

## SOME RESULTS ON $[n, m]$ -PARACOMPACT AND $[n, m]$ -COMPACT SPACES

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**ABSTRACT.** Let  $n$  and  $m$  be infinite cardinals with  $n \leq m$  and  $n$  be a regular cardinal. We prove certain implications of  $[n, m]$ -strongly paracompact,  $[n, m]$ -paracompact and  $[n, m]$ -metacompact spaces. Let  $X$  be  $[n, \infty]$ -compact and  $Y$  be a  $[n, m]$ -paracompact (resp.  $[n, \infty]$ -paracompact),  $P_n$ -space (resp.  $\omega P_n$ -space). If  $m = \sum_{k < n} m^k$  we prove that  $X \times Y$  is  $[n, m]$ -paracompact (resp.  $[n, \infty]$ -paracompact)

**KEY WORDS AND PHRASES:** Strongly paracompact and metacompact spaces.

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### 1. INTRODUCTION

Throughout this paper  $m$  and  $n$  will denote infinite cardinals with  $n \leq m$  and  $n$  will be a regular cardinal. A space  $X$  is called  $[n, m]$ -compact (see Alexandroff [1]) if every open cover  $\alpha$  of  $X$  with  $|\alpha| \leq m$  has a subcover of cardinality  $< n$ . For a set  $A$ , we denote by  $|A|$ , the cardinality of  $A$ . A family  $\alpha$  of subsets of  $X$  is a *locally- $n$*  (*point- $n$* ) family (Mansfield [2]) if for every  $x \in X$ , there is an open neighborhood of  $x$  in  $X$  which meets  $< n$  members of  $\alpha$  (resp.  $x$  belongs to  $< n$  members of  $\alpha$ ). An *open refinement* of a cover  $\alpha$  of a space  $X$  is an open cover  $\beta$  such that each member of  $\beta$  is contained in some member of  $\alpha$ . A space  $X$  is  $[n, m]$ -paracompact (resp.  $[n, m]$ -metacompact) if every open cover  $\alpha$  of  $X$  with  $|\alpha| \leq m$  has a locally- $n$  (resp. point- $n$ ) open refinement.  $X$  is  $[n, m]$ -strongly paracompact if every open cover of  $X$  with  $|\alpha| \leq m$ , has an open refinement  $\beta$  such that for each  $B \in \beta$ ,

$$|\{C \in \beta : C \cap B \neq \emptyset\}| < n.$$

Originally, Singal and Singal introduced the concept of  $(m, k)$ -paracompactness in [3]. Our notation is slightly different than theirs. However, we note that a space  $X$  is  $(m, k)$ -paracompact, as defined in [3], if and only if  $X$  is  $[k^+, m]$ -paracompact. A space  $X$  is  $[n, \infty]$ -compact (resp.  $[n, \infty]$ -paracompact,  $[n, \infty]$ -metacompact,  $[n, \infty]$ -strongly paracompact) if  $X$  is  $[n, m]$ -compact (resp.  $[n, m]$ -paracompact,  $[n, m]$ -metacompact,  $[n, m]$ -strongly paracompact for each cardinal  $m \geq n$ ). A space  $X$  is a  $P_n$ -space [4] if for every family  $\alpha$  of open subsets of  $X$  with  $|\alpha| < n$ ,  $\bigcap \alpha$  is open in  $X$ . We observe that the class of  $P_{\omega_0}$ -spaces is the class of all topological spaces, where  $\omega_0$  denotes the first infinite cardinal number. Also we observe that if  $P$  is any of "compact", "paracompact", or "metacompact", then the class of  $[\omega_0, \infty] - P$  spaces is the same as the class of  $P$  spaces in the ordinary sense.

