

FUZZY n -NORMED LINEAR SPACE

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The primary purpose of this paper is to introduce the notion of fuzzy n -normed linear space as a generalization of n -normed space. Ascending family of α - n -norms corresponding to fuzzy n -norm is introduced. Best approximation sets in α - n -norms are defined. We also provide some results on best approximation sets in α - n -normed space.

1. Introduction

A satisfactory theory of 2-norm and n -norm on a linear space has been introduced and developed by Gähler in [9, 10]. Following Misiak [16], Kim and Cho [13] and Malčeski [15] developed the theory of n -normed space. In [11], Gunawan and Mashadi gave a simple way to derive an $(n-1)$ -norm from the n -norm and realized that any n -normed space is an $(n-1)$ -normed space. Best approximation theory in 2-normed space can be viewed in the papers [3, 4, 5, 9]. Different authors introduced the definitions of fuzzy norms on a linear space. For reference, one may see [2, 6, 7, 8, 12, 14, 17]. Following Cheng and Mordeson [2], Bag and Samanta [1] introduced the concept of fuzzy norm on a linear space.

In the present paper, we introduce the concept of fuzzy n -normed linear space as a generalization of n -normed space by Gunawan and Mashadi [11]. Bag and Samanta [1] introduced α -norms on a linear space corresponding to the fuzzy norm on a linear space. As an analogue of Bag and Samanta [1], we introduce the notion of α - n -norm on a linear space corresponding to the fuzzy n -norm on a linear space. Based on Elumalai et al. [3] and Elumalai and Souruparani [5], we introduce the notion of best approximation sets in α - n -norms and establish some results on it.

2. Preliminaries

For the sake of completeness, we reproduce the following definitions due to Gähler [9], Gunawan and Mashadi [11], Elumalai et al. [3], and Bag and Samanta [1].

Definition 2.1 [9]. Let X be a real vector space of dimension greater than 1 and let $\|\bullet, \bullet\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

- (1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
- (2) $\|x, y\| = \|y, x\|$,
- (3) $\|\alpha x, y\| = |\alpha| \|x, y\|$, where α is real,
- (4) $\|x, y+z\| \leq \|x, y\| + \|x, z\|$.

$\|\bullet, \bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet, \bullet\|)$ is called a linear 2-normed space.

Definition 2.2 [11]. Let $n \in \mathbb{N}$ (natural numbers) and let X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $\|\bullet, \dots, \bullet\|$ on $\underbrace{X \times \dots \times X}_n$ satisfying the following four properties,

- (1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (2) $\|x_1, x_2, \dots, x_n\|$ is invariant under any permutation,
- (3) $\|x_1, x_2, \dots, \alpha x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$ (real),
- (4) $\|x_1, x_2, \dots, x_{n-1}, y+z\| \leq \|x_1, x_2, \dots, x_{n-1}, y\| + \|x_1, x_2, \dots, x_{n-1}, z\|$,

is called an n -norm on X and the pair $(X, \|\bullet, \dots, \bullet\|)$ is called an n -normed space.

Definition 2.3 [3]. Let $(X, \|\bullet, \bullet\|)$ be a linear 2-normed space and let G be an arbitrary nonempty subset of X and $x_0 \in X$. Then, for every $x \in X$ and for every $z \in X \setminus G$ which is independent of x and x_0 , $d_z(x, G) \leq \|x - x_0, z\| + d_z(x_0, G)$, where $d_z(x, G) = \inf_{g \in G} \|x - g, z\|$. For each $G \subset X$ and $x_0 \in X$, define $D_z(x_0, G) = \{x \in X : d_z(x, G) = \|x - x_0, z\| + d_z(x_0, G)\}$ for any $z \in X \setminus G$ which is independent of x and x_0 .

Also $P_{G,z}(x) = \{g_0 \in G : \|x - g_0, z\| = d_z(x, G)\}$ and $P_{G,z}^{-1}(x_0) = \{x \in X : \|x - x_0, z\| = d_z(x, G)\}$, where $x_0 \in G$.

Definition 2.4 [1]. Let X be a linear space over F (field of real or complex numbers). A fuzzy subset N of $X \times \mathbb{R}$ (\mathbb{R} , set of real numbers) is called a fuzzy norm on X if and only if for all $x, u \in X$ and $c \in F$,

- (N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x, t) = 0$,
- (N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x, t) = 1$ if and only if $x = \underline{0}$,
- (N3) for all $t \in \mathbb{R}$ with $t > 0$, $N(cx, t) = N(x, t/|c|)$, if $c \neq 0$,
- (N4) for all $s, t \in \mathbb{R}$, $x, u \in X$, $N(x+u, s+t) \geq \min\{N(x, s), N(u, t)\}$,
- (N5) $N(x, \circ)$ is a nondecreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1$.

The pair (X, N) will be referred to as a fuzzy normed linear space.

