plate or belt. Similar biorthogonal eigenfunction expansions and biorthogonality conditions are required for the axisymmetric Stokes flow problems in a wedge shaped trench studied by Liu and Joseph [7], the axisymmetric Stokes flow in a cone studied by Liu and Joseph [8] and for the problem of Stokes flow in a trench between concentric cylinders studied by Yoo and Joseph [11].

Most recently, biorthogonality conditions were used by Khuri to solve Stokes flow in a sectorial cavity [5] and by Khuri and Wang for solving Stokes flow around a bend [6].

The previous references are just a small sample of problems arising in Stokes flow and elasticity which can be solved in biorthogonal series of eigenfunctions generated by separating variables. A list of several other problems is given in [4, 9].

In this paper, we derive the biorthogonality condition for axisymmetric Stokes flow in a spherical region by implementing a theorem proved by Khuri [5]. This biorthogonality condition is a property satisfied by the eigenfunctions and adjoint eigenfunctions, which is needed to compute the coefficients of the eigenfunction expansion solution of the corresponding creeping flow problem.

2. Biorthogonality conditions. We state a biorthogonality property satisfied by the eigenfunctions and adjoint eigenfunctions of the following fourth-order boundary value problem:

$$(P_0(r)y''(r))'' + (P_1(r;\alpha)y'(r))' + P_2(r;\alpha)y(r) = 0 \quad r \in [r_1, r_2] \quad (2.1)$$

The boundary conditions are given by

$$y(r_1) = y(r_2) = y'(r_1) = y'(r_2) = 0. \quad (2.2)$$

This biorthogonality condition, given in Theorem 2.1, which was proved by Khuri [5], gives the biorthogonality property for the boundary value problem given in equations (2.1) and (2.2) with certain restrictions imposed on the coefficients.

THEOREM 2.1 (biorthogonality condition). *Consider the boundary value problem given in (2.1) and (2.2), where $P_0(r)$, $P_1'(r;\alpha)$, $P_2(r;\alpha)$ are continuous and $P_0(r) \neq 0$ on $r_1 \leq r \leq r_2$. P_i in equation (2.1) is a polynomial of degree at most i in the parameter α , in particular, let $P_1(r;\alpha) = p_{11}(r)\alpha + p_{12}(r)$, and we require*

$$P_1^2(r;\alpha) - 4P_0(r)P_2(r;\alpha) = p_{31}(r)\alpha + p_{32}(r), \quad (2.3)$$

$$p_{11}^2(r) + p_{31}^2(r) \neq 0.$$

Then with P_n^* defined by

$$P_n^* = \int_{r_1}^{r_2} \left[\phi_2^{(n)}(r), \phi_1^{(n)}(r) \right] B(r) \begin{bmatrix} \phi_1^{(n)}(r) \\ \phi_2^{(n)}(r) \end{bmatrix} dr, \quad (2.4)$$

we have the following biorthogonality condition:

$$\int_{r_1}^{r_2} \left[\phi_2^{(m)}(r), \phi_1^{(m)}(r) \right] B(r) \begin{bmatrix} \phi_1^{(n)}(r) \\ \phi_2^{(n)}(r) \end{bmatrix} dr = P_n^* \delta_{mn}, \quad (2.5)$$

where δ_{mn} is the Kronecker's delta,

$$B(r) = \begin{pmatrix} -\frac{1}{2} \frac{p_{11}(r)}{P_0(r)} & 0 \\ \frac{1}{2} p_{11}''(r) + \frac{1}{4} \frac{p_{31}(r)}{P_0(r)} & -\frac{1}{2} \frac{p_{11}(r)}{P_0(r)} \end{pmatrix} \quad (2.6)$$

with

$$\begin{aligned} \phi_1^{(n)}(r) &= y_n(r), \\ \phi_2^{(n)}(r) &= P_0(r)y_n''(r) + \frac{1}{2}P_1(r;\alpha_n)y_n(r). \end{aligned} \quad (2.7)$$

Here y_i is an eigenfunction of equation (2.1) corresponding to the eigenvalue α_i . Assume the eigenvalues α_i are simple.

