

ON STRONG FORM OF ARZELA CONVERGENCE

JANINA EWERT

Department of Mathematics
Pedagogical University
Arciszewskiego 22 b
76-200 Slupsk, POLAND

(Received November 30, 1995 and in revised form April 23, 1996)

ABSTRACT. We define some new type of convergence of nets of functions which is formulated in terms of open covers. It preserves continuity and under some assumptions implies (or coincides with) the Arzela quasi-uniform convergence. Furthermore, the introduced strong convergence is used for characterization of compactness and regularity of a topological space.

KEY WORDS AND PHRASES: Arzela convergence, almost compact space, uniform space

1991 AMS SUBJECT CLASSIFICATION CODES: 54A20.

1. INTRODUCTION

Let $f_n : X \rightarrow Y$, $n \geq 1$, be continuous functions and let $f : X \rightarrow Y$ be a limit of the sequence $\{f_n : n \geq 1\}$. Which assumptions on the convergence guarantee the continuity of f ? This question led to defining various types of convergences for nets of functions with values in metric or uniform spaces [1,2,3,4,5]. But initial notions in this problem—continuity and pointwise convergence—are depending on topologies only and they can be considered even then if spaces are not uniformizable. This is the motivation of our paper. We define some new form of convergence formulated in terms of open covers; it preserves continuity and under some assumptions implies (or coincides with) the Arzela quasi-uniform convergence.

Let X, Y be topological spaces; the symbols $F(X, Y)$ and $C(X, Y)$ are used to denote the class of all functions or all continuous functions from X to Y , respectively. For any set A its closure is denoted by ClA .

A net $\{f_j : j \in J\} \subset F(X, Y)$ is said to be strongly convergent to $f \in F(X, Y)$ if:

- this net is pointwise convergent to f , and
- for each open cover \mathcal{A} of Y and each $j_0 \in J$ there exists a finite set $J_0 \subset J$ such that $j \geq j_0$ for $j \in J_0$ and for each $x \in X$ there are $j \in J_0$ and $W \in \mathcal{A}$ with $\{f(x), f_j(x)\} \subset W$

THEOREM 1. Let X be a topological space and Y a regular one. If a net $\{f_j : j \in J\} \subset C(X, Y)$ is strongly convergent to a function $f : X \rightarrow Y$, then $f \in C(X, Y)$.

PROOF. Let $x_0 \in X$ and let W, V be open sets in Y such that $f(x_0) \in V \subset ClV \subset W$, then $\mathcal{A} = \{W, Y \setminus ClV\}$ is an open cover of Y . Now we can choose $j_0 \in J$ and a finite subset $J_0 \subset J$ such that

$$\begin{aligned} j &\geq j_0 && \text{for } j \in J_0, \\ f_j(x_0) &\in V && \text{for } j \in J, j \geq j_0, \end{aligned}$$

and for each $z \in X$ there is $j_z \in J_0$ with $\{f(z), f_{j_z}(z)\} \subset W$ or $\{f(z), f_{j_z}(z)\} \subset Y \setminus ClV$. There exists a neighborhood U of x_0 such that $f_j(U) \subset V$ for $j \in J_0$, thus $f_j(U) \cap (Y \setminus ClV) = \emptyset$ for $j \in J_0$

Hence for each $x \in U$ we have $\{f(x), f_{j_x}(x)\} \subset W$ which implies $f(U) \subset V$ and the proof is completed

Now we are going to compare the strong convergence with other ones. To begin with let us observe that the uniform convergence and the strong one are independent as shown in the following.

EXAMPLES. In the space R of real numbers with the usual metric let us consider functions $f_n, f: R \rightarrow R$ defined by $f_n(x) = x + \frac{1}{n}$, $f(x) = x$ for each $x \in R, n \geq 1$; so the sequence $\{f_n: n \geq 1\}$ is uniformly convergent to f . Now, taking open intervals $U_n = (n - \frac{1}{2n}, n + \frac{1}{2n})$ and $V_n = (n + \frac{1}{3n}, n + 1 - \frac{1}{3n})$ for $n \geq 1$ we obtain the open cover $\mathcal{A} = \{U_n, V_n: n \geq 1\} \cup \{(-\infty, 3)\}$ of R . For fixed n, k with $k > n > 1$ and for each j with $n < j < k$ we have that U_k is the unique set in \mathcal{A} containing $f(k)$ and $|f_j(k) - f(k)| = \frac{1}{j} > \frac{1}{2k}$, so $f_j(k) \notin U_k$. Hence the sequence $\{f_n: n \geq 1\}$ is not strongly convergent to f .