THEOREM 2.5 [1]. Let (X, N) be a fuzzy normed linear space. Assume further that

- (N6) $N(x, t) > 0$ for all $t > 0$ implies $x = \underline{0}$.

Define $\|x\|_\alpha = \inf\{t : N(x, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

Then $\{\|\bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms on X (or) α -norms on X corresponding to the fuzzy norm on X .

3. Fuzzy n -normed linear space

By generalizing Definition 2.2, we obtain a satisfactory notion of fuzzy n -normed linear space as follows.

Definition 3.1. Let X be a linear space over a real field F . A fuzzy subset N of $\underbrace{X \times \cdots \times X}_n \times \mathbb{R}$ (\mathbb{R} , set of real numbers) is called a fuzzy n -norm on X if and only if

- (N1) for all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, \dots, x_n, t) = 0$,
- (N2) for all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, \dots, x_n, t) = 1$ if and only if x_1, x_2, \dots, x_n are linearly dependent,
- (N3) $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of x_1, x_2, \dots, x_n ,
- (N4) for all $t \in \mathbb{R}$ with $t > 0$,

$$N(x_1, x_2, \dots, cx_n, t) = N\left(x_1, x_2, \dots, x_n, \frac{t}{|c|}\right), \quad \text{if } c \neq 0, c \in F \text{ (field)}, \quad (3.1)$$

- (N5) for all $s, t \in \mathbb{R}$,

$$N(x_1, x_2, \dots, x_n + x'_n, s + t) \geq \min\{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x'_n, t)\}, \quad (3.2)$$

- (N6) $N(x_1, x_2, \dots, x_n, \circ)$ is a nondecreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x_1, x_2, \dots, x_n, t) = 1$.

Then (X, N) is called a fuzzy n -normed linear space or in short f- n -NLS.

Remark 3.2. From (N3), it follows that in an f- n -NLS,

- (N4) for all $t \in \mathbb{R}$ with $t > 0$,

$$N(x_1, x_2, \dots, cx_i, \dots, x_n, t) = N\left(x_1, x_2, \dots, x_i, \dots, x_n, \frac{t}{|c|}\right), \quad \text{if } c \neq 0, \quad (3.3)$$

- (N5) for all $s, t \in \mathbb{R}$,

$$\begin{aligned} & N(x_1, x_2, \dots, x_i + x'_i, \dots, x_n, s + t) \\ & \geq \min\{N(x_1, x_2, \dots, x_i, \dots, x_n, s), N(x_1, x_2, \dots, x'_i, \dots, x_n, t)\}. \end{aligned} \quad (3.4)$$

The following example agrees with our notion of f- n -NLS.

Example 3.3. Let $(X, \|\bullet, \bullet, \dots, \bullet\|)$ be an n -normed space as in Definition 2.2. Define

$$\begin{aligned} & N(x_1, x_2, \dots, x_n, t) \\ & = \begin{cases} \frac{t}{t + \|\|x_1, x_2, \dots, x_n\|\|}, & \text{when } t > 0, t \in \mathbb{R}, (x_1, x_2, \dots, x_n) \in \underbrace{X \times \cdots \times X}_n, \\ 0, & \text{when } t \leq 0. \end{cases} \quad (3.5) \end{aligned}$$

Then (X, N) is an f- n -NLS.

Proof. (N1) for all $t \in \mathbb{R}$ with $t \leq 0$, we have by our definition

$$N(x_1, x_2, \dots, x_n, t) = 0. \quad (3.6)$$

(N2) for all $t \in \mathbb{R}$ with $t > 0$, we have $N(x_1, x_2, \dots, x_n, t) = 1$

- (i) if and only if $t/(t + \|x_1, x_2, \dots, x_n\|) = 1$,
- (ii) if and only if $t = t + \|x_1, x_2, \dots, x_n\|$,
- (iii) if and only if $\|x_1, x_2, \dots, x_n\| = 0$,
- (iv) if and only if x_1, x_2, \dots, x_n are linearly dependent.

(N3) for all $t \in \mathbb{R}$ with $t > 0$,

$$\begin{aligned} N(x_1, x_2, \dots, x_n, t) &= \frac{t}{t + \|x_1, x_2, \dots, x_n\|} \\ &= \frac{t}{t + \|x_1, x_2, \dots, x_{n-1}\|} = N(x_1, x_2, \dots, x_{n-1}, t) = \dots \end{aligned} \quad (3.7)$$

(N4) For all $t \in \mathbb{R}$ with $t > 0$ and $c \in F$, $c \neq 0$,

$$\begin{aligned} N\left(x_1, x_2, \dots, x_n, \frac{t}{|c|}\right) &= \frac{t/|c|}{(t/|c|) + \|x_1, x_2, \dots, x_n\|} \\ &= \frac{t/|c|}{(t + |c|\|x_1, x_2, \dots, x_n\|)/|c|} \\ &= \frac{t}{t + |c|\|x_1, x_2, \dots, x_n\|} \\ &= \frac{t}{t + \|x_1, x_2, \dots, cx_n\|} = N(x_1, x_2, \dots, cx_n, t). \end{aligned} \quad (3.8)$$