Morita [5] studied  $m$ -paracompact spaces. A space  $X$  is  $m$ -paracompact if and only if  $X$  is  $[\omega_0, m]$ -paracompact. Morita proved that if  $Y$  is an  $m$ -paracompact space and  $X$  is a compact space, then  $X \times Y$  is  $m$ -paracompact. In case  $m = \sum_{k < n} m^k$ , we generalize Morita's result by showing that if  $X$  is an  $[n, \infty]$ -compact space and  $Y$  is  $[n, m]$ -paracompact,  $P_n$ -space, then  $X \times Y$  is  $[n, m]$ -paracompact. We note that for  $n = \omega_0$  this result implies Morita's result. A subset  $W$  of a topological space  $Y$  is called  $n$ -open (Hdeib [6]) if for each  $y \in W$  there exists an open set  $V$  in  $Y$  such that  $y \in V$  and  $|V \setminus W| < n$ . A subset  $F$  of  $Y$  is called  $n$ -closed if  $Y \setminus F$  is  $n$ -open. A space  $Y$  is called a *weak  $P_n$ -space* [6] or  $wP_n$ -space if  $\cap \alpha$  is  $n$ -open for every family  $\alpha$  of open subsets of  $Y$  with  $|\alpha| < n$ . We prove that if  $X$  is a  $[n, \infty]$ -compact space and  $Y$  is an  $[n, \infty]$ -paracompact,  $wP_n$ -space, then  $X \times Y$  is  $[n, \infty]$ -paracompact. This result is a variation of our generalization of Morita's result.

It is well known (Dungundji [7]) that if a space  $X$  is locally compact and Hausdorff, then  $X$  is paracompact if and only if  $X$  is a disjoint topological sum of  $\sigma$ -compact spaces. We prove that if  $n > \omega_0$ , then a locally  $[n, \infty]$ -compact, regular space  $X$  is  $[n, \infty]$ -paracompact if and only if  $X$  is a disjoint topological sum of  $[n, \infty]$ -compact spaces. A space  $X$  is, by definition, locally  $[n, \infty]$ -compact if for each point  $x \in X$  and an open neighborhood  $G$  of  $x$ , there exists an  $[n, \infty]$ -compact neighborhood  $H$  of  $x$  such that  $H \subseteq G$ .

In this paper we also prove certain implications concerning  $[n, m]$ -paracompact, metacompact, strongly paracompact spaces.

For a space  $X$ , the *density*  $d(X)$  of  $X$  is defined as the smallest cardinal number that is the cardinal number of a dense subset of  $X$ . For terminology not defined here see Engelking [8].

## 2. $[n, m]$ -PARACOMPACT SPACES

It is clear that each  $[n, m]$ -strongly paracompact space is  $[n, m]$ -paracompact which in turn is  $[n, m]$ -metacompact. However, in general, the converses of these implications do not hold.

The following two theorems are interesting in this respect.

**THEOREM 2.1.** Let  $\gamma$  be an open cover of a space  $X$  such that  $|\gamma| \leq m$  and  $d(A) < n$  for each  $A \in \gamma$ . Then  $X$  is  $[n, m]$ -strongly paracompact if and only if  $X$  is  $[n, m]$ -metacompact.

**PROOF.** We only need to prove "if" part. Let  $X$  be  $[n, m]$ -metacompact. Let  $\alpha$  be an open cover of  $X$  with  $|\alpha| \leq m$ . Let  $\beta = \{A \cap W : A \in \gamma \text{ and } W \in \alpha\}$ . Then  $|\beta| \leq m$ ,  $\beta$  is an open refinement of  $\alpha$  and  $d(B) < n$  for each  $B \in \beta$ . Since  $X$  is  $[n, m]$ -metacompact, then there exists a point- $n$  open refinement  $\lambda$  of  $\beta$ . Each  $L \in \lambda$  is contained in some  $B_L \in \beta$ . Since  $L$  is open and  $d(B_L) < n$ , then  $d(L) < n$ . Let  $L \in \lambda$  and  $D$  be a dense set in  $L$  such that  $|D| < n$ . Let  $\Delta = \{A \in \lambda : A \cap L \neq \emptyset\}$ . Since  $D$  is dense in  $L$ , then  $A \in \Delta$  if and only if  $A \cap D \neq \emptyset$ . Thus  $\Delta = \{A \in \lambda : A \cap D \neq \emptyset\}$ . For  $d \in D$  let us set  $\Delta_d = \{K \in \lambda : d \in K\}$ . Then  $|\Delta_d| < n$  since  $\lambda$  is point- $n$ . Hence

$$|\Delta| \leq \sum_{d \in D} |\Delta_d| < n.$$

Since  $|D| < n$  and  $n$  is a regular cardinal, it follows that  $X$  is  $[n, m]$ -strongly paracompact.

