MULTIPLE PERIODIC SOLUTIONS TO A CLASS OF SECOND-ORDER NONLINEAR MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Received 22 October 2004 and in revised form 23 August 2005

By means of variational structure and Z_2 group index theory, we obtain multiple periodic solutions to a class of second-order mixed-type differential equations $x''(t-\tau)+f(t,x(t),x(t-\tau),x(t-2\tau))=0$ and $x''(t-\tau)+\lambda(t)f_1(t,x(t),x(t-\tau),x(t-2\tau))=x(t-\tau)$.

1. Introduction

Recently, the existence and multiplicity of periodic solutions for second-order functional differential equations has received a great deal of attention (e.g., see [8, 9, 12]). In [9], Wang and Yan studied the second-order functional differential equation

$$[x(t)+cx(t-\tau)]''+g(t,x(t-\sigma))=p(t), \tag{1.1}$$

where τ, σ, and c are constants in R with $\tau\geq 0$, $\sigma\geq 0$, $|c|<1$, $g(t,x)$ is a $T(>0)$-periodic function in $t>0$, and for an arbitrary bounded domain $E\subset R$, $g(t,x)$ is a Lipschitz function in $[0,T]\times E$, $p\in C(R,R)$, $p(t+T)=p(t)$, and $\int_0^T p(t)dt=0$. They obtained some sufficient conditions to guarantee the existence, at least a T-periodic solution, for this system.

But, for the existence of periodic solutions of functional differential equations, one commonly uses methods of fixed point theory, coincidence degree theory, Fourier analysis, and so forth. Critical point theory has rarely been used. In [10, 11], the authors obtained multiple periodic solutions for a class retarded differential equations by means of critical point theory and Z_p group index theory. Nevertheless, we noted that these results were obtained by reducing retarded differential equations to related ordinary differential equations.

The purpose of our paper is to establish a kind of variational framework with delayed variables for a class of mixed-type differential equations. Unlike [10, 11], our approach enables us to obtain, by critical point theory and Z_2 group index theory, the existence of nontrivial periodic solutions to such equations without reducing it to the one of ordinary differential equations. Subsequently, we introduce Z_2 group index theory and knowledge about critical points.

International Journal of Mathematics and Mathematical Sciences 2005:17 (2005) 2689–2702
DOI: 10.1155/IJMMS.2005.2689

Definition 1.1. A critical point of f is a point where $f'(x) = 0$. A critical value of f is a number c such that $f(x) = c$ for some critical points x. K is critical set where $K = \{x \in E \mid f'(x) = 0\}$, $K_c = \{x \in E \mid f'(x) = 0,\ f(x) = c\}$. f_c is a level set if $f_c = \{x \in E \mid f(x) \le c\}$.

Definition 1.2. Let E be real Banach space, and $f \in C^1(E,R)$, we say that f satisfies the Palais-Smale condition if every sequence $\{x_n\} \subset E$ such that $\{f(x_n)\}$ is bounded and $f'(x_n) \to \theta (n \to \infty)$ has a converging subsequence.

Definition 1.3. Let E be real Banach space, and $\sum = \{A \mid A \subset E \setminus \{\theta\}$ is closed, symmetric set$\}$. Define $\gamma : \sum \to Z^+ \bigcup \{+\infty\}$ as following:

$$\gamma(A) = \begin{cases} \min & \{n \in Z : \text{there exists an odd continuous map } \varphi : A \longrightarrow R^n \setminus \{\theta\}\}; \\ 0 & \text{if } A = \emptyset; \\ +\infty & \text{if there is no odd continuous map } \varphi : A \longrightarrow R^n \setminus \{\theta\} \text{ for any } n \in Z. \end{cases} \tag{1.2}$$

Then we say γ is the genus of $\sum$. Denote $i_1(f) = \lim_{a \to -0} \gamma(f_a)$ and $i_2(f) = \lim_{a \to -\infty} \gamma(f_a)$.

Lemma 1.4 (Rabinowitz [7]). *Let $f \in C^1(X,R^1)$ be an even functional which satisfies the Palais-Smale condition and $f(\theta) = 0$. If*

(i) *there exists $\rho > 0$, $\alpha > 0$, and a finite dimensional subspace E of X, such that $f|_{E^\perp \bigcap S_\rho} \ge \alpha$.*

(ii) *for all finite dimensional subspaces $\widetilde{E}$ of X, there is an $r = r(\widetilde{E}) > 0$, such that $f(x) \le 0$ for $x \in \widetilde{E} \setminus B_r$.*

Then, f possesses an unbounded sequence of critical values.

Lemma 1.5 (Chang [1]). *Let $f \in C^1(E,R^1)$ be an even functional which satisfies the Palais-Smale condition and $f(\theta) = 0$. Then,*

(P_1) *If there exists an m-dimensional subspace X of E and $\rho > 0$ such that*

$$\sup_{x \in X \cap S_\rho} f(x) < 0, \tag{1.3}$$

then we have $i_1(f) \ge m$;

(P_2) *If there exists an j-dimensional subspace $\widetilde{X}$ of E such that*

$$\inf_{x \in \widetilde{X}^\perp} f(x) > -\infty, \tag{1.4}$$

we have $i_2(f) \le j$.

If $m \ge j$, and (P_1) and (P_2) hold, then f at least has $2(m-j)$ distinct critical points.

