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EXISTENCE AND UNIQUENESS THEOREM FOR A SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS

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ABSTRACT. By using the method of successive approximation, we prove the existence and uniqueness of a solution of the fuzzy differential equation $x'(t) = f(t,x(t))$, $x(t_0) = x_0$. We also consider an ϵ-approximate solution of the above fuzzy differential equation.

Keywords and phrases. Fuzzy set-valued mapping, levelwise continuous, fuzzy derivative, fuzzy integral, fuzzy differential equation, fuzzy solution, fuzzy ϵ-approximate solution.

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1. Introduction. The differential equation

$$x'(t) = f(t,x(t)), \qquad x(t_0) = x_0 \tag{1.1}$$

has a solution provided f is continuous and satisfies a Lipschitz condition by C. Corduneanu [2]. The definition given here generalizes that of Aumann [1] for set-valued mappings. Kaleva [3] discussed the properties of differentiable fuzzy set-valued mappings and gave the existence and uniqueness theorem for a solution of the fuzzy differential equation $x'(t) = f(t,x(t))$ when f satisfies the Lipschitz condition. Also, in [4], he dealt with fuzzy differential equations on locally compact spaces. Park [6, 7] showed existence of solutions for fuzzy integral equations and a fixed point theorem for a pair of generalized nonexpansive fuzzy mappings.

In this paper, we prove the existence and uniqueness theorem of a solution to the fuzzy differential equation (1.1), where $f : I \times E^n \to E^n$ is levelwise continuous and satisfies a generalized Lipschitz condition.

Under some hypotheses, we consider an ϵ-approximate solution of the above fuzzy differential equation.

2. Preliminaries. Let $P_K(R^n)$ denote the family of all nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $P_K(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric

$$d(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\}, \tag{2.1}$$

where $\|\cdot\|$ denotes the usual Euclidean norm in R^n. Then it is clear that $(P_K(R^n),d)$ becomes a metric space.

THEOREM 2.1 [8]. *The metric space $(P_K(R^n),d)$ is complete and separable.*

Let $T = [c,d] \subset R$ be a compact interval and denote

$$E^n = \{u : R^n \longrightarrow [0,1] \mid u \text{ satisfies (i)–(iv) below}\}, \tag{2.2}$$

where

(i) u is normal, i.e., there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex,
(iii) u is upper semicontinuous,
(iv) $[u]^0 = \mathrm{cl}\{x \in R^n \mid u(x) > 0\}$ is compact.

For $0 < \alpha \le 1$, denote $[u]^\alpha = \{x \in R^n \mid u(x) \ge \alpha\}$, then from (i)–(iv), it follows that the α-level set $[u]^\alpha \in P_K(R^n)$ for all $0 \le \alpha \le 1$.

If $g : R^n \times R^n \to R^n$ is a function, then, according to Zadeh's extension principle, we can extend g to $E^n \times E^n \to E^n$ by the equation

$$g(u,v)(z) = \sup_{z=g(x,y)} \min\{u(x), v(y)\}. \tag{2.3}$$

It is well known that

$$[g(u,v)]^\alpha = g([u]^\alpha, [v]^\alpha) \tag{2.4}$$

for all $u, v \in E^n$, $0 \le \alpha \le 1$ and g is continuous. Especially for addition and scalar multiplication, we have

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha, \tag{2.5}$$

where u, $v \in E^n$, $k \in R$, $0 \le \alpha \le 1$.

