

## THE $GL(n, F_p)$ — INVARIANCE OF THE POTTS HAMILTONIAN

MIHAI CARAGIU

Department of Mathematics  
Penn State University  
218 McAllister Building  
University Park, PA 16802 USA

MELLITA CARAGIU

Department of Physics  
"Babes-Bolyai" University  
Str. M. Kogalniceanu 1  
3400 Cluj-Napoca, ROMANIA

(Received June 8, 1995 and in revised form August 8, 1995)

**ABSTRACT.** After defining a meanfield by arithmetic means, using multiplicative characters of finite fields, its Potts Hamiltonian is exactly computed. Moreover, it proves to be invariant with respect to every change of basis in  $F_q$  over the prime field  $F_p$ .

**KEY WORDS AND PHRASES:** Finite fields, characters, Hamiltonian invariance

**1991 AMS SUBJECT CLASSIFICATION CODES:** 11L40, 82B20, 11T24, 51E25.

### 1. INTRODUCTION

Let us consider the finite field  $F_q$ ,  $q = p^n$ , as an  $n$ -dimensional vector space over its prime subfield  $F_p$ . This is, of course, done by choosing a basis  $e = (e_1, e_2, \dots, e_n)$  — an (ordered)  $n$ -tuple of "vectors" linearly independent over  $F_p$ . Then, we can see  $F_q$  in a natural way as an  $n$ -dimensional hypercubic lattice with size  $p$ . Every element  $z$  of this lattice can be expressed in a unique way as  $z = z_1 e_1 + \dots + z_n e_n$ , the coefficients  $z_i$  satisfying periodic boundary conditions modulo  $p$  (that is, if we substitute  $z_i + p$  instead of  $z_i$ , the same point  $z$  is obtained). That means that the above defined hypercubic lattice can be seen as a discrete model of the  $n$ -dimensional torus  $T^n = S^1 \times S^1 \times \dots \times S^1$  we obtain thus a finite field "wrapped" on a torus.

Suppose now that  $r > 1$  is a natural number, and  $q \equiv 1 \pmod{r}$ . This happens, for example, if  $p \equiv 1 \pmod{r}$ . We may consider then a character (multiplicative) of  $F_q$ , that is a surjective morphism of multiplicative groups  $f : F_q^* \rightarrow U_r$ , where  $U_r$  is the group of complex  $r$ -th roots of unity. Then, to every point  $z$  of our lattice we associate an  $r$ -th root of unity, namely  $f(z)$ . We obtain thus a discrete analogue for a section of a  $U(1)$ -bundle over the  $n$ -dimensional torus. Also we may see this as a physical field,  $f(z)$  being the "spin" attached to site  $z$ . Our aim is to study this field using some ideas of statistical mechanics ([1]-basic reference), with the hope of obtaining more information about such pure mathematical entities as characters of finite fields.

## 2. NEAREST NEIGHBORS

Once we have chosen a basis  $e = (e_1, e_2, \dots, e_n)$  as above, a natural concept of nearest neighbor sites pops up — namely, we say that the sites  $z$  and  $w$  are  $e$ -nearest neighbors, and we denote this by  $z \sim w \pmod{e}$ , if the difference  $z - w$  belongs to the set  $\{e_1, \dots, e_n, -e_1, \dots, -e_n\}$ . Of course, this concept is strongly dependent on the choice of the basis  $e$ . If we change the basis, it may happen that we obtain a totally different set of nearest neighbor pairs. Obviously, if one changes the basis only by permuting the  $e_i$ 's between themselves, and/or by changing the sign of some of them, the set of nearest neighbor pairs remains the same. Ignoring the action of the symmetric group, this is just  $Z/2Z \times \dots \times Z/2Z$ -invariance, but most of the transformations are not so simple. We obtain thus a model for a strange chaotic medium, in which the very idea of nearest neighbor is strongly dependent on the action of  $GL(n, F_p)$  — the group which takes care about all base changes in  $F_q$  over  $F_p$ .

