ON A SUBCLASS OF n-STARLIKE FUNCTIONS

MUGUR ACU AND SHIGEYOSHI OWA

Received 21 February 2005

In 1999, Kanas and Rønning introduced the classes of starlike and convex functions, which are normalized with $f(w) = f'(w) - 1 = 0$ and w a fixed point in U. In 2005, the authors introduced the classes of functions close to convex and α-convex, which are normalized in the same way. All these definitions are somewhat similar to the ones for the uniform-type functions and it is easy to see that for $w = 0$, the well-known classes of starlike, convex, close-to-convex, and α-convex functions are obtained. In this paper, we continue the investigation of the univalent functions normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in U.

1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, and $S = \{f \in A : f \text{ is univalent in } U\}$.

We recall here the definitions of the well-known classes of starlike and convex functions:

$$S^* = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U \right\},$$
$$S^c = \left\{ f \in A : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0,\ z \in U \right\}. \tag{1.1}$$

Let w be a fixed point in U and $A(w) = \{f \in \mathcal{H}(U) : f(w) = f'(w) - 1 = 0\}$.

In [3], Kanas and Rønning introduced the following classes:

$$S(w) = \{ f \in A(w) : f \text{ is univalent in } U \},$$
$$ST(w) = S^*(w) = \left\{ f \in S(w) : \operatorname{Re} \frac{(z-w)f'(z)}{f(z)} > 0,\ z \in U \right\}, \tag{1.2}$$
$$CV(w) = S^c(w) = \left\{ f \in S(w) : 1 + \operatorname{Re} \frac{(z-w)f''(z)}{f'(z)} > 0,\ z \in U \right\}.$$

International Journal of Mathematics and Mathematical Sciences 2005:17 (2005) 2841–2846
DOI: 10.1155/IJMMS.2005.2841

It is obvious that a natural "Alexander relation" exists between the classes $S^*(w)$ and $S^c(w)$:

$$g \in S^c(w) \qquad \text{iff } f(z) = (z-w)g'(z) \in S^*(w). \tag{1.3}$$

Denote with $\mathcal{P}(w)$ the class of all functions $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$ that are regular in U and satisfy $p(w) = 1$ and $\operatorname{Re}\, p(z) > 0$ for $z \in U$.

2. Preliminary results

If is easy to see that a function $f_{(z)} \in A(w)$ has the series of expansions:

$$f(z) = (z-w) + a_2(z-w)^2 + \dots. \tag{2.1}$$

In [8], Wald gives the sharp bounds for the coefficients B_n of the function $p \in \mathcal{P}(w)$.

Theorem 2.1. *If $p(z) \in \mathcal{P}(w)$, $p(z) = 1 + \sum_{n=1}^{\infty} B_n \cdot (z-w)^n$, then*

$$|B_n| \leq \frac{2}{(1+d)(1-d)^n}, \quad \text{where } d = |w|,\ n \geq 1. \tag{2.2}$$

Using the above result, Kanas and Rønning obtain the following theorem in [3].

Theorem 2.2. *Let $f \in S^*(w)$ and $f(z) = (z-w) + b_2(z-w)^2 + \dots$. Then*

$$\begin{gathered} |b_2| \leq \frac{2}{1-d^2}, \qquad |b_3| \leq \frac{3+d}{(1-d^2)^2}, \\ |b_4| \leq \frac{2}{3} \cdot \frac{(2+d)(3+d)}{(1-d^2)^3}, \qquad |b_5| \leq \frac{1}{6} \cdot \frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4}, \end{gathered} \tag{2.3}$$

where $d = |w|$.

Remark 2.3. It is clear that the above theorem also provides bounds for the coefficients of functions in $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.

In [1], are also defined the following sets:

$$\begin{gathered} D(w) = \left\{ z \in U : \operatorname{Re}\left[\frac{w}{z}\right] < 1,\ \operatorname{Re}\left[\frac{z(1+z)}{(z-w)(1-z)}\right] > 0 \right\} \quad \text{for } w \neq 0,\ D(0) = U; \\ s(w) = \{f : D(w) \longrightarrow \mathbb{C}\} \cap S(w); \qquad s^*(w) = S^*(w) \cap s(w), \end{gathered} \tag{2.4}$$

where w is a fixed point in U.