Theorem 2.2 [5]. *If $u \in E^n$, then*

(1) $[u]^\alpha \in P_K(R^n)$ *for all* $0 \le \alpha \le 1$,
(2) $[u]^\alpha \subset [u]^{\alpha_1}$ *for all* $0 \le \alpha_1 \le \alpha_2 \le 1$,
(3) *if* $\{\alpha_k\} \subset [0,1]$ *is a nondecreasing sequence converging to* $\alpha > 0$, *then*

$$[u]^\alpha = \bigcap_{k \ge 1} [u]^{\alpha_k}. \tag{2.6}$$

Conversely, if $\{A^\alpha \mid 0 \le \alpha \le 1\}$ is a family of subsets of R^n satisfying (1)–(3), then there exists $u \in E^n$ such that

$$[u]^\alpha = A^\alpha \quad \text{for } 0 < \alpha \le 1 \tag{2.7}$$

and

$$[u]^0 = \overline{\bigcup_{0<\alpha\le 1} A^\alpha} \subset A^0. \tag{2.8}$$

Define $D : E^n \times E^n \to R^+ \cup \{0\}$ by the equation

$$D(u,v) = \sup_{0 \le \alpha \le 1} d([u]^\alpha, [v]^\alpha), \tag{2.9}$$

where d is the Hausdorff metric defined in $P_K(R^n)$.

The following definitions and theorems are given in [3].

Definition 2.1. A mapping $F : T \to E^n$ is *strongly measurable* if, for all $\alpha \in [0,1]$, the set-valued mapping $F_\alpha : T \to P_K(R^n)$ defined by

$$F_\alpha(t) = [F(t)]^\alpha \tag{2.10}$$

is Lebesgue measurable, when $P_K(R^n)$ is endowed with the topology generated by the Hausdorff metric d.

Definition 2.2. A mapping $F : T \to E^n$ is called *levelwise continuous* at $t_0 \in T$ if the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is continuous at $t = t_0$ with respect to the Hausdorff metric d for all $\alpha \in [0,1]$.

A mapping $F : T \to E^n$ is called *integrably bounded* if there exists an integrable function h such that $\|x\| \le h(t)$ for all $x \in F_0(t)$.

Definition 2.3. Let $F : T \to E^n$. The integral of F over T, denoted by $\int_T F(t)$ or $\int_c^d F(t)dt$, is defined levelwise by the equation

$$\begin{aligned}\left(\int_T F(t)dt\right)^\alpha &= \int_T F_\alpha(t)dt \\ &= \left\{\int_T f(t)dt \mid f : T \to R^n \text{ is a measurable selection for } F_\alpha\right\}\end{aligned} \tag{2.11}$$

for all $0 < \alpha \le 1$.

A strongly measurable and integrably bounded mapping $F : T \to E^n$ is said to be *integrable* over T if $\int_T F(t)dt \in E^n$.

Theorem 2.3. *If $F : T \to E^n$ is strongly measurable and integrably bounded, then F is integrable.*

It is known that $[\int_T F(t)dt]^0 = \int_T F_0(t)dt$.

Theorem 2.4. *Let $F, G : T \to E^n$ be integrable, and $\lambda \in R$. Then*
(i) $\int_T (F(t) + G(t))dt = \int_T F(t)dt + \int_T G(t)dt$.
(ii) $\int_T \lambda F(t)dt = \lambda \int_T F(t)dt$.
(iii) $D(F,G)$ *is integrable.*
(iv) $D\left(\int_T F(t)dt, \int_T G(t)dt\right) \le \int_T D(F,G)(t)dt$.

Definition 2.4. A mapping $F : T \to E^n$ is called *differentiable* at $t_0 \in T$ if, for any $\alpha \in [0,1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) \mid \alpha \in [0,1]\}$ define a fuzzy number $F(t_0) \in E^n$.

If $F : T \to E^n$ is differentiable at $t_0 \in T$, then we say that $F'(t_0)$ is the *fuzzy derivative* of $F(t)$ at the point t_0.

Theorem 2.5. *Let $F : T \to E^1$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then f_α and g_α are differentiable and* $[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)]$.