## 3. THE MEANFIELD AND ITS POTTS HAMILTONIAN

We return now to our field  $f$  defined on  $F_q$ , viewed as an  $n$ -dimensional lattice wrapped on  $T^n$ . We need, in one way or another, to prove that this is a representative field from the randomness point of view, i.e., that it is a "meanfield", in the physical use of the term. This follows from a result [2] about the distribution of the values taken by a multiplicative character of  $F_q$ . Namely, let us consider an integer  $k < p$ . To every  $z$  in  $F_q$ , we can associate a function  $h_z : \{0, 1, \dots, k-1\}^n \rightarrow U_r$ , given as follows

$$h_z(a_1, \dots, a_n) := f(z + a_1 e_1 + \dots + a_n e_n).$$

The functions  $h_z$  can be seen as a means of testing the local state of the field in the  $e$ -neighborhood of  $z$  represented by a hypercube of size  $k$ , with the point  $z$  as one of its corners: namely,  $h_z$  is only one out of a total of  $r^{k^n}$  functions  $h : \{0, 1, \dots, k-1\}^n \rightarrow U_r$ . The main result in [2] states that all the functions  $h$  have an "almost equal" probability of occurrence as an  $h_z$ .

Namely, if we denote by  $N_h$  the number of those  $z$  in  $F_q$  for which  $h_z = h$  ( $k$  is fixed), we have the following estimate, where we have denoted  $k^n$  by  $b$ :

$$N_h = (q+1)/r^b + O(b \cdot q^{1/2}), \quad \text{when } q \text{ is large.}$$

The proof uses the Lang-Weil estimates applied to an appropriate algebraic curve, whose genus is computed using the Riemann-Hurwitz formula.

Moreover, the constant implied by 0 is smaller than 1.

Now, we can assume that our field is indeed a meanfield, in the most natural sense. We are thus ready to define its Potts Hamiltonian:

$$H = \sum_{z \sim w \pmod{e}} \delta_{(f(z), f(w))}$$

where  $\delta_{(i,j)}$  is 1 if  $i = j$  and 0 otherwise (we assume that the elementary energy of interaction [1] is  $J = 1$ ). By definition, it seems that  $H$  is dependent on both the basis  $e$  and the character choice. Of course  $GL(n, F_p)$  acts as well on the Hamiltonian  $H$ , by an action induced by its action on the basis  $e$  (on which the neighbor system is strongly dependent). However, we shall see that the action of  $GL(n, F_p)$  on the Hamiltonian is trivial.

## 4. THE $GL(n, F_p)$ -INVARIANCE

We prove now the following theorem.

**THEOREM.**  $H$  does not depend on the choice of the basis  $e$ , nor on the character  $f$ .

**PROOF.** First, we may note that one may extend the definition of the character to the whole field by  $f(0) := 0$ . As a physical interpretation one may consider either the spin 0 at the site  $z = 0$ , or, maybe better, one may consider in fact a field theory on a punctured torus.

The proof is actually very simple. First, let us consider an apparently more general situation if  $a$  is an arbitrary element of  $F_q^*$ , we count first the number of those  $z$  in  $F_q$  for which the character takes the same values at  $z$  and  $z + a$

If  $f(z) = f(z + a)$ , then obviously  $z, z + a$  are non-zero, and  $(z + a)/z = y$  is an  $r$ -th power in  $F_q^* - \{1\}$ . Thus,  $y$  can take a number of  $(q - r - 1)/r$  values, and for every such  $y$ ,  $z$  is uniquely determined

Thus we have found  $(q - r - 1)/r$  possible values of  $z$  with the character taking the same values at  $z$  and  $z + a$ . Take now  $a = e_1, e_2, \dots, e_n$ . In each of these  $n$  directions, the number of  $e$ -nearest neighbor pairs to which the same spin is assigned is  $(q - r - 1)/r$ . It follows that the total energy or Potts Hamiltonian is exactly given by

$$H = n(q - r - 1)/r.$$

This computation, in fact, proves the  $GL(n, F_p)$ -invariance, because, as one can see, the above relation for  $H$  does not depend on  $e$  or on  $f$ .

**REMARK 1.** The same invariance property holds also (if one takes  $r = 2$ ) for the Ising Hamiltonian [1].

**REMARK 2.** One may work also with more general extensions of finite fields and obtain in a similar way a concept of  $e$ -nearest neighbor for every basis of  $F_{q^n}$  over  $F_q$  but the prime field  $F_p$  is more suitably to be taken as a base field, due to its interpretation as a linear spatial array

**REMARK 3.** For  $n = 1$ , the change of basis is, in fact, a rescaling. Thus, we get, in this particular case, the "scale-invariance" of the Potts Hamiltonian. If one chooses another scale  $e$ , the elements of the  $F_p^*$  are reordered as  $e, 2e, \dots, (p - 1)e$ , and the worst that can happen is the interchange of quadratic residues and nonresidues.