Now let $g_n, g: R \rightarrow R$ be functions given by:

$$g(x) = 0 \quad \text{for each } x \in R, \quad \text{and}$$

$$g_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x \in (-\infty, n-1] \cup [n+1, \infty) \\ (1 - \frac{1}{n})x - n + 2, & \text{if } x \in [n-1, n] \\ (\frac{1}{n} - 1)x + n, & \text{if } x \in [n, n+1]. \end{cases}$$

The sequence $\{g_n: n \geq 1\}$ strongly converges to g but not uniformly.

In a uniform space (Y, \mathcal{V}) for any $V \in \mathcal{V}$ and $y \in Y$ we will write $V[y] = \{z \in Y: (y, z) \in V\}$. A net $\{f_j: j \in J\} \subset F(X, Y)$ is said to be \mathcal{V} -quasi-uniformly in the sense of Arzela convergent to $f \in F(X, Y)$ if

- it is pointwise convergent to f , and
- for each $V \in \mathcal{V}$ and $j_0 \in J$ there exists a finite set $J_0 \subset J$ with $j \geq j_0$ for $j \in J_0$ and for each $x \in X$ there is $j \in J_0$ such that $(f(x), f_j(x)) \in V[1, 2]$.

Let us remark that the quasi-uniform convergence is strictly connected with a uniformity (or metric). For instance let $X = Y = (0, \infty)$, $d_1(x, y) = |x - y|$ and $d_2(x, y) = |x^2 - y^2|$; then d_1 and d_2 are topologically equivalent metrics on Y . Putting $f_n(x) = x + \frac{1}{n}$, $f(x) = x$ for $x \in X, n \geq 1$ we have the sequence $\{f_n: n \geq 1\}$ which converges to f d_1 -uniformly but it is not d_2 -quasi-uniformly convergent to f .

PROPOSITION. Yet Y be a completely regular space. If a net $\{f_j: j \in J\} \subset F(X, Y)$ is strongly convergent to a function $f: X \rightarrow Y$, then for each compatible uniformity \mathcal{V} on Y this net \mathcal{V} -quasi-uniformly converges to f .

PROOF. Let \mathcal{V} be a compatible uniformity on Y and $V \in \mathcal{V}$. We take an open set $W \in \mathcal{V}$ with $W = W^{-1}$ and $W^2 \subset V$. Then it suffices to consider the open cover $\mathcal{A} = \{W[y]: y \in Y\}$.

A uniform space (Y, \mathcal{V}) (or simply a uniformity \mathcal{V}), is said to have the Lebesgue property if for each open cover \mathcal{A} of Y there is $V \in \mathcal{V}$ such that $\{V[y]: y \in Y\}$ is a refinement of \mathcal{A} [6].

It is easy to see that if \mathcal{V} has the Lebesgue property, then for each X the strong convergence in $F(X, Y)$ coincides with the \mathcal{V} -quasi-uniform one.

The Lebesgue property is closely related to the paracompactness, namely we have the following

THEOREM 2 [6]. Let Y be a Tychonoff space. Then Y possesses a uniformity \mathcal{V} compatible with the given topology for which (Y, \mathcal{V}) has the Lebesgue property if and only if Y is paracompact. Under this condition, this uniformity is the finest of all compatible uniformities, and is unique.

So, as a simple consequence we have:

COROLLARY. Let Y be a paracompact space and let \mathcal{V} be the finest compatible uniformity on Y . Then for each X the strong convergence in $F(X, Y)$ is equivalent to the \mathcal{V} -quasi-uniform one

A topological space X is called almost compact if for each open cover \mathcal{A} of X there is a finite number $A_1, A_2, \dots, A_n \in \mathcal{A}$ such that $X = \bigcup_{i=1}^n Cl A_i$ [7, p. 239].

THEOREM 3. Let X be an almost compact space, Y a Tychonoff one and let $\{f_j : j \in J\} \subset F(X, Y)$, $f \in C(X, Y)$. Then the following conditions are equivalent:

- (a) the net $\{f_j : j \in J\}$ is strongly convergent to f ;
- (b) for each compatible uniformity \mathcal{V} on Y the net $\{f_j : j \in J\}$ is \mathcal{V} -quasi-uniformly convergent to f ,
- (c) there exists a compatible uniformity \mathcal{V} on Y such that the net $\{f_j : j \in J\}$ is \mathcal{V} -quasi-uniformly convergent on f .