Theorem 2.6. *Let $F : T \to E^n$ be differentiable and assume that the derivative F' is integrable over T. Then, for each $s \in T$, we have*

$$F(s) = F(a) + \int_a^s F'(t)dt. \tag{2.12}$$

Definition 2.5. A mapping $f : T \times E^n \to E^n$ is called *levelwise continuous* at point $(t_0, x_0) \in T \times E^n$ provided, for any fixed $\alpha \in [0,1]$ and arbitrary $\epsilon > 0$, there exists a

$\delta(\epsilon,\alpha) > 0$ such that

$$d\Big([f(t,x)]^{\alpha},[f(t_0,x_0)]^{\alpha}\Big) < \epsilon \tag{2.13}$$

whenever $|t-t_0| < \delta(\epsilon,\alpha)$ and $d([x]^{\alpha},[x_0]^{\alpha}) < \delta(\epsilon,\alpha)$ for all $t \in T$, $x \in E^n$.

3. Fuzzy differential equations. Assume that $f : I \times E^n \to E^n$ is levelwise continuous, where the interval $I = \{t : |t-t_0| \le \delta \le a\}$. Consider the fuzzy differential equation (1.1) where $x_0 \in E^n$. We denote $J_0 = I \times B(x_0,b)$, where $a > 0$, $b > 0$, $x_0 \in E^n$,

$$B(x_0,b) = \{x \in E^n \mid D(x,x_0) \le b\}. \tag{3.1}$$

Definition 3.1. A mapping $x : I \to E^n$ is a solution to the problem (1.1) if it is levelwise continuous and satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))ds \quad \text{for all } t \in I. \tag{3.2}$$

According to the method of successive approximation, let us consider the sequence $\{x_n(t)\}$ such that

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s,x_{n-1}(s))ds, \quad n = 1,2,\dots, \tag{3.3}$$

where $x_0(t) \equiv x_0$, $t \in I$.

Theorem 3.1. *Assume that*

(i) *a mapping $f : J_0 \to E^n$ is levelwise continuous,*

(ii) *for any pair $(t,x),(t,y) \in J_0$, we have*

$$d\Big([f(t,x)]^{\alpha},[f(t,y)]^{\alpha}\Big) \le Ld([x]^{\alpha},[y]^{\alpha}), \tag{3.4}$$

where $L > 0$ is a given constant and for any $\alpha \in [0,1]$.

Then there exists a unique solution $x = x(t)$ of (1.1) defined on the interval

$$|t-t_0| \le \delta = \min\left\{a, \frac{b}{M}\right\}, \tag{3.5}$$

where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ such that $\hat{o}(t) = 1$ for $t = 0$ and 0 otherwise and for any $(t,x) \in J_0$.

Moreover, there exists a fuzzy set-valued mapping $x : I \to E^n$ such that $D(x_n(t),x(t)) \to 0$ on $|t-t_0| \le \delta$ as $n \to \infty$.

Proof. Let $t \in I$, from (3.3), it follows that, for $n = 1$,

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s,x_0)ds \tag{3.6}$$

which proves that $x(t)$ is levelwise continuous on $|t-t_0| \le a$ and, hence on $|t-t_0| \le \delta$. Moreover, for any $\alpha \in [0,1]$, we have

$$d([x_1(t)]^{\alpha},[x_0]^{\alpha}) = d\left(\left[\int_{t_0}^{t} f(s,x_0)ds\right]^{\alpha},0\right) \le \int_{t_0}^{t} d([f(s,x_0)]^{\alpha},0)ds \tag{3.7}$$

and by the definition of D, we get

$$D(x_1(t),x_0) \le M|t-t_0| \le M\delta = b \tag{3.8}$$

if $|t-t_0| \leq \delta$, where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ and for any $(t,x) \in J_0$.

Now, assume that $x_{n-1}(t)$ is levelwise continuous on $|t-t_0| \leq \delta$ and that

$$D(x_{n-1}(t),x_0) \leq M|t-t_0| \leq M\delta = b \tag{3.9}$$

if $|t-t_0| \leq \delta$, where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ and for any $(t,x) \in J_0$.