The authors consider the integral operator $L_a : A(w) \to A(w)$ defined by

$$f(z) = L_a F(z) = \frac{1+a}{(z-w)^a} \cdot \int_w^z F(t) \cdot (t-w)^{a-1} dt, \quad a \in \mathbb{R},\ a \geq 0. \tag{2.5}$$

The next theorem is a result of the so called "admissible functions method" introduced by Mocanu and Miller (see [3, 4, 6]).

Theorem 2.4. *Let h be convex in U and* $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in U$. *If* $p \in \mathcal{H}(U)$ *with* $p(0) = h(0)$ *and p satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \tag{2.6}$$

then $p(z) \prec h(z)$.

3. Main results

Deffinition 3.1. Let w be a fixed point in U, $n \in \mathbb{N}$. D_w^n denotes the differential operator:

$$\begin{gathered} D_w^n : A(w) \longrightarrow A(w) \ \text{ with },\\ D_w^0 f(z) = f(z),\\ D_w^1 f(z) = D_w f(z) = (z-w) \cdot f'(z),\\ D_w^n f(z) = D_w\left(D_w^{n-1} f(z)\right). \end{gathered} \tag{3.1}$$

Remark 3.2. For $f \in A(w)$, $f(w) = (z-w) + \sum_{j=2}^{\infty} a_j (z-w)^j$, we have

$$D_w^n f(z) = (z-w) + \sum_{j=2}^{\infty} j^n \cdot a_j \cdot (z-w)^j. \tag{3.2}$$

It easy to see that if we take $w = 0$, we obtain the Sălăgean differential operator (see [7]).

Deffinition 3.3. Let w be a fixed point in U, $n \in \mathbb{N}$ and $f \in S(w)$. f is said to be an n-w-starlike function if

$$\operatorname{Re} \frac{D_w^{n+1} f(z)}{D_w^n f(z)} > 0, \quad z \in U. \tag{3.3}$$

The class of all these functions is denoted by $S_n^*(w)$.

Remark 3.4. (1) $S_0^*(w) = S^*(w)$ and $S_n^*(0) = S_n^*$, where S_n^* is the class of n-starlike functions introduced by Sălăgean in [7].

(2) If $f(z) \in S_n^*(w)$ and we denote $D_w^n f(z) = g(z)$, we obtain $g(z) \in S^*(w)$.

(3) Using the class $s(w)$, we obtain $s_n^*(w) = S_n^*(w) \cap s(w)$.

Theorem 3.5. *Let w be a fixed point in U and* $n \in \mathbb{N}$. *If* $f(z) \in s_{n+1}^*(w)$ *then* $f(z) \in s_n^*(w)$. *This means*

$$s_{n+1}^*(w) \subset s_n^*(w). \tag{3.4}$$

Proof. From $f(z)\in s_{n+1}^*(w)$, we have $\operatorname{Re}(D_w^{n+2}f(z)/D_w^{n+1}f(z))>0, z\in U$. We denote $p(z)=(D_w^{n+1}f(z)/D_w^n f(z))$, where $p(0)=1$ and $p(z)\in\mathcal{H}(U)$. We obtain

$$\frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)}=\frac{D_w(D_w^{n+1}f(z))}{D_w(D_w^n f(z))}=\frac{(z-w)(D_w^{n+1}f(z))'}{(z-w)(D_w^n f(z))'}=\frac{(D_w^{n+1}f(z))'}{(D_w^n f(z))'},$$
$$p'(z)=\frac{(D_w^{n+1}f(z))'\cdot(D_w^n f(z))-(D_w^{n+1}f(z))\cdot(D_w^n f(z))'}{(D_w^n f(z))^2}$$
$$=\frac{(D_w^{n+1}f(z))'}{(D_w^n f(z))'}\cdot\frac{(D_w^n f(z))'}{D_w^n f(z)}-p(z)\cdot\frac{(D_w^n f(z))'}{D_w^n f(z)}. \tag{3.5}$$