3. Axisymmetric Stokes flow in spherical regions. In this section, the biorthogonality condition for the axisymmetrical creeping flow in a region between two concentric spheres is derived. The flow region is

$$\nu = \{r, \theta : 0 < r_1 \leq r \leq r_2, -\theta_1 \leq \theta \leq \theta_1\}. \quad (3.1)$$

The Stokes flow equation in spherical coordinates (r, θ, ϕ) in ν is given by

$$E^4 \Psi(r, \theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} \right)^2 \Psi(r, \theta) = 0. \quad (3.2)$$

The velocity components in the (r, θ) direction in terms of the stream function are given by

$$v_r = -\frac{1}{r^2 \sin \theta} \frac{\partial \Psi}{\partial \theta}, \quad v_\theta = \frac{1}{r \sin \theta} \frac{\partial \Psi}{\partial r}. \quad (3.3)$$

Requiring the velocity to vanish on $r = r_1, r_2$, (3.3) gives

$$\Psi(r_1, \theta) = \Psi(r_2, \theta) = \frac{\partial \Psi}{\partial r}(r_1, \theta) = \frac{\partial \Psi}{\partial r}(r_2, \theta) = 0. \quad (3.4)$$

Separable solutions of (3.2) and (3.4) in the form

$$\Psi(r, \theta) \sim T(\cos \theta) \gamma(r) \quad (3.5)$$

exist (see [1]) when $\gamma(r)$ satisfies the following equation:

$$\gamma^{(4)} + \frac{2}{r^2} p(1-p) \gamma^{(2)} - \frac{4}{r^3} p(1-p) \gamma^{(1)} + p(1-p)(2+p)(3-p) \frac{1}{r^4} \gamma = 0 \quad (3.6)$$

and the boundary conditions

$$\gamma(r_1) = \gamma(r_2) = \gamma'(r_1) = \gamma'(r_2) = 0. \quad (3.7)$$

Seeking an eigenfunction solution in r direction it is necessary that the function $T(\eta)$, where $\eta = \cos \theta$ be required to satisfy the following equation:

$$(1 - \eta^2) T''(\eta) - p(1-p) T(\eta) = 0. \quad (3.8)$$

Equation (3.8) is Gegenbauer's equation of degree $-1/2$ where p could be complex. The two independent solutions of (3.8) are $C_p^{-1/2}(\eta)$ and $D_p^{-1/2}(\eta)$ that are termed as Gegenbauer functions of the first and second kind, respectively. Clearly, equation (3.6) can be written in the following form:

$$(\gamma'')'' + 2p(1-p) \left(\frac{1}{r^2} \gamma' \right)' + p(1-p)(2+p)(3-p) \frac{1}{r^4} \gamma = 0. \quad (3.9)$$

The hypothesis of Theorem 2.1 is satisfied when $\alpha_n \neq \alpha_m$ with

$$P_0(r) = 1; \quad P_1(r; \alpha) = \frac{2}{r^2} \alpha; \quad P_2(r; \alpha) = \frac{1}{r^4} \alpha(\alpha+6), \quad (3.10)$$

where

$$\alpha = p(1 - p). \quad (3.11)$$

Since

$$P_1^2(r; \alpha) - 4P_0(r)P_2(r; \alpha) = -\frac{24}{r^4}\alpha \quad (3.12)$$

so

$$p_{31}(r) = -\frac{24}{r^4}; \quad p_{32}(r) = 0. \quad (3.13)$$

Clearly,

$$p_{11}(r) = \frac{2}{r^2}; \quad p_{12}(r) = 0. \quad (3.14)$$

Thus using Theorem 2.1. The biorthogonality condition is given by

$$\int_{r_1}^{r_2} \frac{-1}{r^2} [\psi_1^{(m)}(r), \psi_2^{(m)}(r)] \begin{bmatrix} \phi_1^{(n)}(r) \\ \phi_2^{(n)}(r) \end{bmatrix} dr = P_n^* \delta_{mn}, \quad p_n(1 - p_n) \neq p_m(1 - p_m) \quad (3.15)$$

upon using

$$B(r) = -\frac{1}{r^2} I_{2 \times 2} = \begin{pmatrix} -\frac{1}{r^2} & 0 \\ 0 & -\frac{1}{r^2} \end{pmatrix}, \quad (3.16)$$