PROOF. In virtue of the proposition it suffices to show the implication (c) \Rightarrow (a). Let \mathcal{V} be a compatible uniformity on Y such that the net $\{f_j : j \in J\}$ is \mathcal{V} -quasi-uniformly convergent to f . We fix an open cover \mathcal{A} of Y and $j_0 \in J$. Then for each $x \in X$ we choose a set $W_x \in \mathcal{A}$ and $V_x \in \mathcal{V}$ with $V_x = V_x^{-1}$ and $V_x[f(x)] \subset W_x$. Since f is continuous for each $x \in X$ there is a neighborhood U_x of x for which $f(U_x) \subset V_x[f(x)]$. From the assumption on X the open cover $\{U_x : x \in X\}$ contains a finite family $\{U_{x_1}, U_{x_2}, \dots, U_{x_n}\}$ such that $X = \bigcup_{k=1}^n Cl U_{x_k}$. We put $V = \bigcap_{k=1}^n V_{x_k}$. Now, by the \mathcal{V} -quasi-uniform convergence there exists a finite set $J_1 \subset J$ such that $j \geq j_0$ for each $j \in J_1$ and for each $z \in X$ we can take $j_z \in J_1$ with $(f(z), f_{j_z}(z)) \in V$. Furthermore, $z \in Cl U_{x_k}$ for some k , $k \leq n$. Hence

$$f(z) \in f(Cl U_{x_k}) \subset Cl V_{x_k}[f(x_k)] \subset V_{x_k}^2[f(x_k)] \subset W_{x_k}$$

and

$$f_{j_z}(z) \in V[f(z)] \subset V_{x_k}[f(z)] \subset V_{x_k}^3[f(x_k)] \subset W_{x_k}.$$

So we have shown that for each $z \in X$ it holds $\{f(z), f_{j_z}(z)\} \subset W_{x_k}$ for some $j_z \in J_1$ and $W_{x_k} \in \mathcal{A}$, which finishes the proof.

THEOREM 4. For a topological space X let us consider the following properties:

- (a) the strong convergence in $F(X, [0, 1])$ coincides with the pointwise one;
- (b) X is an almost compact space;
- (c) for each regular space Y the strong convergence in $C(X, Y)$ is equivalent to the pointwise one;

then (a) \Rightarrow (b) \Rightarrow (c).

PROOF. We will show the implication (a) \Rightarrow (b). Let \mathcal{A} be an open cover of X and let $\{U_s : s \in S\}$ be the family of all finite sums of sets belonging to \mathcal{A} . We define a relation " \leq " in S assuming $s_1 \leq s_2$ iff $U_{s_1} \subset U_{s_2}$; so (S, \leq) is a directed set. Now let us consider functions $f_s, f : X \rightarrow [0, 1]$, $s \in S$, given by $f(x) = 1$ for $x \in X$ and

$$f_s(x) = \begin{cases} 1, & \text{if } x \in Cl U_s \\ 0, & \text{if } x \in X \setminus Cl U_s. \end{cases}$$

It is easy to see that f is the pointwise limit of the net $\{f_s : s \in S\}$; thus, by the assumption this net strongly converges to f . Hence for the open cover $\mathcal{G} = \{[0, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4}), (\frac{1}{2}, 1]\}$ of $[0, 1]$ and a fixed $s_0 \in S$ there exists a finite set $S_1 \subset S$ with $s \geq s_0$ for $s \in S_1$ and for each $x \in X$ some $s \in S_1$ can be taken such that both $f(x)$ and $f_s(x)$ are contained in the same set from \mathcal{G} . This implies that for each $x \in X$ there is $s \in S_1$ for which $f_s(x) = 1$, which means $X = \bigcup \{Cl U_s : s \in S_1\}$. From the definition of the sets U_s it follows that \mathcal{A} contains a finite family \mathcal{A}_1 with $X = \bigcup \{Cl U : U \in \mathcal{A}_1\}$, so X is almost compact.