## 5. FURTHER COMMENTS

The most natural step forward would be to define a partition function  $Z$ .

It is known [1] that all the information about a statistical system is hidden in the partition function. Our hope is to define an appropriate  $Z$  in order to get, via its analytic properties, new insights concerning this strange connection between the arithmetic of finite fields and field theory.

Recall that in order to define  $Z$ , we need a set of states. Then, we attach a Boltzmann weight [1] to every state in this set.  $Z$  will be the sum of all these weights. Hopefully, in the case of a field associated with a character  $f$  of  $F_q^*$  the Boltzmann weight has a simple form:

$$W(f) = \exp(-\beta n(q - r - 1)/r)$$

where  $r$  is the order of  $f$ ,  $n$  the dimension of the space, and  $\beta$  a parameter playing the role of inverse temperature. Note that  $q$  is the lattice volume

In order to obtain a proper  $Z$ , one has to select with great care a set of states satisfying two basic requirements. First, it should not be a "small" set (otherwise the formalism of statistical mechanics will be useless). Second, the selected states must be meaningful from an arithmetical point of view, that is, closely related to the multiplicative characters.

However, the total number of multiplicative characters of  $F_q$  is  $q - 1$ , which is rather small even if we fix  $r$ , the number of  $r$ -valued spin distributions on  $F_q^*$  is  $r^{q-1}$ . In order to deal with a larger number of states, one may choose to take into account the meanfield property (see section 3). Namely, for every divisor  $r$  of  $q - 1$ , we will select a character  $f_r$  of order  $r$  (we shall call it a "basic state"). By the meanfield property we will agree to consider every such basic state  $f_r$  as an average field out of a total number of  $r^{q-1}$   $r$ -valued random spin distributions on our punctured torus. Thus, in the expression of  $Z$  we have to multiply the Boltzmann weight  $W(f_r)$  with a corresponding factor of  $r^{q-1}$ . If we restrict, for example, the multiplicity of the spin to a single value  $r$ , then for every  $q \equiv 1 \pmod{r}$  we get

$$Z = r^{q-1} W(f_r) = \exp((q-1) \log r - \beta n(q-1-r)/\tau).$$

One may note that the function  $G(\beta) := -\lim_q (\log Z)/\beta q$  — that is, the so-called Gibbs free energy density (in thermodynamical limit) presents in this case a sign change at  $\beta = (\tau/n) \log r$

On the other hand, if the multiplicity of  $r$  of the spin is unrestricted, that is if  $r$  can be every divisor of  $q-1$ , the partition function will be given by a sum over the divisors of  $q-1$ , namely

$$Z = \sum_{r|q-1} r^{q-1} W(f_r).$$

More generally, one may consider some special class of divisors of  $q-1$ . Then, the partition function  $Z$  will be the associated divisor sum. This could lead us to some deep arithmetical problems

For an explicit selection of new states, without making use of the meanfield approach, one may consider the set of all " $GL(n, F_p)$ -gauge invariant functions:  $g : F_q^* \rightarrow U_r$ , that is, functions for which the Potts Hamiltonian is well defined (does not depend on the choice of the basis of  $F_q$  over  $F_p$ ). The partition function can be written then as the sum over all such  $g$  of the Boltzmann weights  $\exp(-\beta H(g))$ , where  $H(g)$  is the Hamiltonian associated to the function  $g$

## REFERENCES

- [1] BAXTER, J., *Exactly Solved Models in Statistical Mechanics*, London, New York, Academic Press, 1982.
- [2] CARAGIU, M. and MULLEN, G., L., The distribution of  $r$ -th powers in finite fields, Penn State University Preprint, PM 161/1993.

## Special Issue on Singular Boundary Value Problems for Ordinary Differential Equations

### Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/bvp/guidelines.html>. Authors should follow the Boundary Value Problems manuscript format described at the journal site <http://www.hindawi.com/journals/bvp/>. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	May 1, 2009
First Round of Reviews	August 1, 2009
Publication Date	November 1, 2009

### Lead Guest Editor

**Juan J. Nieto**, Departamento de Análisis Matemático,  
Facultad de Matemáticas, Universidad de Santiago de

Compostela, Santiago de Compostela 15782, Spain;  
[juanjose.nieto.roig@usc.es](mailto:juanjose.nieto.roig@usc.es)

### Guest Editor

**Donal O'Regan**, Department of Mathematics, National  
University of Ireland, Galway, Ireland;  
[donal.oregan@nuigalway.ie](mailto:donal.oregan@nuigalway.ie)