From (3.3), we deduce that $x_n(t)$ is levelwise continuous on $|t-t_0| \leq \delta$ and that

$$D(x_n(t),x_0) \leq M|t-t_0| \leq M\delta = b \tag{3.10}$$

if $|t-t_0| \leq \delta$, where $M = D(f(t,x),\hat{o})$, $\hat{o} \in E^n$ and for any $(t,y) \in J_0$.

Consequently, we conclude that $\{x_n(t)\}$ consists of levelwise continuous mappings on $|t-t_0| \leq \delta$ and that

$$(t,x_n(t)) \in J_0, \qquad |t-t_0| \leq \delta, \quad n = 1,2,\dots. \tag{3.11}$$

Let us prove that there exists a fuzzy set-valued mapping $x : I \to E^n$ such that $D(x_n(t), x(t)) \to 0$ uniformly on $|t-t_0| \leq \delta$ as $n \to \infty$. For $n = 2$, from (3.3),

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s,x_1(s))ds. \tag{3.12}$$

From (3.6) and (3.12), we have

$$\begin{aligned} d\Big([x_2(t)]^\alpha,[x_1(t)]^\alpha\Big) &= d\left(\left[\int_{t_0}^{t} f(s,x_1(s))ds\right]^\alpha,\left[\int_{t_0}^{t} f(s,x_0)ds\right]^\alpha\right) \\ &\leq \int_{t_0}^{t} d\Big([f(s,x_1(s))]^\alpha,[f(s,x_0)]^\alpha\Big)ds \end{aligned} \tag{3.13}$$

for any $\alpha \in [0,1]$.

According to the condition (3.4), we obtain

$$d([x_2(t)]^\alpha,[x_1(t)]^\alpha) \leq \int_{t_0}^{t} Ld([x_1(s)]^\alpha,[x_0]^\alpha)ds \tag{3.14}$$

and by the definition of D, we obtain

$$D(x_2(t),x_1(t)) \leq L\int_{t_0}^{t} D(x_1(s),x_0(s))ds. \tag{3.15}$$

Now, we can apply the first inequality (3.8) in the right-hand side of (3.15) to get

$$D(x_2(t),x_1(t)) \leq ML\frac{|t-t_0|^2}{2!} \leq ML\frac{\delta^2}{2!}. \tag{3.16}$$

Starting from (3.8) and (3.16), assume that

$$D(x_n(t),x_{n-1}(t)) \leq ML^{n-1}\frac{|t-t_0|^n}{n!} \leq ML^{n-1}\frac{\delta^n}{n!} \tag{3.17}$$

and let us prove that such an inequality holds for $D(x_{n+1}(t),x_n(t))$.

Indeed, from (3.3) and condition (3.4), it follows that

$$\begin{aligned} d\Big([x_{n+1}(t)]^\alpha,[x_n(t)]^\alpha\Big) &= d\left(\left[\int_{t_0}^t f(s,x_n(s))ds\right]^\alpha,\left[\int_{t_0}^t f(s,x_{n-1}(s))ds\right]^\alpha\right) \\ &\le \int_{t_0}^t d\Big([f(s,x_n(s))]^\alpha,[f(s,x_{n-1}(s))]^\alpha\Big)ds \\ &\le \int_{t_0}^t Ld\Big([x_n(s)]^\alpha,[x_{n-1}(s)]^\alpha\Big)ds \end{aligned} \tag{3.18}$$

for any $\alpha \in [0,1]$ and from the definition of D, we have

$$D(x_{n+1}(t),x_n(t)) \le L\int_{t_0}^t D(x_n(s),x_{n-1}(s))ds. \tag{3.19}$$

According to (3.17), we get

$$D(x_{n+1}(t),x_n(t)) \le ML^n\int_{t_0}^t \frac{|s-t_0|^n}{n!}ds = ML^n\frac{|t-t_0|^{n+1}}{(n+1)!} \le ML^n\frac{\delta^{n+1}}{(n+1)!}. \tag{3.20}$$

Consequently, inequality (3.17) holds for $n = 1,2,\dots$. We can also write

$$D(x_n(t),x_{n-1}(t)) \le \frac{M}{L}\frac{(L\delta)^n}{n!} \tag{3.21}$$

for $n = 1,2,\dots$, and $|t-t_0| \le \delta$.