(N5) We have to prove

$$N(x_1, x_2, \dots, x_n + x'_n, s + t) \geq \min \{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x'_n, t)\}. \quad (3.9)$$

If

- (a) $s + t < 0$,
- (b) $s = t = 0$,
- (c) $s + t > 0$; $s > 0$, $t < 0$; $s < 0$, $t > 0$, then the above relation is obvious.
- (d) $s > 0$, $t > 0$, $s + t > 0$, then

$$\begin{aligned} N(x_1, x_2, \dots, x_n + x'_n, s + t) &= \frac{s + t}{s + t + \|x_1, x_2, \dots, x_n + x'_n\|} \\ &\geq \frac{s + t}{s + t + \|x_1, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x'_n\|}. \end{aligned} \quad (3.10)$$

If

$$\frac{s}{s + \|x_1, x_2, \dots, x_n\|} \geq \frac{t}{t + \|x_1, x_2, \dots, x'_n\|}, \quad (3.11)$$

then

$$\frac{s}{s + \|x_1, x_2, \dots, x_n\|} - \frac{t}{t + \|x_1, x_2, \dots, x'_n\|} \geq 0, \quad (3.12)$$

which implies

$$s(t + \|x_1, x_2, \dots, x_n\|) - t(s + \|x_1, x_2, \dots, x_n\|) \geq 0, \quad (3.13)$$

which in turn implies

$$s\|x_1, x_2, \dots, x_n\| - t\|x_1, x_2, \dots, x_n\| \geq 0. \quad (3.14)$$

So

$$\begin{aligned} & \frac{s+t}{s+t+\|x_1, x_2, \dots, x_n\|+\|x_1, x_2, \dots, x_n'\|} - \frac{t}{t+\|x_1, x_2, \dots, x_n'\|} \\ &= \frac{s\|x_1, x_2, \dots, x_n\| - t\|x_1, x_2, \dots, x_n\|}{(s+t+\|x_1, x_2, \dots, x_n\|+\|x_1, x_2, \dots, x_n'\|)(t+\|x_1, x_2, \dots, x_n'\|)}. \end{aligned} \quad (3.15)$$

By (3.14),

$$\frac{s+t}{s+t+\|x_1, x_2, \dots, x_n\|+\|x_1, x_2, \dots, x_n'\|} - \frac{t}{t+\|x_1, x_2, \dots, x_n'\|} \geq 0, \quad (3.16)$$

which implies

$$\frac{s+t}{s+t+\|x_1, x_2, \dots, x_n\|+\|x_1, x_2, \dots, x_n'\|} \geq \frac{t}{t+\|x_1, x_2, \dots, x_n'\|}. \quad (3.17)$$

Similarly, if

$$\frac{t}{t+\|x_1, x_2, \dots, x_n'\|} \geq \frac{s}{s+\|x_1, x_2, \dots, x_n\|}, \quad (3.18)$$

then

$$\frac{s+t}{s+t+\|x_1, x_2, \dots, x_n\|+\|x_1, x_2, \dots, x_n'\|} \geq \frac{s}{s+\|x_1, x_2, \dots, x_n\|}. \quad (3.19)$$

Thus,

$$N(x_1, x_2, \dots, x_n + x_n', s+t) \geq \min \{N(x_1, x_2, \dots, x_n, s), N(x_1, x_2, \dots, x_n', t)\}. \quad (3.20)$$

(N6) For all $t_1, t_2 \in \mathbb{R}$, if $t_1 < t_2 \leq 0$, then, by our definition,

$$N(x_1, x_2, \dots, x_n, t_1) = N(x_1, x_2, \dots, x_n, t_2) = 0. \quad (3.21)$$

Suppose $t_2 > t_1 > 0$, then

$$\begin{aligned} & \frac{t_2}{t_2 + \|x_1, x_2, \dots, x_n\|} - \frac{t_1}{t_1 + \|x_1, x_2, \dots, x_n\|} \\ &= \frac{\|x_1, x_2, \dots, x_n\|(t_2 - t_1)}{(t_2 + \|x_1, x_2, \dots, x_n\|)(t_1 + \|x_1, x_2, \dots, x_n\|)} \geq 0, \end{aligned} \quad (3.22)$$

for all $(x_1, x_2, \dots, x_n) \in \underbrace{X \times \dots \times X}_n$, implies

$$\frac{t_2}{t_2 + \|x_1, x_2, \dots, x_n\|} \geq \frac{t_1}{t_1 + \|x_1, x_2, \dots, x_n\|}, \quad (3.23)$$

which in turn implies $N(x_1, x_2, \dots, x_n, t_2) \geq N(x_1, x_2, \dots, x_n, t_1)$.