Now, we suppose that X is almost compact, Y is a regular space and $\{f_j : j \in J\} \subset C(X, Y)$ is a net of functions which pointwise converges to some $f \in C(X, Y)$. Let \mathcal{A} be an open cover of Y and $j_0 \in J$. For each set $W \in \mathcal{A}$ and each point $y \in W$ we choose an open set $V_{y,W}$ with $y \in V_{y,W} \subset Cl V_{y,W} \subset W$. Then $\{V_{y,W} : W \in \mathcal{A}, y \in W\}$ is an open cover of Y , so

$$\{f^{-1}(V_{y,W}) \cap f_j^{-1}(V_{y,W}) : W \in \mathcal{A}, y \in W, j \in J, j \geq j_0\}$$

is an open cover of X . Since X is almost compact there exists a finite family $\{f^{-1}(V_{y_k, W_k}) \cap f_{j_k}^{-1}(V_{y_k, W_k}) : k = 1, 2, \dots, n\}$ such that

$$X = \bigcup_{k=1}^n Cl(f^{-1}(V_{y_k, W_k}) \cap f_{j_k}^{-1}(V_{y_k, W_k})).$$

Let us put $J_1 = \{j_1, j_2, \dots, j_n\}$; then $j_k \geq j_0$ for each $j_k \in J_1$. Furthermore for each $x \in X$ there is $j_k \in J_1$ and V_{y_k, W_k} for which it holds

$$x \in Cl(f^{-1}(V_{y_k, W_k}) \cap f_{j_k}^{-1}(V_{y_k, W_k})) \subset f^{-1}(Cl V_{y_k, W_k}) \cap f_{j_k}^{-1}(Cl V_{y_k, W_k}),$$

thus $f(x) \in W_k$ and $f_{j_k}(x) \in W_k$ which finishes the proof.

Let us observe that in the above theorem the implication (a) \Rightarrow (b) is not reversible; moreover (b) \Rightarrow (c) is not true if in (c) the $C(X, Y)$ is replaced by $F(X, Y)$. It suffices to take into account $X = Y = [0, 1]$ with the usual metric and functions $f_n, f : [0, 1] \rightarrow [0, 1], n \geq 1$, given by $f_n(x) = x^n$ for $n \geq 1, x \in [0, 1]$ and $f(1) = 1, f(x) = 0$ if $x \in [0, 1)$.

THEOREM 5. For a Tychonoff space X the following conditions are equivalent:

- (a) for each topological space Y the strong convergence is equivalent to the pointwise one on $C(X, Y)$;
- (b) the strong convergence coincides on $C(X[0, 1])$ with the pointwise one;
- (c) X is a compact space.

PROOF. The implication (a) \Rightarrow (b) is evident. Let (b) be satisfied and let \mathcal{V} be the uniformity on $[0, 1]$ determined by the usual metric. Since $([0, 1], \mathcal{V})$ has the Lebesgue property, the \mathcal{V} -quasi-uniform convergence coincides with the strong convergence. Thus, according to (b), the \mathcal{V} -quasi-uniform convergence is equivalent to the pointwise one on $C(X, [0, 1])$. So, from [2, Th. 6] it means that X is compact. Finally we assume that X is compact. Let Y be a topological space and let $\{f_j : j \in J\}$ be a net in $C(X, Y)$ which pointwise converges to $f \in C(X, Y)$. Now, for an open cover \mathcal{A} of Y and for fixed $j_0 \in J$ it suffices to consider the open cover $\{f^{-1}(W) \cap f_{j_0}^{-1}(W) : W \in \mathcal{A}, j \in J, j \geq j_0\}$ of X , and the proof is completed.

THEOREM 6. Let Y be a Hausdorff space. If for each topological space X the strong convergence coincides with the pointwise one on $C(X, Y)$, then Y is a regular space

PROOF. Assume that Y is not regular and let T denote the topology on Y . There exists an open set U_0 and a point $a \in U_0$ such that for each neighborhood V of a we have $Cl V \not\subset U_0$; hence $Cl U_0 \neq Y$ and we can fix a point $b \in Y \setminus Cl U_0$. By $N(a)$ we denote the family of all neighborhoods of a which are contained in U_0 . For any $V_1, V_2 \in N(a)$ we write $V_1 \leq V_2$ iff $V_2 \subset V_1$; thus $(N(a), \leq)$ is a directed set. If $V \in N(a)$, then $\emptyset \neq (Cl V) \setminus U_0$; so for each $V \in N(a)$ we choose a point $x_V \in (Cl V) \setminus U_0$. We denote

$$A_V = \{x_W : W \in N(a), W \geq V, W \neq V\}, \quad \text{for } V \in N(a).$$