where $I_{2 \times 2}$ is the identity matrix. The eigenfunctions satisfy

$$\phi_1^{(n)}(r) = y_n(r), \quad \phi_2^{(n)}(r) = y_n''(r) + \frac{\alpha_n}{r^2} y_n(r) \quad (3.17)$$

and the adjoint eigenfunctions satisfy

$$\psi_1^{(m)}(r) = y_m''(r) + \frac{\alpha_m}{r^2} y_m(r), \quad \psi_2^{(m)}(r) = y_m(r), \quad (3.18)$$

where

$$\alpha_n = p_n(1 - p_n). \quad (3.19)$$

REFERENCES

- [1] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics with Special Applications to Particulate Media*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1965. MR 33#3562.
- [2] D. D. Joseph, *Slow motion and viscometric motion; stability and bifurcation of the rest state of a simple fluid*, Arch. Rational Mech. Anal. **56** (1974/75), 99-157. MR 50#3744. Zbl 295.76035.

- [3] D. D. Joseph and L. Sturges, *The convergence of biorthogonal series for biharmonic and Stokes flow edge problems. II*, SIAM J. Appl. Math. **34** (1978), no. 1, 7–26. MR 57#14765. Zbl 379.31001.
- [4] D. D. Joseph, L. D. Sturges, and W. H. Warner, *Convergence of biorthogonal series of biharmonic eigenfunctions by the method of Titchmarsh*, Arch. Rational Mech. Anal. **78** (1982), no. 3, 223–274. MR 84i:35108. Zbl 486.35061.
- [5] S. A. Khuri, *Biorthogonal series solution of Stokes flow problems in sectorial regions*, SIAM J. Appl. Math. **56** (1996), no. 1, 19–39. MR 96i:76035. Zbl 844.76019.
- [6] S. A. Khuri and C. Y. Wang, *Stokes flow around a bend*, Quart. Appl. Math. **55** (1997), no. 3, 573–600. MR 98d:76042. Zbl 887.76020.
- [7] C. H. Liu and D. D. Joseph, *Stokes flow in wedge-shaped trenches*, J. Fluid Mech. **80** (1977), 443–463. Zbl 355.76027.
- [8] ———, *Stokes flow in conical trenches*, SIAM J. Appl. Math. **34** (1978), no. 2, 286–296. MR 80d:76034. Zbl 376.76018.
- [9] T. N. Phillips, *Singular matched eigenfunction expansions for Stokes flow around a corner*, IMA J. Appl. Math. **42** (1989), no. 1, 13–26. MR 90f:76044. Zbl 674.76020.
- [10] R. C. T. Smith, *The bending of a semi-infinite strip*, Australian J. Sci. Research. Ser. A. **5** (1952), 227–237. MR 15,842a.
- [11] J. Y. Yoo and D. D. Joseph, *Stokes flow in a trench between concentric cylinders*, SIAM J. Appl. Math. **34** (1978), no. 2, 247–285. MR 80d:76033. Zbl 376.76019.

KHURI: DEPARTMENT OF COMPUTER SCIENCE, MATHEMATICS AND STATISTICS, AMERICAN UNIVERSITY OF SHARJAH, P.O. BOX 26666-739, SHARJAH, UNITED ARAB EMIRATES.

E-mail address: skhoury@aus.ac.ae

Special Issue on Space Dynamics

Call for Papers

Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

The main objective of this Special Issue is to publish topics that are under study in one of those lines. The idea is to get the most recent researches and published them in a very short time, so we can give a step in order to help scientists and engineers that work in this field to be aware of actual research. All the published papers have to be peer reviewed, but in a fast and accurate way so that the topics are not outdated by the large speed that the information flows nowadays.

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/mpe/guidelines.html>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	July 1, 2009
First Round of Reviews	October 1, 2009
Publication Date	January 1, 2010

Lead Guest Editor

Antonio F. Bertachini A. Prado, Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; prado@dem.inpe.br

Guest Editors

Maria Cecilia Zanardi, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; cecilia@feg.unesp.br

Tadashi Yokoyama, Universidade Estadual Paulista (UNESP), Rio Claro, 13506-900 São Paulo, Brazil; tadashi@rc.unesp.br

Silvia Maria Giuliatti Winter, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; silvia@feg.unesp.br