Thus we have

$$(z-w)\cdot p'(z)=\frac{(D_w^{n+1}f(z))'}{(D_w^n f(z))'}\cdot\frac{(z-w)\cdot(D_w^n f(z))'}{D_w^n f(z)}-p(z)\cdot\frac{(z-w)\cdot(D_w^n f(z))'}{D_w^n f(z)},$$
$$(z-w)\cdot p'(z)=\frac{(D_w^{n+1}f(z))'}{(D_w^n f(z))'}\cdot p(z)-[p(z)]^2,$$
$$\frac{(D_w^{n+1}f(z))'}{(D_w^n f(z))'}=p(z)+\frac{1}{p(z)}\cdot(z-w)\cdot p'(z). \tag{3.6}$$

From $\operatorname{Re}(D_w^{n+2}f(z)/D_w^{n+1}f(z))>0$ we obtain $p(z)+(1/p(z))\cdot(z-w)\cdot p'(z)\prec((1+z)/(1-z))$ or

$$p(z)+\frac{zp'(z)}{1/(1-(w/z))\cdot p(z)}\prec\frac{1+z}{1-z}\equiv h(z),\quad \text{with } h(0)=1. \tag{3.7}$$

By hypothesis, we have $\operatorname{Re}[1/(1-(w/z))\cdot h(z)]>0$, and thus from Theorem 2.4 we obtain $p(z)\prec h(z)$ or $\operatorname{Re}p(z)>0$. This means $f\in s_n^*(w)$. □

Remark 3.6. From Theorem 3.5, we obtain $s_n^*(w)\subset s_0^*(w)\subset S^*(w)$, $n\in\mathbb{N}$.

Theorem 3.7. *If $F(z)\in s_n^*(w)$ then $f(z)=L_aF(z)\in S_n^*(w)$, where L_a is the integral operator defined by (2.5).*

Proof. From (2.5) we obtain

$$(1+a)\cdot F(z)=a\cdot f(z)+(z-w)\cdot f'(z). \tag{3.8}$$

By means of the application of the operator D_w^{n+1} we obtain

$$(1+a)\cdot D_w^{n+1}F(z)=a\cdot D_w^{n+1}f(z)+D_w^{n+1}[(z-w)\cdot f'(z)] \tag{3.9}$$

or

$$(1+a)\cdot D_w^{n+1}F(z) = a\cdot D_w^{n+1}f(z) + D_w^{n+2}f(z). \tag{3.10}$$

Similarly, by means of the application of the operator D_w^n, we obtain

$$(1+a)\cdot D_w^{n}F(z) = a\cdot D_w^{n}f(z) + D_w^{n+1}f(z). \tag{3.11}$$

Thus

$$\frac{D_w^{n+1}F(z)}{D_w^nF(z)} = \frac{\left(D_w^{n+2}f(z)/D_w^{n+1}f(z)\right)\cdot\left(D_w^{n+1}f(z)/D_w^nf(z)\right) + a\cdot\left(D_w^{n+1}f(z)/D_w^nf(z)\right)}{\left(D_w^{n+1}f(z)/D_w^nf(z)\right)+a}. \tag{3.12}$$

Using the notation $D_w^{n+1}f(z)/D_w^nf(z) = p(z)$, with $p(0)=1$, we have

$$\frac{(z-w)\cdot p'(z)}{p(z)} = \frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} - p(z) \tag{3.13}$$

or

$$\frac{D_w^{n+2}f(z)}{D_w^{n+1}f(z)} = p(z) + \frac{(z-w)\cdot p'(z)}{p(z)}. \tag{3.14}$$

Thus

$$\begin{aligned}\frac{D_w^{n+1}F(z)}{D_w^nF(z)} &= \frac{p(z)\left[p(z)+\left((z-w)p'(z)/p(z)\right)+a\right]}{p(z)+a}\\ &= p(z) + \frac{zp'(z)}{\left(1/\left(1-(w/z)\right)\right)p(z) + \left(a/\left(1-(w/z)\right)\right)}.\end{aligned} \tag{3.15}$$