**COROLLARY 2.1** (Traylor [9]). Let  $X$  be a regular space with an open cover  $\gamma$  such that  $d(G) \leq \omega_0$  for all  $G \in \gamma$ . Then  $X$  is strongly paracompact if and only if  $X$  is metalindelöf.

**PROOF.** The proof follows from Theorem 2.1 and Theorem 3, page 229 in [8].

**THEOREM 2.3.** Let  $X$  be a locally  $[n, \infty]$ -compact space. Then  $X$  is  $[n, \infty]$ -paracompact if and only if  $X$  is  $[n, \infty]$ -strongly paracompact.

**PROOF.** We only need to prove "only if" part. Let  $X$  be  $[n, \infty]$ -paracompact. Let  $\alpha$  be an open cover of  $X$ . Since  $X$  is locally  $[n, \infty]$ -compact then there exists a cover  $\sigma$  of  $X$  such that

- (i)  $\sigma$  refines  $\alpha$
- (ii)  $\beta = \{\text{int } H : H \in \sigma\}$  is a cover of  $X$ ,

(iii) if  $H \in \sigma$ , then  $H$  is  $[n, \infty]$ -compact

Since  $X$  is  $[n, \infty]$ -paracompact, then  $\beta$  has a locally- $n$  open refinement  $\gamma$ . Now, let  $G \in \gamma$  and

$$\Delta = \{L \in \gamma : G \cap L \neq \emptyset\}.$$

Since  $\gamma$  refines  $\beta$ , then  $G \subseteq \text{int } H \subseteq H$  for some  $H \in \sigma$ . For each  $x \in H$ , there is an open set  $W_x$  containing  $x$  such that  $W_x$  meets  $< n$  members of  $\gamma$ . We have

$$H = \bigcup \{W_x \cap H : x \in H\}.$$

Since  $H$  is  $[n, \infty]$ -compact, then there exists a subset  $T$  of  $H$  such that  $|T| < n$  and

$$H = \bigcup \{W_x \cap H : x \in T\}.$$

For  $x \in T$ . Let us set

$$\Delta_x = \{L \in \gamma : W_x \cap L \neq \emptyset\}.$$

We see that

$$\Delta \subseteq \{\Delta_x : x \in T\}.$$

Hence

$$|\Delta| \leq \sum_{x \in T} |\Delta_x| < n.$$

Since  $|T| < n$ ,  $|\Delta_x| < n$  for each  $x \in T$  and  $n$  is a regular cardinal.

**COROLLARY 2.4.** Let  $X$  be a regular, locally Lindelöf space. Then  $X$  is strongly paracompact if and only if  $X$  is paralindelöf

**PROOF.** The proof follows from Theorem 2.3 and Theorem 3, page 229 in [8].

It is well known in [7] that if  $X$  is a locally compact Hausdorff space, then  $X$  is paracompact if and only if  $X$  is a disjoint topological sum of  $\sigma$ -compact spaces. It is natural to ask when  $X$  is a locally  $[n, \infty]$ -compact,  $[n, \infty]$ -paracompact space, whether  $X$  is a disjoint topological sum of  $\sigma$ - $[n, \infty]$ -compact spaces. The result above is the answer to the case when  $n = \omega_0$  and  $X$  is Hausdorff. So we are only interested in the case when  $n > \omega_0$ . The following theorem provides the answer to this question