Thus $N(x_1, x_2, \dots, x_n, t)$ is a nondecreasing function.

Also,

$$\begin{aligned} \lim_{t \rightarrow \infty} N(x_1, x_2, \dots, x_n, t) &= \lim_{t \rightarrow \infty} \frac{t}{t + \|x_1, x_2, \dots, x_n\|} \\ &= \lim_{t \rightarrow \infty} \frac{t}{t(1 + (1/t)\|x_1, x_2, \dots, x_n\|)} = 1. \end{aligned} \quad (3.24)$$

Thus (X, N) is an f- n -NLS. \square

As a consequence of Theorem 2.5, we introduce an interesting notion of ascending family of α - n -norms corresponding to the fuzzy n -norm in the following theorem.

THEOREM 3.4. *Let (X, N) be an f- n -NLS. Assume the condition that*

(N7) $N(x_1, x_2, \dots, x_n, t) > 0$ for all $t > 0$ implies x_1, x_2, \dots, x_n are linearly dependent.

Define $\|x_1, x_2, \dots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

Then $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n -norms on X . These n -norms are called α - n -norms on X corresponding to the fuzzy n -norm on X .

Proof. (1) $\|x_1, x_2, \dots, x_n\|_\alpha = 0$. This

- (i) implies $\inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\} = 0$,
- (ii) implies, for all $t \in \mathbb{R}$, $t > 0$, $N(x_1, x_2, \dots, x_n, t) \geq \alpha > 0$, $\alpha \in (0, 1)$,
- (iii) implies, by (N7), x_1, x_2, \dots, x_n are linearly dependent.

Conversely assume that x_1, x_2, \dots, x_n are linearly dependent. This

- (i) implies, by (N2), $N(x_1, x_2, \dots, x_n, t) = 1$ for all $t > 0$,
- (ii) implies, for all $\alpha \in (0, 1)$, $\inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\} = 0$,
- (iii) implies $\|x_1, x_2, \dots, x_n\|_\alpha = 0$.

(2) As $N(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation, it follows that $\|x_1, x_2, \dots, x_n\|_\alpha$ is invariant under any permutation.

(3) If $c \neq 0$, then

$$\begin{aligned} \|x_1, x_2, \dots, cx_n\|_\alpha &= \inf\{s : N(x_1, x_2, \dots, cx_n, s) \geq \alpha\} \\ &= \inf\left\{s : N\left(x_1, x_2, \dots, x_n, \frac{s}{|c|}\right) \geq \alpha\right\}. \end{aligned} \quad (3.25)$$

Let $t = s/|c|$, then

$$\begin{aligned} \|x_1, x_2, \dots, cx_n\|_\alpha &= \inf \{|c|t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\} \\ &= |c| \inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\} \\ &= |c| \|x_1, x_2, \dots, x_n\|_\alpha. \end{aligned} \quad (3.26)$$

If $c = 0$, then

$$\begin{aligned} \|x_1, x_2, \dots, cx_n\|_\alpha &= \|x_1, x_2, \dots, 0\|_\alpha \\ &= 0 = 0 \|x_1, x_2, \dots, x_n\|_\alpha \\ &= |c| \|x_1, x_2, \dots, x_n\|_\alpha, \quad \forall c \in F \text{ (field)}. \end{aligned} \quad (3.27)$$

(4)

$$\begin{aligned} &\|x_1, x_2, \dots, x_n\|_\alpha + \|x_1, x_2, \dots, x'_n\|_\alpha \\ &= \inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\} + \inf \{s : N(x_1, x_2, \dots, x'_n, s) \geq \alpha\} \\ &= \inf \{t+s : N(x_1, x_2, \dots, x_n, t) \geq \alpha, N(x_1, x_2, \dots, x'_n, s) \geq \alpha\} \\ &\geq \inf \{t+s : N(x_1, x_2, \dots, x_n + x'_n, t+s) \geq \alpha\} \\ &\geq \inf \{r : N(x_1, x_2, \dots, x_n + x'_n, r) \geq \alpha\}, \quad r = t+s \\ &= \|x_1, x_2, \dots, x_n + x'_n\|_\alpha. \end{aligned} \quad (3.28)$$

Therefore, $\|x_1, x_2, \dots, x_n + x'_n\|_\alpha \leq \|x_1, x_2, \dots, x_n\|_\alpha + \|x_1, x_2, \dots, x'_n\|_\alpha$.

Thus $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an α -n-norm on X .

Let $0 < \alpha_1 < \alpha_2$. Then

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_{\alpha_1} &= \inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1\}, \\ \|x_1, x_2, \dots, x_n\|_{\alpha_2} &= \inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2\}. \end{aligned} \quad (3.29)$$

As $\alpha_1 < \alpha_2$,

$$\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2\} \subset \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1\} \quad (3.30)$$

implies

$$\inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_2\} \geq \inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha_1\} \quad (3.31)$$

which implies

$$\|x_1, x_2, \dots, x_n\|_{\alpha_2} \geq \|x_1, x_2, \dots, x_n\|_{\alpha_1}. \quad (3.32)$$

Hence, $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of α -n-norms on X corresponding to the fuzzy n-norm on X . \square

4. Best approximation sets in α - n -normed space

Inspired by this α - n -norm on X , we introduce the notion of two subsets of X , namely, $D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $P_{G, x_2, x_3, \dots, x_n}(x)$.

Definition 4.1. Let $(X, \|\bullet, \bullet, \dots, \bullet\|_\alpha)$ be an α - n -normed space corresponding to the fuzzy n -norm N on X . Let G be an arbitrary nonempty subset of X and $x_0 \in X$. Then for every $x \in X$ and for every $x_2, x_3, \dots, x_n \in X \setminus G$ which is independent of x and x_0 ,

$$d_{x_2, x_3, \dots, x_n}(x, G) \leq \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \quad (4.1)$$

where

$$d_{x_2, x_3, \dots, x_n}(x, G) = \inf_{g \in G} \|x - g, x_2, x_3, \dots, x_n\|_\alpha. \quad (4.2)$$

For each $G \subset X$ and $x_0 \in X$, we define

$$\begin{aligned} & D_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \{x \in X : d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)\} \end{aligned} \quad (4.3)$$

for any $x_2, x_3, \dots, x_n \in X \setminus G$ which is independent of x and x_0 .

We denote

$$\begin{aligned} P_{G, x_2, x_3, \dots, x_n}(x) &= \{g_0 \in G : \|x - g_0, x_2, x_3, \dots, x_n\|_\alpha = d_{x_2, x_3, \dots, x_n}(x, G)\}, \\ P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0) &= \{x \in X : \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha = d_{x_2, x_3, \dots, x_n}(x, G)\}, \end{aligned} \quad (4.4)$$

where $x_0 \in G$.

We give the following examples in the α -2-normed linear space and α - n -normed linear space for the sets $D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $P_{G, x_2, x_3, \dots, x_n}(x)$. It is easy to find the set $P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$.

Example 4.2. Let $X = \mathbb{R}^3$ be a linear space over \mathbb{R} .

Define $\|\bullet, \bullet\| : X \times X \rightarrow \mathbb{R}$ by

$$\begin{aligned} \|x_1, x_2\|_1 &= \max \{ |a_1 b_2 - a_2 b_1|, |b_1 c_2 - b_2 c_1|, |a_1 c_2 - a_2 c_1| \}, \\ \|x_1, x_2\|_2 &= \frac{1}{2} \{ \max \{ |a_1 b_2 - a_2 b_1|, |b_1 c_2 - b_2 c_1|, |a_1 c_2 - a_2 c_1| \} \}, \end{aligned} \quad (4.5)$$

where $x_i = (a_i, b_i, c_i) \in \mathbb{R}^3$, $i = 1, 2$. Then $(X, \|\bullet, \bullet\|_1)$ and $(X, \|\bullet, \bullet\|_2)$ are 2-normed linear spaces.

Define $N : X \times X \times \mathbb{R} \rightarrow [0, 1]$ by

$$N(x_1, x_2, t) = \begin{cases} 1, & \text{if } t > \|x_1, x_2\|_1, \\ 0.5, & \text{if } \|x_1, x_2\|_2 < t \leq \|x_1, x_2\|_1, \\ 0, & \text{if } t \leq \|x_1, x_2\|_2. \end{cases} \quad (4.6)$$

Then (X, N) is a fuzzy 2-normed linear space. Define $\|x_1, x_2\|_\alpha = \inf\{t : N(x_1, x_2, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

The α -2-norms are given by

$$\begin{aligned}\|x_1, x_2\|_\alpha &= \|x_1, x_2\|_1, \quad \text{when } 1 > \alpha > 0.5, \\ &= \|x_1, x_2\|_2, \quad \text{when } 0 < \alpha \leq 0.5.\end{aligned}\quad (4.7)$$

Let $G = \{(a, 0, 0) : a \in \mathbb{R}\}$ be a subset of X .

Choose $x_0 = (0, 1, 1)$ and $x_2 \in K = \{(0, 0, k) : k \in \mathbb{R} \setminus \{0\}\}$.

Then

$$\begin{aligned}D_{x_2}(x_0, G) &= \{x = (0, b, 0), b \in R^+ \setminus \{0\} : d_{x_2}(x, G) = \|x - x_0, x_2\|_\alpha + d_{x_2}(x_0, G)\}, \\ P_{G, x_2}(x) &= \{g' = (a, 0, 0) : -1 \leq a \leq 1\}.\end{aligned}\quad (4.8)$$

Example 4.3. Let $X = \mathbb{R}^{n+1}$ be a linear space over \mathbb{R} .

Define $\|\bullet, \bullet, \dots, \bullet\| : \underbrace{X \times \dots \times X}_n \rightarrow \mathbb{R}$ by

$$\begin{aligned}\|x_1, x_2, \dots, x_n\|_1 &= \max \{\Delta_1, \Delta_2, \dots, \Delta_n\}, \\ \|x_1, x_2, \dots, x_n\|_2 &= \frac{1}{2} \{ \max \{\Delta_1, \Delta_2, \dots, \Delta_n\} \},\end{aligned}\quad (4.9)$$

where

$$\begin{aligned}\Delta_1 &= \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{n(n+1)} \end{vmatrix}, \\ \Delta_2 &= \begin{vmatrix} a_{13} & \cdots & a_{1(n+1)} & a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n3} & \cdots & a_{n(n+1)} & a_{n1} \end{vmatrix}, \\ &\vdots \\ \Delta_n &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}\end{aligned}\quad (4.10)$$

and $x_i = (a_{i1}, a_{i2}, \dots, a_{i(n+1)}) \in \mathbb{R}^{n+1}$, $i = 1, 2, \dots, n$.

Then $(X, \|\bullet, \bullet, \dots, \bullet\|_1)$ and $(X, \|\bullet, \dots, \bullet, \bullet\|_2)$ are n -normed linear spaces. Define $N : \underbrace{X \times \dots \times X}_n \times \mathbb{R} \rightarrow [0, 1]$ by

$$N(x_1, x_2, \dots, x_n, t) = \begin{cases} 1, & \text{if } t > \|x_1, x_2, \dots, x_n\|_1, \\ 0.5, & \text{if } \|x_1, x_2, \dots, x_n\|_2 < t \leq \|x_1, x_2, \dots, x_n\|_1, \\ 0, & \text{if } t \leq \|x_1, x_2, \dots, x_n\|_2. \end{cases}\quad (4.11)$$

Then (X, N) is a fuzzy n -normed linear space. Define $\|x_1, x_2, \dots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

The α - n -norms are given by

$$\begin{aligned} & \|x_1, x_2, \dots, x_n\|_{\alpha} \\ &= \|x_1, x_2, \dots, x_n\|_1, \quad \text{when } 1 > \alpha > 0.5, \\ &= \|x_1, x_2, \dots, x_n\|_2, \quad \text{when } 0 < \alpha \leq 0.5. \end{aligned} \quad (4.12)$$

Let $G = \{(a, 0, 0, \dots, n \text{ times } 0) : a \in \mathbb{R}\}$ be a subset of X .

Choose $x_0 = (0, 1, 1, \dots, n \text{ times } 1)$ and

$$x_2, x_3, \dots, x_n \in K = \left\{ \left(0, 0, k_3^{(i)}, \dots, k_{n+1}^{(i)} \right) : k_3^{(i)}, \dots, k_{n+1}^{(i)} \in \mathbb{R} \setminus \{0\} \right\}. \quad (4.13)$$

That is,

$$\begin{aligned} x_2 &= \left(0, 0, k_3^{(2)}, \dots, k_{n+1}^{(2)} \right), \\ x_3 &= \left(0, 0, k_3^{(3)}, \dots, k_{n+1}^{(3)} \right), \\ &\vdots \\ x_n &= \left(0, 0, k_3^{(n)}, \dots, k_{n+1}^{(n)} \right). \end{aligned} \quad (4.14)$$

Then

$$\begin{aligned} & D_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \{x = (0, b, 0, \dots, (n-1) \text{ times } 0), b \in \mathbb{R}^+ \setminus \{0\} : \\ & \quad d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_{\alpha} + d_{x_2, x_3, \dots, x_n}(x_0, G)\}, \end{aligned} \quad (4.15)$$

where $d_{x_2, x_3, \dots, x_n}(x, G) = \max \{|b| \Delta, |a| \Delta\}$,

$$\Delta = \begin{vmatrix} k_3^{(2)} & k_4^{(2)} & \cdots & k_{n+1}^{(2)} \\ k_3^{(3)} & k_4^{(3)} & \cdots & k_{n+1}^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ k_3^{(n)} & k_4^{(n)} & \cdots & k_{n+1}^{(n)} \end{vmatrix}, \quad (4.16)$$

$$\|x - x_0, x_2, \dots, x_n\|_{\alpha} = |b - 1| \Delta, \quad d_{x_2, x_3, \dots, x_n}(x_0, G) = \max \{\Delta, |a| \Delta\}$$

and also $P_{G, x_2, x_3, \dots, x_n}(x) = \{g' = (a, 0, \dots, n \text{ times } 0) : -1 \leq a \leq 1\}$.

By routine calculation the following theorems are validate from the Examples 4.2 and 4.3.

THEOREM 4.4. For $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $y \in D_{x_2, x_3, \dots, x_n}(x, G)$,

- (i) $\|y - x_0, x_2, x_3, \dots, x_n\|_{\alpha} = \|y - x, x_2, x_3, \dots, x_n\|_{\alpha} + \|x - x_0, x_2, x_3, \dots, x_n\|_{\alpha}$,
- (ii) $y - x + x_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $y \in D_{x_2, x_3, \dots, x_n}(x, G)$.

Then by (4.3) we have

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \\ d_{x_2, x_3, \dots, x_n}(y, G) &= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G). \end{aligned} \quad (4.17)$$

Consider

$$\begin{aligned} &\|y - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|y - x_0 - x + x, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|(y - x) + (x - x_0), x_2, x_3, \dots, x_n\|_\alpha \\ &\leq \|y - x, x_2, x_3, \dots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= (d_{x_2, x_3, \dots, x_n}(y, G) - d_{x_2, x_3, \dots, x_n}(x_0, G)) \\ &\quad + (d_{x_2, x_3, \dots, x_n}(x, G) - d_{x_2, x_3, \dots, x_n}(x_0, G)) \\ &= d_{x_2, x_3, \dots, x_n}(y, G) - d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &\leq \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha. \end{aligned} \quad (4.18)$$

Therefore,

$$\|y - x_0, x_2, x_3, \dots, x_n\|_\alpha = \|y - x, x_2, x_3, \dots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha. \quad (4.19)$$

(ii) By (4.2), we have

$$\begin{aligned} &d_{x_2, x_3, \dots, x_n}(y - x + x_0, G) \\ &\geq d_{x_2, x_3, \dots, x_n}(y, G) - \|y - (y - x + x_0), x_2, x_3, \dots, x_n\|_\alpha \\ &= d_{x_2, x_3, \dots, x_n}(y, G) - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= (\|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G)) - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + (\|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)) \\ &\quad - \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha \\ &= \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \|(y - x + x_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G). \end{aligned} \quad (4.20)$$

Again by (4.2), it follows that

$$d_{x_2, x_3, \dots, x_n}(y - x + x_0, G) = \|(y - x + x_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \quad (4.21)$$

which implies $y - x + x_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. \square

THEOREM 4.5. Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. Then

- (i) $[x_0, x] = \{\lambda x_0 + (1 - \lambda)x : 0 \leq \lambda \leq 1\} \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$,
- (ii) $D_{x_2, x_3, \dots, x_n}(x, G) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. (i) Let $y = \lambda x_0 + (1 - \lambda)x$ such that $0 \leq \lambda \leq 1$. Then

$$\begin{aligned}
 & d_{x_2, x_3, \dots, x_n}(y, G) \\
 & \geq d_{x_2, x_3, \dots, x_n}(x, G) - \|x - y, x_2, x_3, \dots, x_n\|_\alpha \\
 & = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) - \|x - y, x_2, x_3, \dots, x_n\|_\alpha \\
 & = \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
 \end{aligned} \tag{4.22}$$

By (4.2), we have

$$d_{x_2, x_3, \dots, x_n}(y, G) = \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \tag{4.23}$$

which implies $y \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

(ii) Let $y \in D_{x_2, x_3, \dots, x_n}(x, G)$. Then, by (4.3) and Theorem 4.4(i),

$$\begin{aligned}
 & d_{x_2, x_3, \dots, x_n}(y, G) \\
 & = \|y - x, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x, G) \\
 & = \|y - x, x_2, x_3, \dots, x_n\|_\alpha + (\|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G)) \\
 & = \|y - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G),
 \end{aligned} \tag{4.24}$$

which implies $y \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Therefore, $D_{x_2, x_3, \dots, x_n}(x, G) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$. \square

THEOREM 4.6. Let $x_0, y_0 \in X$ and $\lambda \neq 0$. Then

- (i) $D_{x_2, x_3, \dots, x_n}(x_0, G) + y_0 = D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$,
- (ii) $D_{x_2, x_3, \dots, x_n}(x_0, \lambda G) = \lambda D_{x_2, x_3, \dots, x_n}(x_0/\lambda, G)$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. Then

$$\begin{aligned}
 d_{x_2, x_3, \dots, x_n}(x + y_0, G + y_0) & = d_{x_2, x_3, \dots, x_n}(x, G) \\
 & = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\
 & = \|x + y_0 - (x_0 + y_0), x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
 \end{aligned} \tag{4.25}$$

Therefore, $x + y_0 \in D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$.