From $F(z)\in s_n^*(w)$ we obtain $(D_w^{n+1}F(z)/D_w^nF(z)) \prec ((1+z)/(1-z)) \equiv h(z)$ or

$$p(z) + \frac{zp'(z)}{\left(1/\left(1-(w/z)\right)\right)p(z) + \left(a/\left(1-(w/z)\right)\right)} \prec h(z). \tag{3.16}$$

By hypothesis, we have $\operatorname{Re}[(1/(1-(w/z)))\cdot h(z) + (a/(1-(w/z)))] > 0$ and from Theorem 2.4 we obtain $p(z)\prec h(z)$ or $\operatorname{Re}\{D_w^{n+1}f(z)/D_w^nf(z)\} > 0$, $z\in U$. This means $f(z) = L_aF(z)\in S_n^*(w)$. □

Remark 3.8. If we consider $w=0$ in Theorem 3.7 we obtain that the integral operator defined by (2.5) preserves the class of n-starlike functions, and if we consider $w=0$ and $n=0$ in the above theorem we obtain that the integral operator defined by (2.5) preserves the well-known class of starlike functions.

Theorem 3.9. *Let w be a fixed point in U and $f \in S_n^*(w)$ with $f(z) = (z-w) + \sum_{j=2}^{\infty} a_j \cdot (z-w)^j$. Then*

$$\begin{aligned} |a_2| &\le \frac{1}{2^{n-1}\cdot(1-d^2)},\\ |a_3| &\le \frac{3+d}{3^n\cdot(1-d^2)^2},\\ |a_4| &\le \frac{(2+d)(3+d)}{2^{2n-1}\cdot 3\cdot(1-d^2)^3},\\ |a_5| &\le \frac{(2+d)(3+d)(3d+5)}{5^n\cdot 6\cdot(1-d^2)^4}, \end{aligned} \tag{3.17}$$

where $d = |w|$.

Proof. From Remark 3.4 for $f \in S_n^*(w)$ we obtain

$$D_w^n f(z) = g(z) \in S^*(w). \tag{3.18}$$

If we consider $g(z) = (z-w) + \sum_{j=2}^{\infty} b_j \cdot (z-w)^j$, using Remark 3.2, from (3.18) we obtain $j^n \cdot a_j = b_j$, $j = 2,3,\ldots$.

Thus we have $a_j = 1/j^n \cdot b_j$, $j = 2,3,\ldots$, and from the estimates (2.3) we get the result. □

References

[1] M. Acu and S. Owa, *On some subclasses of univalent functions*, JIPAM. J. Inequal. Pure Appl. Math. **6** (2005), no. 3, article 70, 1–6.

[2] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), no. 1, 87–92.

[3] S. Kanas and F. Rønning, *Uniformly starlike and convex functions and other related classes of univalent functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **53** (1999), 95–105.

[4] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), no. 2, 157–172.

[5] ———, *On some classes of first-order differential subordinations*, Michigan Math. J. **32** (1985), no. 2, 185–195.

[6] ———, *Univalent solutions of Briot-Bouquet differential equations*, J. Differential Equations **56** (1985), no. 3, 297–309.

[7] G. Ş. Sălăgean, *Subclasses of univalent functions*, Complex Analysis—Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, pp. 362–372.

[8] J. K. Wald, *On starlike functions*, Ph.D. thesis, University of Delaware, Delaware, 1978.

Mugur Acu: Department of Mathematics, "Lucian Blaga" University of Sibiu, 5–7 Ion Raţiu Street, 550012 Sibiu, Romania

E-mail address: acu_mugur@yahoo.com

Shigeyoshi Owa: Department of Mathematics, School of Science and Engineering, Kinki University, Higashiosaka, Osaka 577-8502, Japan

E-mail address: owa@math.kindai.ac.jp

Special Issue on
Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	May 1, 2009
First Round of Reviews	August 1, 2009
Publication Date	November 1, 2009

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O'Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie