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JOINS OF EUCLIDEAN ORBITAL TOPOLOGIES

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ABSTRACT. This paper is concerned with joins of orbital topologies especially on the orbit of the reals with the usual topology.

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The importance of comparing two different topologies on the same set was noted by Garrett Birkhoff in 1936 [1]. Let X be a set and L(X) be the lattice of all topologies on X. If f is a bijective function from X to X and τ is a fixed topology on X, then we can define $\tau_f = \{f(U) \mid U \in \tau\}$. Note that τ_f is a topology. Let $\mathfrak{B}$ be the set of all bijections from X to X. Define $\{\tau_f \mid f \in \mathfrak{B}\}$ to be the orbit of τ in L(X). The topologies in this orbit are homeomorphic to each other. Also note that for all bijective functions f and g, there exists a bijection h such that $\tau_f \vee \tau_g{}'$ is homeomorphic to $\tau \vee \tau_h$.

Throughout this paper we will refer to the orbit of the usual topology on the reals as the <u>Euclidean Orbit</u>. All functions will be bijective, and $\{(x, f(x)) \mid x \in X\}$, the graph of f, will be denoted G(f).

Bourbaki [2] showed $(X, \tau \vee \tau_f)$ is homeomorphic to $\{(x, x) \mid x \in X\}$ with the relative topology of $\tau \times \tau_f$ via $h(x) = (x, x)$. Clearly, $(X \times X, \tau \times \tau_f)$ is homeomorphic to $(X \times X, \tau \times \tau)$ via $F(x, y) = (x, f^{-1}(y))$. Hence $(X, \tau \vee \tau_f)$ is homeomorphic to $(G(f^{-1}), \tau \times \tau)$. It is this graph which will help us discover properties of $(X, \tau \vee \tau_f)$.

Note that if X is a metric space, it is trivial to see that $\tau \vee \tau_f$ is metric. But locally compact is not so clear. Given a locally compact Hausdorff space, we have the following:

THEOREM 1. Let $G^{\#}(f) = cl(G(f)) - G(f)$. $\tau \vee \tau_{f^{-1}}$ is locally compact if and only if $cl(G^{\#}(f)) \cap G(f) = \varnothing$

PROOF. If $cl(G^{\#}(f)) \cap G(f) \neq \varnothing$, then let $p \in cl(G^{\#}(f)) \cap G(f)$. Then $p \notin G^{\#}(f)$; hence p is in the derived set. Let C be a compact neighborhood of p in G (f); then there exists an open $V \subset X^2$ such that $V \cap G(f) \subset C$ and $V \cap G(f)$ is compact. Since cl(V) is a neighborhood of p in X^2, there exists a point $q \in V$ such that $q \in G^{\#}(f)$. Let $\{V_\alpha\}$ be a basis at q. Since X is regular, we can assume there is a basis element V_β such that $cl(V_\beta) \subseteq V_\alpha$ Let $U_\alpha = X - cl(V_\beta)$; then $\{U_\alpha\}$ covers X - q. Hence $\{U_\alpha\}$ covers $G(f) \cap cl(V)$. But since $G(f) \cap cl(V)$ is compact, there exists a finite subcover $\{U_{\alpha_1},\dots,U_{\alpha_m}\}$ which covers $G(f) \cap cl(V)$. Let U be the union of the subcover. Then U covers $G(f) \cap cl(V)$. This is a contradiction since $q \in cl(G(f))$, but $q \notin U$.

Now suppose $cl(G^{\#}(f)) \cap G(f) = \varnothing$ and let $p \in G(f)$. Then there is an open U containing p such that $U \cap G^{\#}(f) = \varnothing$. Also we can find an open neighborhood V of p such that $cl(V) \subset U$. Since $cl(V) \cap cl(G^{\#}(f)) = \varnothing$, $G(f) \cap cl(V)$ is closed. Therefore, G (f) is locally compact.

For the remainder of this paper, we restrict ourselves to the Euclidean orbit. In the Euclidean orbit we know that $\tau = \tau_f$ only if f is continuous and that since τ is connected, τ_f is also, but what about $\tau \vee \tau_f$?

THEOREM 2. $\tau \vee \tau_f$ is connected if and only if $\tau = \tau_f$.