**THEOREM 2.5.** Let  $n > \omega_0$  and  $X$  be a locally  $[n, \infty]$ -compact regular space. Then  $X$  is  $[n, \infty]$ -paracompact if and only if  $X$  is a disjoint topological sum of  $[n, \infty]$ -compact spaces

**PROOF.** It is obvious that if  $X$  is a disjoint topological sum of  $[n, \infty]$ -compact spaces, then  $X$  is  $[n, \infty]$ -paracompact. Thus let us assume that  $X$  is  $[n, \infty]$ -paracompact. Let

$$\alpha = \{U : U \subseteq X \text{ and } U \text{ is } [n, \infty]\text{-compact}\}.$$

Then  $\beta = \{\text{int } U : U \in \alpha\}$  is an open cover of  $X$  since  $X$  is locally  $[n, \infty]$ -compact. Since  $X$  is regular, then there is an open cover  $\gamma$  of  $X$  such that  $\bar{\gamma} = \{c\ell G : G \in \gamma\}$  refines  $\beta$ . Since  $X$  is a locally  $[n, \infty]$ -compact,  $[n, \infty]$ -paracompact space, then by Theorem 2.3,  $X$  is  $[n, \infty]$ -strongly paracompact. Hence there exists an open refinement  $\sigma$  of  $\gamma$  such that for each  $L \in \sigma$  the set  $\Delta_L = \{H \in \sigma : L \cap H \neq \emptyset\}$  has cardinality  $n$ . For a positive integer  $t$ , a chain of length  $t$  in  $\sigma$  is a sequence  $L_1, \dots, L_t$  in  $\sigma$  such that  $L_i \cap L_{i+1} \neq \emptyset$  for  $1 \leq i \leq t-1$ . If  $t = 1$  we simply require  $L_1 \neq \emptyset$ . For  $x, y \in X$  we define  $x \sim y$  if there is a chain  $L_1, \dots, L_t$  in  $\sigma$  such that  $x \in L_1$  and  $y \in L_t$ . Clearly " $\sim$ " is an equivalence relation since  $\sigma$  is an open cover of  $X$ . Let  $R$  be an equivalence class and  $a \in R$ . If  $y \in R$ , then there is a chain  $L_1, \dots, L_t$  in  $\sigma$  such that  $a \in L_1$  and  $y \in L_t$ . Clearly each point in  $L_t$  is equivalent to  $a$  with respect to " $\sim$ ", hence  $L_t \subseteq R$ . So  $R$  is open. Let  $z \in c\ell R$ . There exists  $L \in \sigma$  such that  $z \in L$ . Since  $z \in c\ell R$ , then  $L \cap R \neq \emptyset$ . Thus if  $w \in L \cap R$ , then  $z \sim w$ , i.e.,  $z \in R$ . This shows that  $R$  is also closed. Let  $a \in L$  and  $L \in \sigma$ . We know that  $L \subseteq R$ . For a positive integer  $t$ , let

$$\mu_t = \{H \in \gamma : \text{there is a chain } L_1, \dots, L_t \text{ in } \sigma \text{ such that } L = L_1 \text{ and } L_t = H\}.$$

Clearly  $\mu_1 = \{L\}$ . Thus  $|\mu_1| < n$ . Assume that  $|\mu_t| < n$ . If  $K \in \mu_{t+1}$ , then there is a chain  $L_1, L_2, \dots, L_t, L_{t+1}$  in  $\sigma$  such that  $L = L_1$  and  $K = L_{t+1}$ . Then  $L_t \in \mu_t$ . Thus

