

# BAIRE SPACES, $k$ -SPACES, AND SOME PROPERLY HEREDITARY PROPERTIES

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Received 11 October 2004 and in revised form 8 September 2005

A topological property is *properly hereditary property* if whenever every proper subspace has the property, the whole space has the property. In this note, we will study some topological properties that are preserved by proper subspaces; in fact, we will study the following topological properties: Baire spaces, second category, sequentially compact, hemi-compact,  $\delta$ -normal, and spaces having dispersion points. Also, we will solve some open problems raised by Al-Bsoul (2003) and Arenas (1996) and conclude this note by some open problems.

## 1. Introduction

In 1996, Professor M. L. Puertas suggested the following question which was studied by Arenas in his paper [2]:

(\*) *if every proper subspace has the property, the whole space has the property.*

Any topological property satisfying the property (\*) is called *properly hereditary property*. Moreover, if every proper closed (resp., open,  $F_\sigma$ ,  $G_\delta$ , etc.) subspace has the property, then the whole space has the property, and we call such a property *properly closed* (resp., open,  $F_\sigma$ ,  $G_\delta$ , etc.) *hereditary*.

In [2], Arenas studied some topological properties having the property (\*). In fact, Arenas [2] proved that the topological properties  $T_i$ ,  $i = 0, 1, 2, 3$ , are properly hereditary properties. He also studied countability axioms and metrizability. Nevertheless, for Arenas, the property (\*) makes sense only for the topological properties that are hereditary to all subsets. On the other hand, as we saw in [1], there are some topological properties which are properly hereditary properties but are not hereditary properties, for instance, paracompactness. Thus, we may ask the converse.

*Question 1.1.* Is there a topological property  $P$  which is hereditary but is not properly hereditary?

It is obvious to see that if a topological property is properly hereditary, it is not necessary that it is properly open (closed) hereditary. So, it is natural to redefine the following (compare with the definition in [2]).

*Definition 1.2.* A topological property is called strong (resp., closed, open,  $F_\sigma$ ,  $G_\delta$ , etc.) hereditary property provided that it is (resp., closed, open,  $F_\sigma$ ,  $G_\delta$ , etc.) hereditary property and properly (resp., closed, open,  $F_\sigma$ ,  $G_\delta$ , etc.) hereditary property.

Hence,  $T_i$  is a strong hereditary property for  $i = 0, 1/4, 1/2, 1, 1(1/3), 2, 2(1/2), 3$ , but  $T_1, T_{1(1/3)}, T_2, T_{2(1/2)}, T_3$  are neither strong open hereditary properties nor strong closed hereditary properties.

In [1], Al-Bsoul studied some topological properties as some nonfamiliar separation axioms beside some strong separation axioms, and also some covering properties.

Reading [1, 2], the reader may conclude that all classical topological properties are properly hereditary properties. This is not the case; however, in this note we are going to give two topological properties which are not properly (resp., closed, open) hereditary properties. Moreover, we will positively solve two open problems raised in [1, 2]. Also, we will prove that some topological properties, such as sequential compactness, hemicompactness, and  $\delta$ -normality, are properly hereditary properties.

## 2. Baire spaces and $k$ -spaces

In this section we will positively solve the following open problem raised in [1].

*Question 2.1* [1]. Are Baire spaces and second category properly hereditary properties?

**THEOREM 2.2.** *If every proper subspace of  $X$  is a Baire subspace, then  $X$  is a Baire space.*

*Proof.* Let  $\{G_n : n \in \mathbb{N}\}$  be a countable family of open dense subsets of  $X$ . Pick  $x_0 \in X \setminus (\bigcap_{n \in \mathbb{N}} G_n)$ . If such  $x_0$  does not exist, then  $\bigcap_{n \in \mathbb{N}} G_n = X$ , and the proof is finished.

Let  $Y = X \setminus \{x_0\}$ , then  $G_n \setminus \{x_0\} \neq \emptyset$  for all  $n \in \mathbb{N}$ . If  $\overline{G_i \setminus \{x_0\}}^Y \neq Y$  for some  $i \in \mathbb{N}$ , then there exist  $t \in Y \setminus (\overline{G_i \setminus \{x_0\}}^Y)$  and an open set  $V$  in  $Y$  such that  $t \in V \subseteq Y \setminus (\overline{G_i \setminus \{x_0\}}^Y)$ . So, there exists an open set  $U$  in  $X$  such that  $V = U \cap Y$ . Now,  $U \cap G_i \neq \emptyset$  and  $U = V \cup \{x_0\}$ , so  $V \cap G_i = \emptyset$ , and hence  $U \cap G_i = \{x_0\}$ , which is a contradiction. Thus  $G_n \setminus \{x_0\}$  is an open dense set in  $Y$  for all  $n \in \mathbb{N}$ . Therefore,  $(\bigcap_{n \in \mathbb{N}} G_n)$  is dense in  $X$ .  $\square$

**COROLLARY 2.3.** *Let  $X$  be a  $T_1$ -space. If every proper open subspace of  $X$  is a Baire subspace, then  $X$  is a Baire space.*

**COROLLARY 2.4.** *Second category is properly hereditary property.*

*Proof.* Let  $X$  be a space such that every proper subspace of  $X$  is a second category, and let  $\{G_n : n \in \mathbb{N}\}$  be a countable family of open dense subsets of  $X$ . Suppose that  $\bigcap_{n \in \mathbb{N}} G_n = \emptyset$ . Pick  $x_0 \in X$ , and let  $Y = X \setminus \{x_0\}$ , then  $G_n \setminus \{x_0\}$  is an open dense subset of  $Y$  for all  $n \in \mathbb{N}$ . Therefore,  $\bigcap_{n \in \mathbb{N}} G_n \neq \emptyset$ .  $\square$

Also, we will solve the following open problem for  $k$ -spaces positively.

*Question 2.5* [2]. Is being a  $k$ -space a properly (closed) hereditary property?

Recall that a Hausdorff space  $X$  is called a  $k$ -space if and only if the following condition holds:

$$A \subseteq X \text{ is open if } A \cap K \text{ is open in } K \text{ for each compact set } K \text{ in } X. \quad (2.1)$$

In [2, Example 2.2], we see that every proper closed subspace of  $X$  is a  $k$ -subspace, but  $X$  itself is not a  $k$ -space. Also, [1, Example 1.1], shows that every proper open subspace of  $X$  is a  $k$ -subspace, but  $X$  itself is not a  $k$ -space. But we have the following theorem.

**THEOREM 2.6.** *If every proper subspace of  $X$  is a  $k$ -subspace, then  $X$  is a  $k$ -space.*

*Proof.* Let  $U$  be an open subset in  $X$  and let  $K$  be any compact subset in  $X$ . If  $U \subseteq K$  or  $K \subseteq U$ , we are done. Assume that  $U \setminus K \neq \emptyset$  and  $K \setminus U \neq \emptyset$ . Fix a point  $x_0 \in U \setminus K$ , then  $U \cap K = (U \setminus \{x_0\}) \cap K$  is open in  $K$ .

Conversely, Assume that  $U \cap K$  is open in  $K$  for each compact set  $K$  in  $X$ . If  $U = X$  or  $U = \emptyset$ , we are done. Assume that  $U$  is a nonempty proper subset of  $X$ , so choose  $y_0 \in X \setminus U$ , and let  $Y = X \setminus \{y_0\}$ , so  $U \cap K_Y$  is open in  $K_Y$  for each compact set  $K_Y$  in  $Y$ , so  $U$  is open in  $Y$  and hence in  $X$ .  $\square$

**COROLLARY 2.7.** *For Hausdorff spaces, if every proper closed (resp., open) subspace of  $X$  is a  $k$ -subspace, then  $X$  is a  $k$ -space.*

### 3. More topological properties satisfying (\*)

Several topological properties will be studied in this section. Let us start with some types of compactness.

**THEOREM 3.1.**  *$\sigma$ -compactness is a properly hereditary property.*

**THEOREM 3.2.** *Sequential compactness is a properly hereditary property.*

*Proof.* Let  $\{x_n\}$  be a sequence in  $X$ . If  $|\{x_n : n \in \mathbb{N}\}| < \aleph_0$ , then it has a constant subsequence. If the value  $x_{n_0}$  is repeated infinitely many times, that is, the set  $\{n : x_n = x_{n_0}\}$  is infinite, then this sequence has a constant subsequence. If  $x_1$  is repeated only finitely many times, assume that  $x_n \neq x_1$  for all  $n > n_0$ , hence the subsequence  $x_{n_j} = x_{n_0+j}$  has a convergent subsequence in  $X \setminus \{x_1\}$  and hence in  $X$ .  $\square$

Since the following topological property (hemicompact or denumerable at infinity) is not familiar, we give its definition.

**Definition 3.3.** A Hausdorff space  $X$  is said to be hemicompact or denumerable at infinity if and only if there is a sequence  $K_1, K_2, \dots$  of compact subsets of  $X$  such that if  $K$  is any compact subset of  $X$ , then  $K \subseteq K_n$  for some  $n$ .

**THEOREM 3.4.** *If every proper subspace of  $X$  is a hemicompact subspace, then  $X$  is a hemicompact space.*

*Proof.* Let  $x \neq y$  in  $X$ , so there exists an open set  $V$  in  $X$  such that  $x \notin \overline{V}$ . Since  $Y = X \setminus V$  and  $\overline{V}$  are hemicompact, there exist compact subsets  $C_1, C_2, \dots$  of  $Y$ , such that for every compact subset  $K_Y$  of  $Y$  there exists  $C_{n_0}$  such that  $K_Y \subseteq C_{n_0}$ , and there exist compact subsets  $D_1, D_2, \dots$  of  $\overline{V}$ , such that for every compact subset  $K_{\overline{V}}$  of  $\overline{V}$  there exists  $D_{m_0}$  such that  $K_{\overline{V}} \subseteq D_{m_0}$ . Let  $W = \{0\} \cup \mathbb{N}$ , and define a sequence of compact subsets  $K_{m,n}$  of  $X$  as follows:  $K_{m,n} = C_m \cup D_n$ ,  $K_{m,0} = C_m$ , and  $K_{0,n} = D_n$  for all  $(m, n) \in W \times W \setminus \{(0, 0)\}$ .

Now, if  $K$  is any compact subset of  $X$ , then  $K \cap Y$  is compact in  $Y$  and  $K \cap \overline{V}$  is compact in  $\overline{V}$ . Thus, there exist  $C_{m_0}$  and  $D_{n_0}$  such that  $K \cap Y \subseteq C_{m_0}$  and  $K \cap \overline{V} \subseteq D_{n_0}$ , hence  $K = (K \cap Y) \cup (K \cap \overline{V}) \subseteq C_{m_0} \cup D_{n_0} = K_{m_0, n_0}$ .  $\square$

It is easy to construct examples to show that hemicompactness is neither a properly open hereditary property nor a properly closed hereditary property. Recall that a space  $X$  is said to be  $\delta$ -normal if whenever  $A$  and  $B$  are disjoint closed subsets of  $X$ , there exist two disjoint  $G_\delta$ -sets  $H$  and  $K$  such that  $A \subseteq H$  and  $B \subseteq K$ .

Example 1.1 in [1], shows that if every closed proper subspace of  $X$  is a  $\delta$ -normal  $T_1$  subspace, then  $X$  need not be a  $\delta$ -normal space. Moreover, [2, Example 2.2] shows that if every open proper subspace of  $X$  is a  $\delta$ -normal  $T_1$ -subspace, then  $X$  need not be a  $\delta$ -normal space. Fortunately, we have the following result.

**THEOREM 3.5.** *If every proper subspace of  $X$  is a  $\delta$ -normal  $T_1$ -subspace, then  $X$  is a  $\delta$ -normal space.*

*Proof.* Let  $A$  and  $B$  be two nonempty disjoint closed subsets of  $X$ . The case  $X = A \cup B$  is evident. Let  $y_0 \in X \setminus (A \cup B)$ , and let  $Y = X \setminus \{y_0\}$ , so there exist two disjoint  $G_\delta$ -sets  $H_0, K_0$  in  $Y$  such that  $A \subseteq H_0$  and  $B \subseteq K_0$ , thus  $H_0 = \bigcap_{i=1}^{\infty} H_i$  and  $K_0 = \bigcap_{i=1}^{\infty} K_i$ , where  $H_i$  and  $K_i$  are open in  $Y$  for all  $i \in \mathbb{N}$ . Hence, there exist open sets  $U_i$  and  $V_i$  in  $X$ , such that  $H_i = U_i \cap Y$  and  $K_i = V_i \cap Y$  for all  $i \in \mathbb{N}$ . Thus,  $U = \bigcap_{i=1}^{\infty} U_i$  and  $V = \bigcap_{i=1}^{\infty} V_i$  are  $G_\delta$ -sets. If  $U \cap V = \emptyset$ , the proof is completed, otherwise,  $U \cap V = \{y_0\}$ . Take  $z_0 \in X \setminus (A \cup B \cup \{y_0\})$ , and let  $Z = X \setminus \{z_0\}$ , so there exist two disjoint  $G_\delta$ -sets  $H^0, K^0$  in  $Z$  such that  $A \subseteq H^0$  and  $B \subseteq K^0$ , thus  $H^0 = \bigcap_{i=1}^{\infty} H^i$  and  $K^0 = \bigcap_{i=1}^{\infty} K^i$ , where  $H^i$  and  $K^i$  are open in  $Z$  for all  $i \in \mathbb{N}$ . Hence, there exist open sets  $U^i$  and  $V^i$  in  $X$ , such that  $H^i = U^i \cap Z$  and  $K^i = V^i \cap Z$  for all  $i \in \mathbb{N}$ . Let  $L_i = U_i \cap U^i$  and  $M_i = V_i \cap V^i$ , and define the required  $G_\delta$ -sets  $L = \bigcap_{i=1}^{\infty} L_i$  and  $M = \bigcap_{i=1}^{\infty} M_i$ .  $\square$

It is easy to prove the following.

**THEOREM 3.6.** *Pseudocompactness is a properly hereditary property.*

Recall that a space  $X$  is *extremely disconnected* if and only if the closure of every open set is open. This topological property is not a properly (resp., closed, open) hereditary property, as we will see in the next example.

*Example 3.7.* Let  $X = \{u, x, y\}$  and let  $\tau = \{\{u\}, \{y\}, \{u, y\}, \emptyset, X\}$ , hence every proper subspace of  $X$  is an extremely disconnected subspace, but  $X$  itself is not.

**THEOREM 3.8.** *The topological property “extremely disconnected” is a properly hereditary property in the class of  $T_1$ -spaces.*

*Remark 3.9.* If every proper open subspace of  $X$  is an extremely disconnected  $T_1$ -subspace, then  $X$  is not necessarily an extremely disconnected space.

#### 4. Open problems

In this section, we will study a special kind of topological properties, namely, *having a dispersion point*. For this, let us state its definition. Recall that a connected space  $X$  is said to have a *dispersion point*  $p$  if and only if  $X \setminus \{p\}$  is totally disconnected. Thus, no space  $X$  of cardinality greater than 2 has the property that every subspace of  $X$  has a dispersion point because, if  $x, y \in X \setminus \{p\}$ , where  $p$  is the dispersion point of  $X$ , then the subspace  $\{x, y\}$  has the discrete topology, and hence has no dispersion point. Thus, it is natural to ask the following question.

*Question 4.1.* Is there a connected space for which every proper open (closed) subspace of  $X$  has a dispersion point?

Indeed, take [2, Example 2.2] under consideration for the case of proper open subspaces, and take [1, Example 1.1] for the case of proper closed subspaces. Thus, one may ask the following.

*Question 4.2.* Is the property of having a dispersion point a properly open (closed) hereditary property?

The next example is a modification of [1, Example 1.1], which shows that the answer of Question 4.2 is negative in general.

*Example 4.3.* Let  $X = \mathbb{Z} \cup \{\pi, \sqrt{2}\}$ , and topologize  $X$  as

$$\tau = \{A \cup \{\pi\} : A \subseteq \mathbb{Z}\} \cup \{X, \emptyset\}. \quad (4.1)$$

So,  $\pi$  is a dispersion point of every nonempty proper open subspace of  $X$ , and  $\sqrt{2}$  is a dispersion point of every nonempty proper closed subspace of  $X$ . But  $X$  itself has no dispersion points.

Note that the space in Example 4.3 is not a  $T_1$ -space, but it is easy to see that there is no  $T_1$ -space for which every proper closed (open) subspace has a dispersion point. A topological space  $X$  is called  $T_{1/2}$  if and only if every singleton is open or closed. Also, a topological space  $X$  is called  $T_{1/4}$  if and only if for every finite subset  $F$  of  $X$  and for every  $y \in X \setminus F$ , there exists a set  $A$  containing  $F$  and not containing  $y$  which is either open or closed in  $X$ . Hence we can restate the previous questions as follows.

*Question 4.4.* Is there a  $T_{1/4}$  ( $T_{1/2}$ )-space for which every proper open (closed) subspace of  $X$  has a dispersion point? If so, does  $X$  have a dispersion point?

Questions 1.1 and 4.4 are still open.

For the following two questions, the reader may consult [3].

*Question 4.5.* Is maximal compactness (minimal Hausdorffness, realcompactness, absolutely countably compactness) a properly (resp., closed, open,  $F_\sigma$ ,  $G_\delta$ , etc.) hereditary property?

*Question 4.6.* Is  $H$ -closedness a properly (resp., closed, open,  $F_\sigma$ ,  $G_\delta$ , etc.) hereditary property?

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