Let us mention now that

$$x_n(t) = x_0 + [x_1(t)-x_0] + \cdots + [x_n(t)-x_{n-1}(t)], \tag{3.22}$$

which implies that the sequence $\{x_n(t)\}$ and the series

$$x_0 + \sum_{n=1}^{\infty}[x_n(t)-x_{n-1}(t)] \tag{3.23}$$

have the same convergence properties.

From (3.21), according to the convergence criterion of Weierstrass, it follows that the series having the general term $x_n(t)-x_{n-1}(t)$, so $D(x_n(t),x_{n-1}(t)) \to 0$ uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

Hence, there exists a fuzzy set-valued mapping $x : I \to E^n$ such that $D(x_n(t),x(t)) \to 0$ uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

From (3.4), we get

$$d\Big([f(t,x_n(t))]^\alpha,[f(t,x(t))]^\alpha\Big) \le Ld\Big([x_n(t)]^\alpha,[x(t)]^\alpha\Big) \tag{3.24}$$

for any $\alpha \in [0,1]$. By the definition of D,

$$D(f(t,x_n(t)),f(t,x(t))) \le LD(x_n(t),x(t)) \longrightarrow 0 \tag{3.25}$$

uniformly on $|t-t_0| \le \delta$ as $n \to \infty$.

Taking (3.25) into account, from (3.3), we obtain, for $n \to \infty$,

$$x(t) = x_0 + \int_{t_0}^t f(s,x(s))ds. \tag{3.26}$$

Consequently, there is at least one levelwise continuous solution of (1.1).

We want to prove now that this solution is unique, that is, from

$$y(t) = x_0 + \int_{t_0}^{t} f(s, y(s))\,ds \tag{3.27}$$

on $|t - t_0| \le \delta$, it follows that $D(x(t), y(t)) \equiv 0$. Indeed, from (3.3) and (3.27), we obtain

$$\begin{aligned} d\Big([y(t)]^{\alpha}, [x_n(t)]^{\alpha}\Big) &= d\left(\Big[\int_{t_0}^{t} f(s, y(s))\,ds\Big]^{\alpha}, \Big[\int_{t_0}^{t} f(s, x_{n-1}(s))\,ds\Big]^{\alpha}\right) \\ &\le \int_{t_0}^{t} d\Big([f(s, y(s))]^{\alpha}, [f(s, x_{n-1}(s))]^{\alpha}\Big)\,ds \\ &\le \int_{t_0}^{t} L d\Big([y(s)]^{\alpha}, [x_{n-1}(s)]^{\alpha}\Big)\,ds \end{aligned} \tag{3.28}$$

for any $\alpha \in [0,1]$, $n = 1, 2, \ldots$.

By the definition of D, we obtain

$$D(y(t), x_n(t)) \le L \int_{t_0}^{t} D(y(s), x_{n-1}(s))\,ds, \quad n = 1, 2, \ldots. \tag{3.29}$$