$$\mu_{t+1} \subseteq \bigcup \{\Delta_H : H \in \mu_t\}.$$

Hence

$$|\mu_{t+1}| \leq \sum_{H \in \mu_t} |\Delta_H| < n,$$

since  $|\mu_t| < n$  and  $n$  is a regular cardinal. This inductive argument shows that  $|\mu_t| < n$  for all  $t \geq 1$ . We show that  $R = \bigcup \{R_t : t \geq 1\}$  where  $R_t = \bigcup \{clH : H \in \mu_t\}$ . If  $H \in \mu_t$ , then by the definition of " $\sim$ " we get  $H \subseteq R$ . Since  $R$  is closed, then  $clH \subseteq R$ . So  $R \supseteq \bigcup R_t$ . Conversely let  $y \in R$ . Then there is a chain  $L_1, \dots, L_t$  in  $\sigma$  such that  $a \in L_1$  and  $y \in L_t$ . Since  $a \in L_1 \cap L$ , then  $L, L_1, \dots, L_t$  is a chain in  $\sigma$ . Thus  $L_t \in \mu_{t+1}$ ; and consequently  $y \in \bigcup R_t$ . This proves the result

Now, if  $H \in \sigma$ , then  $H \subseteq clE \subseteq U$  for some  $G \in \gamma$  and  $U \in \alpha$ . Thus  $clG$  and consequently  $clH$  is  $[n, \infty]$ -compact. Since  $|\mu_t| < n$  when  $t$  is a positive integer, then  $R_t$  is also  $[n, \infty]$ -compact. Since  $n > \omega_0$ , then  $R = \bigcup R_t$  is also  $[n, \infty]$ -compact. This proves the theorem since  $X$  is the disjoint topological sum of the equivalence classes of " $\sim$ ".

### 3. PRODUCT THEOREMS

In this section we prove theorems concerning  $[n, m]$ -paracompact of a product space  $X \times Y$ . Our first theorem is a generalization of a result by Morita [5] which states that if  $X$  is a compact space and  $Y$  is an  $m$ -paracompact space, then  $X \times Y$  is an  $m$ -paracompact space.

**THEOREM 3.1.** Let the cardinal  $m$  satisfy  $m = \sum \{m^k : k \text{ is a cardinal and } k < n\}$ . Let  $X$  be an  $[n, \infty]$ -compact space and  $Y$  be an  $[n, m]$ -paracompact  $P_n$ -space. Then  $X \times Y$  is  $[n, m]$ -paracompact

**PROOF.** Let  $\alpha$  be an open cover of  $X \times Y$  with  $|\alpha| \leq m$ . For each subset  $\beta$  of  $\alpha$  with  $|\beta| < n$ , let  $W_\beta = \{y \in Y : X \times \{y\} \subseteq \bigcup \beta\}$ . Let  $\beta \subseteq \alpha$  and  $|\beta| < n$ . Then  $W_\beta$  is open in  $X$ . For let  $y \in W_\beta$ . Then  $X \times \{y\}$  is contained in  $G = \bigcup \beta$ . For each  $x \in X$ , there exists a basic open set  $B_x \times C_x$  in  $X \times Y$  such that  $(x, y) \in B_x \times C_x \subseteq G$ . Now  $\{B_x : x \in X\}$  is an open cover of  $X$ . Thus there is a subcover  $\{B_x : x \in S\}$  where  $|S| < n$ .  $C = \bigcap \{C_x : x \in S\}$  is open in  $Y$ , since  $Y$  is a  $P_n$ -space and  $y \in G$ . Moreover,  $X \times C \subseteq \bigcup \{B_x \times C : x \in S\} \subseteq G$ . It follows that  $y \in C \subseteq W_\beta$ . So  $W_\beta$  is open. Let us set

$$\Lambda = \{W_\beta : \beta \subseteq \alpha \text{ and } |\beta| < n\}.$$

Let  $y \in Y$ . For each  $x \in X$ , there exists  $A_x \in \alpha$  such that  $(x, y) \in A_x$ . There is a basic open set  $D_x \times E_x$  in  $X \times Y$  such that  $(x, y) \in D_x \times E_x \subseteq A_x$ . Now,  $\{D_x : x \in X\}$  is an open cover of  $X$ . Thus it has a subcover  $\{D_x : x \in T\}$  such that  $|T| < n$ .

Let  $\beta = \{A_x : x \in T\}$ . Then  $|\beta| < n$  and  $X \times \{y\} \subseteq \bigcup_{x \in T} D_x \times \{y\} \subseteq \bigcup \beta$ . Thus  $y \in W_\beta$ . This shows that  $\Lambda$  is an open cover of  $Y$ . Further notice that

$$|\Lambda| \leq \sum_{k < n} m^k = m.$$

Thus there exists a locally- $n$  open refinement  $\mu$  of  $\Lambda$  since  $Y$  is  $[n, m]$ -paracompact. For each  $M \in \mu$  we pick  $\beta_M \subseteq \alpha$  such that  $|\beta_M| < n$  and  $M \subseteq W_{\beta_M}$ . For  $A \in \beta_M$  we define  $G(M, A) = (X \times M) \cap A$ . Let  $\rho = \{G(M, A) : M \in \mu, A \in \beta_M\}$ . If  $(x, y) \in X \times Y$ , then  $y \in M \subseteq W_{\beta_M}$  for some  $M \in \mu$ . Since  $y \in W_{\beta_M}$ , then  $X \times \{y\} \subseteq \bigcup \beta_M$ . Thus  $(x, y) \in A$  for some  $A \in \beta_M$ . Hence  $(x, y) \in G(M, A)$ .

This shows that  $\rho$  is an open cover of  $X \times Y$ . Clearly  $\rho$  refines  $\alpha$ . Let  $(x, y) \in X \times Y$ . There exists an open set  $N$  in  $Y$  such that  $y \in N$  and  $N$  meets  $< n$  members of  $\mu$ . Let  $\mu' = \{M \in \mu : N \cap M \neq \emptyset\}$ . Thus we have  $|\mu'| < n$ . If  $M \notin \mu'$ , then  $(X \times N) \cap G(M, A) = \emptyset$  for all  $A \in \beta_M$ . Thus the open neighborhood  $X \times N$  of  $(x, y)$  can only meet those  $G(M, A)$  with  $M \in \mu'$  and  $A \in \beta_M$ . The cardinality of such  $G(M, A)$ 's is at most  $\sum_{M \in \mu'} |\beta_M|$  which is less than  $n$  since  $|\mu'| < n$ ,  $|\beta_M| < n$  for each  $M \in \mu'$  and  $n$  is a regular cardinal. Hence  $\rho$  is a locally- $n$  family.

In Theorem 3.1 if we assume the stronger condition that  $Y$  is  $[n, \infty]$ -paracompact then we can show that  $X \times Y$  is  $[n, \infty]$ -paracompact if we only assume that  $Y$  is a  $wP_n$ -space. Before we prove this result we first prove two theorems which are interesting in their own rights.

Let  $A$  and  $B$  be topological spaces and  $f : A \rightarrow B$  be a function.  $f$  is called  $n$ -closed if for every closed subset  $F$  of  $A$ ,  $f(F)$  is an  $n$ -closed subset of  $B$ .

**THEOREM 3.2.** Let  $X$  be an  $[n, \infty]$ -compact space and  $Y$  be a  $wP_n$ -space. Then the projection mapping  $P : X \times Y \rightarrow Y$  is an  $n$ -closed map.

**PROOF.** Let  $F$  be closed in  $X \times Y$  and  $y$  be in  $U = Y \setminus P(F)$ . Then  $(x, y) \notin F$  for each  $x \in X$ . Hence there are open sets  $U_x$  in  $X$  and  $V_x$  in  $Y$ , for each  $x \in X$ , such that  $(x, y) \in U_x \times V_x$  and  $F \cap (U_x \times V_x) = \emptyset$ .  $\alpha = \{U_x : x \in X\}$  is an open cover of  $X$ . Since  $X$  is  $[n, \infty]$ -compact, then there exists a subset  $T$  of  $X$  such that  $|T| < n$  and  $\beta = \{U_x : x \in T\}$  covers  $X$ .  $W = \bigcap \{V_x : x \in T\}$  is  $n$ -open in  $Y$  since  $Y$  is a  $wP_n$ -space and  $y \in W$ . Hence there exists an open set  $V$  in  $Y$  such that  $y \in V$  and  $|V \setminus W| < n$ . Now, we have  $X \times W \cap F = \emptyset$ . Hence  $W \subseteq U$ . Thus  $|V \setminus U| < n$ . It follows that  $U$  is  $n$ -open. Thus  $P$  is  $n$ -closed.