Conversely, let $y \in D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0)$. Then

$$\begin{aligned}
 & d_{x_2, x_3, \dots, x_n}(y - y_0, G) \\
 & = d_{x_2, x_3, \dots, x_n}(y, G + y_0) \\
 & = \|y - y_0 - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0) \\
 & = \|(y - y_0) - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G).
 \end{aligned} \tag{4.26}$$

Therefore, $y - y_0 \in D_{x_2, x_3, \dots, x_n}(x_0, G)$, and so

$$D_{x_2, x_3, \dots, x_n}(x_0, G) + y_0 = D_{x_2, x_3, \dots, x_n}(x_0 + y_0, G + y_0). \tag{4.27}$$

(ii) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, \lambda G)$. Then

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}\left(\frac{x}{\lambda}, G\right) &= \frac{1}{|\lambda|} d_{x_2, x_3, \dots, x_n}(x, \lambda G) \\ &= \frac{1}{|\lambda|} (\|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, \lambda G)) \\ &= \left\| \frac{x}{\lambda} - \frac{x_0}{\lambda}, x_2, x_3, \dots, x_n \right\|_\alpha + d_{x_2, x_3, \dots, x_n}\left(\frac{x_0}{\lambda}, G\right). \end{aligned} \quad (4.28)$$

Therefore, $x/\lambda \in D_{x_2, x_3, \dots, x_n}(x_0/\lambda, G)$.

Conversely, let $x \in D_{x_2, x_3, \dots, x_n}(x_0/\lambda, G)$. Then

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(\lambda x, \lambda G) &= |\lambda| d_{x_2, x_3, \dots, x_n}(x, G) \\ &= |\lambda| \left(\left\| x - \frac{x_0}{\lambda}, x_2, x_3, \dots, x_n \right\|_\alpha + d_{x_2, x_3, \dots, x_n}\left(\frac{x_0}{\lambda}, G\right) \right) \\ &= \left\| \lambda x - x_0, x_2, x_3, \dots, x_n \right\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, \lambda G). \end{aligned} \quad (4.29)$$

Therefore, $\lambda x \in D_{x_2, x_3, \dots, x_n}(x_0, \lambda G)$.

Thus, $D_{x_2, x_3, \dots, x_n}(x_0, \lambda G) = \lambda D_{x_2, x_3, \dots, x_n}(x_0/\lambda, G)$. \square

THEOREM 4.7. Let $G \subset G_1$ and $x_0 \in X$, where G_1 is a subset of X such that

$$d_{x_2, x_3, \dots, x_n}(x_0, G) = d_{x_2, x_3, \dots, x_n}(x_0, G_1). \quad (4.30)$$

Then $D_{x_2, x_3, \dots, x_n}(x_0, G_1) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Proof. Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G_1)$. Then, by (4.30), we have

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &\geq d_{x_2, x_3, \dots, x_n}(x, G_1) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G_1) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G). \end{aligned} \quad (4.31)$$

By (4.2), it follows that

$$d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \quad (4.32)$$

which implies $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$.

Hence, $D_{x_2, x_3, \dots, x_n}(x_0, G_1) \subset D_{x_2, x_3, \dots, x_n}(x_0, G)$. \square

THEOREM 4.8. (i) $P_{G, x_2, x_3, \dots, x_n}(x_0) \subset P_{G, x_2, x_3, \dots, x_n}(x)$ for every $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$,
(ii) $D_{x_2, x_3, \dots, x_n}(x_0, G) = P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$ for every $x_0 \in \overline{G}$.

Proof. (i) Let $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$ and $g \in P_{G, x_2, x_3, \dots, x_n}(x_0)$.

Now,

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + \|x_0 - g_0, x_2, x_3, \dots, x_n\|_\alpha. \end{aligned} \quad (4.33)$$

By Theorem 4.4(i), we have

$$d_{x_2, x_3, \dots, x_n}(x, G) = \|x - g_0, x_2, x_3, \dots, x_n\|_\alpha, \quad (4.34)$$

which implies $g_0 \in P_{G, x_2, x_3, \dots, x_n}(x)$, which in turn implies $P_{G, x_2, x_3, \dots, x_n}(x_0) \subset P_{G, x_2, x_3, \dots, x_n}(x)$.

(ii) Let $x_0 \in \overline{G}$ and $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$. Then

$$\begin{aligned} d_{x_2, x_3, \dots, x_n}(x, G) &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G) \\ &= \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha, \quad \text{where } x_0 \in \overline{G}, \end{aligned} \quad (4.35)$$

which implies $x \in P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$. So

$$D_{x_2, x_3, \dots, x_n}(x_0, G) \subset P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0). \quad (4.36)$$

Conversely, let $x \in P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0)$.

Then $x_0 \in P_{G, x_2, x_3, \dots, x_n}(x)$.

Since $x_0 \in \overline{G}, d_{x_2, x_3, \dots, x_n}(x_0, G) = 0$.

Hence, we have

$$d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, x_3, \dots, x_n\|_\alpha + d_{x_2, x_3, \dots, x_n}(x_0, G), \quad (4.37)$$

which implies $x \in D_{x_2, x_3, \dots, x_n}(x_0, G)$, which in turn implies

$$P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0) \subset D_{x_2, x_3, \dots, x_n}(x_0, G). \quad (4.38)$$

From (4.36) and (4.38), we have

$$D_{x_2, x_3, \dots, x_n}(x_0, G) = P_{G, x_2, x_3, \dots, x_n}^{-1}(x_0). \quad (4.39)$$

□

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