But $D(y(t), x_0) \le b$ on $|t - t_0| \le \delta$, $y(t)$ being a solution of (3.27). It follows from (3.29) that

$$D(y(t), x_1(t)) \le bL|t - t_0| \tag{3.30}$$

on $|t - t_0| \le \delta$. Now, assume that

$$D(y(t), x_n(t)) \le bL^n \frac{|t - t_0|^n}{n!} \tag{3.31}$$

on the interval $|t - t_0| \le \delta$. From

$$D(y(t), x_{n+1}(t)) \le L \int_{t_0}^{t} D(y(s), x_n(s))\,ds \tag{3.32}$$

and (3.31), one obtains

$$D(y(t), x_{n+1}(t)) \le bL^{n+1} \frac{|t - t_0|^{n+1}}{(n+1)!}. \tag{3.33}$$

Consequently, (3.31) holds for any n, which leads to the conclusion

$$D(y(t), x_n(t)) = D(x(t), x_n(t)) \longrightarrow 0 \tag{3.34}$$

on the interval $|t - t_0| \le \delta$ as $n \to \infty$.

This proves the uniqueness of the solution for (1.1). □

Definition 3.2. A mapping $x : L \to E^n$ is an *ϵ-approximate solution* of (1.1) if the following properties hold

(a) $x(t)$ is levelwise continuous on $|t - t_0| \le \delta$,

(b) the derivative $x'(t)$ exists and it is levelwise continuous,

(c) for all t for which $x'(t)$ is defined, we have

$$D\left(x'(t), f(t, x(t))\right) < \epsilon. \tag{3.35}$$

Theorem 3.2. *A mapping $f : J_0 \to E^n$ is levelwise continuous, and let $\epsilon > 0$ be arbitrary. Then there exists at least one ϵ-approximate solution of (1.1), defined on $|t - t_0| \le \delta = \min\{a, b/M\}$, where $M = D(f(t,x), \hat{o})$, $\hat{o} \in E^n$ and for any $(t,x) \in J_0$.*

Proof. In as much as a mapping $f : J_0 \to E^n$ is a levelwise continuous on a compact set J_0, it follows that $f(t,x)$ is uniformly levelwise continuous.

Consequently, for any $\alpha \in [0,1]$, we can find $\delta > 0$ such that $d([f(t,x)]^\alpha, [f(s,y)]^\alpha) < \epsilon$.

Now, we construct the approximate solution for $t \in [t_0, t_0 + \delta]$, the construction being completely similar for $t \in [t_0 - \delta, t_0]$.

Let us consider a division

$$t_0 < t_1 < \cdots < t_n = t_0 + \delta \tag{3.36}$$

of $[t_0, t_0 + \delta]$ such that

$$\max_k (t_k - t_{k-1}) < \lambda = \min\left\{\delta, \frac{\delta}{M}\right\}. \tag{3.37}$$

We define a mapping $x : I \to E^n$ as follows

$$x(t_0) = x_0, \tag{3.38}$$

$$x(t) = x(t_k) + f(t_k, x(t_k))(t - t_k) \tag{3.39}$$

on $t_k < t \le t_{k+1}$, $k = 0, 1, \ldots, n-1$.

It is obvious that a mapping $x : I \to E^n$ satisfies the first two properties from the definition of an ϵ-approximate solution.

Now, we want to prove that the last property is also fulfilled. Indeed, $x'(t) = f(t_k, x(t_k))$ on (t_k, t_{k+1}) and for any $\alpha \in [0,1]$,

$$d\left([x'(t)]^\alpha, [f(t,x(t))]^\alpha\right) = d\left([f(t_k, x(t_k))]^\alpha, [f(t,x(t))]^\alpha\right) < \epsilon \tag{3.40}$$

since $|t - t_k| < \lambda \le \delta$,

$$d([x(t)]^\alpha, [x(t_k)]^\alpha) \le d([f(t_k, x(t_k))]^\alpha, 0)|t - t_k| < M\lambda \le \delta. \tag{3.41}$$

Thus, by the definition of D, we have

$$D(x'(t), f(t,x(t))) < \epsilon \tag{3.42}$$

on $|t - t_0| < \delta$ and $(t,x) \in J_0$.

Theorem 3.2 is completely proved. □

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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Manuscript Due	March 1, 2009
First Round of Reviews	June 1, 2009
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