**THEOREM 3.3.** Let  $f : Z \rightarrow Y$  be a continuous,  $n$ -closed mapping such that  $f^{-1}(y)$  is  $[n, \infty]$ -compact for such  $y \in Y$ . If  $Y$  is  $[n, \infty]$ -paracompact (resp.  $[n, \infty]$ -compact) then  $Z$  is also  $[n, \infty]$ -paracompact (resp.  $[n, \infty]$ -compact).

**PROOF.** We will only prove the case when  $Y$  is  $[n, \infty]$ -paracompact. The  $[n, \infty]$ -compact case can be proved similarly.

Let  $\alpha$  be an open cover of  $Z$ . For each  $y \in Y$  let  $\alpha_y$  be a subcollection of  $\alpha$  such that  $|\alpha_y| < n$  and  $f^{-1}(y) \subseteq \bigcup \alpha_y$ . Such a subcollection exists since  $f^{-1}(y)$  is  $[n, \infty]$ -compact. For  $y \in Y$ , let  $G_y = \bigcup \alpha_y$ , and  $W_y = Y \setminus f(Z \setminus G_y)$ . Then  $y \in W_y$  and  $W_y$  is  $n$ -open since  $f$  is an  $n$ -closed map. Thus for each  $y \in Y$ , there is an open set  $V_y$  in  $Y$  such that  $y \in V_y$  and  $|V_y \setminus W_y| < n$ .  $\gamma = \{V_y : y \in Y\}$  is an open cover of  $Y$  and  $Y$  is  $[n, \infty]$ -paracompact. Hence there exists a locally- $n$  open refinement  $\{T_i : i \in I\}$  of  $\gamma$ . For each  $i \in I$ , pick  $y_i \in Y$  such that  $T_i \subseteq V_{y_i}$ . For  $y \in Y$  let

$$\beta_y = \alpha_y \cup (\bigcup (\alpha_t : t \in V_y \setminus W_y)).$$

Then

$$|\beta_y| \leq |\alpha_y| + \sum \{|\alpha_t| : t \in V_y \setminus W_y\} < n,$$

since  $n$  is a regular cardinal. Moreover  $f^{-1}(T_i) \subseteq \bigcup \beta_{y_i}$  since  $T_i \subseteq V_{y_i}$ . Let

$$\sigma = \{H \cap f^{-1}(T_i) : H \in \beta_{y_i}, i \in I\}.$$

Then clearly  $\sigma$  is an open refinement of  $\alpha$ . Let  $x \in Z$  and  $y = f(x)$ . There is an open set  $N$  in  $Y$  and a subset  $J$  of  $I$  such that  $|J| < n$ ,  $y \in N$  and  $N \cap T_i = \emptyset$  for all  $i \in I \setminus J$ . Let  $M = f^{-1}(N)$  and  $\Lambda = \{H \cap f^{-1}(T_i) : H \in \beta_{y_i}, i \in J\}$ . Then  $x \in M$  and  $|\Lambda| \leq \sum_{i \in J} |\beta_{y_i}| < n$  since  $n$  is a regular cardinal. Moreover, if  $L \in \sigma \setminus \Lambda$ , then  $L \cap M = \emptyset$ . Hence  $\sigma$  is a locally- $n$  family.

As a corollary of Theorem 3.2 and Theorem 3.3 we obtain the following variation of Theorem 3.1

**THEOREM 3.4.** Let  $X$  be an  $[n, \infty]$ -compact space and  $Y$  be an  $[n, \infty]$ -paracompact (resp  $[n, \infty]$ -compact)  $wP_n$ -space, then  $X \times Y$  is  $[n, \infty]$ -paracompact (resp  $[n, \infty]$ -compact)

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