PROOF. If $\tau = \tau_f$, then $\tau \vee \tau_f = \tau$, hence it is connected. Now, if $\tau \neq \tau_f$, then f is not continuous. But f is bijective so neither is the inverse of f. Let x_0 be a point of discontinuity of f^{-1}. Then there is a sequence $\{x_n\}$ such that $\{x_n\} \to x_0$, but $\{f^{-1}(x_n)\} \not\to f^{-1}(x_0)$. Suppose $\{f^{-1}(x_n)\}$ is bounded. Then there exists a convergent subsequence $\{f^{-1}(x_{n_k})\}$. Let $\lim \{f^{-1}(x_{n_k})\} = y$. Without loss of generality, let $y > f^{-1}(x_0)$. Then there is an $M > 0$ such that for every $n_k > M$, $f^{-1}(x_{n_k}) > f^{-1}(x_0)$. Let $n_j > M$ then $f^{-1}(x_{n_j}) > f^{-1}(x_0)$. Now consider the vertical ray $A = \{(a,b) \mid a = x_0$ and $b > f^{-1}(x_{n_j})$ and let $x_{n_\ell} \in \mathbf{R}$ such that $| f^{-1}(x_{n_\ell}) - y| \leq | f^{-1}(x_{n_j}) - y|$ and without loss of generality, let $x_0 < x_{n_\ell}$. Consider the horizontal line segment $B = \{(a,b) \mid x_0 \leq a < x_{n_\ell}$ and $b = f^{-1}(x_{n_j})\}$. Also, consider the vertical ray $C = \{(a,b) \mid a = x_{n_\ell}$ and $b \leq f^{-1}(x_{n_j})\}$.

Since f^{-1} is an injective function, $(A \cup B \cup C) \cap G(f^{-1}) = \varnothing$. Now

$(x_{n_i}, f^{-1}(x_{n_i}))$ and $(x_0, f^{-1}(x_0))$ lie in separate components of $\mathbf{R} - (A \cup B \cup C)$. So in the bounded case, $\tau \vee \tau_f$ is not connected. The unbounded case is similar.

COROLLARY 3. $\tau \vee \tau_f$ is path-connected if and only if it is connected.

THEOREM 4. Let $D(f) = \{x \mid f \text{ is discontinuous at } x\}$. If $D(f^{-1})$ is a discrete subset of $\mathbf{R}$, then $\tau \vee \tau_f$ is locally connected.

The proof is very similar to that of Theorem 2 and hence is omitted.

COROLLARY 5. $\tau \vee \tau_f$ is locally path connected if and only if $\tau \vee \tau_f$ is locally connected.

THEOREM 6. If $\tau \vee \tau_f$ is locally connected, then $\tau \vee \tau_f$ is locally compact.

PROOF. Since $\tau \vee \tau_f$ is locally connected, each component C of $(G(f^{-1}), \tau \vee \tau)$ is open. Now $\pi_1(C)$ and $\pi_2(C)$ are connected subsets of the reals, therefore intervals. Now $f^{-1} \mid \pi_1(C)$ must be monotone, otherwise we would have points $a,b,c \in \pi_1(C)$ with $a < b < c$ such that $f^{-1}(a) \in \pi_2(C)$ and without loss of generality $f^{-1}(b) > f^{-1}(a)$. Now suppose $f^{-1}(c) < f^{-1}(b)$. If $f^{-1}(c) > f^{-1}(a)$, then the set $\{(a,y) \mid y \geq f^{-1}(c)\} \cup \{(x, f^{-1}(c)) \mid a \leq x \leq b\} \cup \{(b,y) \mid y \leq f^{-1}(c)\}$ disconnects C. If $f^{-1}(c) < f^{-1}(a)$, then the set $\{(c,y) \mid y \geq f^{-1}(a)\} \cup \{(x,f^{-1}(a)) \mid b \leq x \leq c\} \cup \{(b,y) \mid y \leq f^{-1}(a)\}$ disconnects C. This shows that a function which increases from a to b must continue to increase, the decreasing case is similar. So we have $f^{-1} \mid \pi_1(C)$ is a monotonic function from $\pi_1(C)$ to $\pi_2(C)$, hence f^{-1} is continuous on $\pi_1(C)$. Therefore $G(f^{1} \mid \pi_1(C))$ is homeomorphic to an interval, thus locally compact. Hence $\tau \vee \tau_f$ is locally compact.

The converse of this theorem is not, however, true. The following counter example illustrates this.

$$f(x) = \begin{cases} 1 & x = -1 \\ -1 & x = 0 \\ x - 1 & \frac{1}{x} \in Z^{+} \\ x + 1 & \frac{1}{x} \in \{Z^{-} - \{-1\}\} \\ x + 1 & x \in (-1,0) \text{ and } \frac{1}{x+1} \in Z^{+} \\ x - 1 & x \in (0,1) \text{ and } \frac{1}{x-1} \in Z^{-} \\ x & \text{Otherwise} \end{cases}$$

The graph of f is locally compact, but there is no connected neighborhood about (0,-1).

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