Furthermore let

$$\tau = \{A \subset Y : a \notin A \text{ or } A \text{ contains some } V \in N(a)\}.$$

Then τ is a topology on $Y, T \subset \tau$ and for each point $y \in Y, y \neq a$ we have $\{y\} \in \tau$. Now we define functions $f_V, f : (Y, \tau) \rightarrow (Y, T)$ by the following

$$f(x) = x \quad \text{for } x \in Y;$$

$$f_V(x) = \begin{cases} b, & \text{if } x \in A_V; \\ x, & \text{if } x \in Y \setminus A_V. \end{cases}$$

Evidently f is continuous; we are going to show that all f_V are continuous. Let f_V be fixed. If $x \in A_V$, then $x = x_W$ with $V \leq W \neq V$ and $f_V(x_W) = b$. For a T -neighborhood G of $f_V(x_W)$, $G \subset Y \setminus Cl U_0$ we can take the τ -neighborhood $\{x_W, b\}$ of x_W for which $f_V(\{x_W, b\}) \subset G$, so f_V is continuous on

A_V . Now, let $x \in Y \setminus A_V$, $x \neq a$. If G is a T -neighborhood of $f_V(x)$, then it suffices to take the τ -neighborhood $\{x\}$ of x ; thus f_V is continuous on $Y \setminus (A_V \cup \{a\})$. Finally, let $x = a$ and let G be a T -neighborhood of $f_V(a)$. Then $G \cap U_0 \in N(a)$, so $G \cap U_0$ is a τ -neighborhood of a such that $f_V(G \cap U_0) \subset G$. Hence we have shown the continuity of all functions f_V . It is easy to see that the net $\{f_V : V \in N(a)\}$ pointwise converges to f . For each point $x \in (Cl U_0) \setminus U_0$ we choose a T -neighborhood U_x of x such that $\{a, b\} \cap U_x = \emptyset$; then $\mathcal{A} = \{U_0, Y \setminus Cl U_0\} \cup \{U_x : x \in (Cl U_0) \setminus U_0\}$ is a T -open cover of Y . For a finite set $\{V_1, V_2, \dots, V_n\} \subset N(a)$ with $V_i \neq V_j$ for $i \neq j$; $i, j \leq n$ we can take $V \in N(a)$ such that $V_j \leq V$ and $V_j \neq V$ for $j \leq n$. Then we have:

$$f(x_V) = x_V \in U_{x_V}, \quad f(x_V) \notin Y \setminus Cl U_0,$$

$$f_{V_j}(x_V) = b \in Y \setminus Cl U_0 \quad \text{and} \quad f_{V_j}(x_V) \notin U_{x_V} \quad \text{for} \quad j \leq n.$$

This means that the sets of the cover \mathcal{A} do not contain none of sets $\{f_{V_j}(x_V), f(x_V)\}$ for $j \leq n$, hence the net $\{f_V : V \in N(a)\}$ is not strongly convergent to f which finishes the proof.

REFERENCES

- [1] ARZELÀ, C., Sulle serie di funzioni, *Mem. R. Accad. Sci. Inst. Bologna ser. 5* (8), (1899-1900), 130-186, 701-744.
- [2] BUKOVSKÁ, Z., BUKOVSKÝ, L. and EWERT, J., Quasi-uniform convergence and L-spaces, *Real Anal. Exchange* 18 (1992-93), 321-329.
- [3] EWERT, J., Almost uniform convergence, *Periodica Math. Hungarica* 26 (1) (1993), 77-84.
- [4] POPPE, H., Compactness in function spaces with a generalized uniform structure II, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 18 (1970), 567-573.
- [5] PREDOI, M., Sur la convergence quasi-uniforme, *Periodica Math. Hungarica* 10 (1979), 31-40.
- [6] KASAHARA, S. and KASAHARA, K., Note on the Lebesgue property in uniform spaces, *Proc. Japan Acad.* 31 (1955), 1615-1617.
- [7] CSÁSZÁR, A., *General Topology*, Budapest, 1978.

Special Issue on Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/jamds/guidelines.html>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

Manuscript Due	June 1, 2009
First Round of Reviews	September 1, 2009
Publication Date	December 1, 2009

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be