Lemma 1.6. *Let E be Hilbert space, if the weak convergence sequence $\{x_n\} \subset E$ satisfies $\|x_n\| \to \|x_0\| (n \to \infty)$, then $\{x_n\}$ is convergent in E, that is, $x_n \to x_0$.*

Proof. By

$$\begin{aligned}||x_n - x_0||^2 &= (x_n - x_0, x_n - x_0) \\ &= ||x_n||^2 - (x_0, x_n) - (x_n, x_0) + ||x_0|| \quad (n = 1, 2, 3, \dots)\end{aligned} \tag{1.5}$$

and continuity of inner product, it is easy to see that

$$\lim_{n\to\infty} ||x_n - x_0||^2 = ||x_0||^2 - 2(x_0, x_0) + ||x_0||^2 = 0, \tag{1.6}$$

that is, $x_n \to x_0 (n \to \infty)$. □

First, we use Lemma 1.4 to deal with multiple periodic solutions of the following second-order mixed-type delay equations

$$x''(t-\tau) + f(t, x(t), x(t-\tau), x(t-2\tau)) = 0. \tag{1.7}$$

Our basic assumptions are that

(A$_1$) $f(t, u_1, u_2, u_3) \in C(R^4, R)$ and $\partial f(t, u_1, u_2, u_3)/\partial t \neq 0$, as well as there exists a continuous function $g(t, u, v) \in C(R^3, R)$ that satisfies $\partial g/\partial u$, and $\partial g/\partial v$ are well defined such that

$$f(t, u_1, u_2, u_3) = g(t, u_1, u_2) + \int_0^{u_3} g'_{u_2}(t, u_2, \omega)\, d\omega - u_2; \tag{1.8}$$

(A$_2$) $f(t+\tau, u_1, u_2, u_3) = f(t, u_1, u_2, u_3)$,

$$f(t, -u_1, -u_2, -u_3) = -f(t, u_1, u_2, u_3). \tag{1.9}$$

2. Variational structure

Let

$$\begin{aligned}H^1_{2\gamma\tau} = \{&x(t) \in L^2[0, 2\gamma\tau] \mid x'(t) \in L^2[0, 2\gamma\tau] x(t) \text{ is } 2\gamma\tau\text{—periodic functionin } t, \\ &\text{where } \gamma \text{ is a given positive integer}\}.\end{aligned} \tag{2.1}$$

It is obvious that $H^1_{2\gamma\tau}$ is a Sobolev space by defining the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$,

$$\begin{aligned}\langle x, y\rangle_{H^1_{2\gamma\tau}} &= \int_0^{2\gamma\tau} [x(t)y(t) + x'(t)y'(t)]\, dt, \\ \|x\|_{H^1_{2\gamma\tau}} &= \left| \int_0^{2\gamma\tau} \left[|x(t)|^2 + |x'(t)|^2 \right] dt \right|^{1/2}, \quad \forall x, y \in H^1_{2\gamma\tau},\end{aligned} \tag{2.2}$$

as well as $x(t)$ can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\frac{k\pi}{\gamma\tau} t + b_k \sin\frac{k\pi}{\gamma\tau} t \right). \tag{2.3}$$

Let us consider the function defined on $H^1_{2\gamma\tau}$,

$$I(x)=\int_0^{2\gamma\tau}\left[\frac{1}{2}\left(|x'(t)|^2+|x(t)|^2\right)-\int_0^{x(t-\tau)}g(t,x(t),\omega)\,d\omega\right]dt. \tag{2.4}$$

Then, for all $x,y\in H^1_{2\gamma\tau}$ and $\varepsilon>0$, we know that

$$\begin{aligned} I(x+\varepsilon y)=I(x)+\varepsilon\bigg(&\int_0^{2\gamma\tau}[x(t)y(t)+x'(t)y'(t)]dt\\ &-\int_0^{2\gamma\tau}\bigg[\bigg(\int_0^{x(t-\tau)}g'_{u_1}(t,x(t)+\varepsilon\theta(t)y(t),\omega)\,d\omega\bigg)y(t)\\ &\qquad+g(t,x(t),x(t-\tau)+\varepsilon y(t-\tau))\,y(t-\tau)\bigg]dt\bigg)\\ &+\frac{\varepsilon^2}{2}\int_0^{2\gamma\tau}[y^2(t)+|y'(t)|^2]dt, \end{aligned} \tag{2.5}$$

where $0\le\theta(t)\le 1$. It is easy to see that

$$\begin{aligned} \langle I'(x),y\rangle=&\int_0^{2\gamma\tau}\left[x'(t)y'(t)+x(t)y(t)-\int_0^{x(t-\tau)}g'_{u_1}(t,x(t),\omega)\,d\omega\right]y(t)dt\\ &-\int_0^{2\gamma\tau}g(t,x(t),x(t-\tau))\,y(t-\tau)dt. \end{aligned} \tag{2.6}$$

By the periodicity $g(t,u,v)$, $x(t)$, $x(t-\tau)$, and $y(t)$, we get that

$$\begin{aligned} \int_0^{2\gamma\tau}g(t,x(t),x(t-\tau))\,y(t-\tau)dt&=\int_{-\tau}^{(2\gamma-1)\tau}g(t+\tau,x(t+\tau),x(t))\,y(t)dt\\ &=\int_0^{2\gamma\tau}g(t,x(t+\tau),x(t))\,y(t)dt. \end{aligned} \tag{2.7}$$

Hence,

$$\begin{aligned} \langle I'(x),y\rangle=\int_0^{2\gamma\tau}\bigg[&-x''(t)+x(t)-\int_0^{x(t-\tau)}g'_{u_1}(t,x(t),\omega)\,d\omega\\ &-g(t,x(t+\tau),x(t))\bigg]y(t)dt. \end{aligned} \tag{2.8}$$

Therefore, the Euler equation corresponding to the function $I(x)$ is as follows:

$$x''(t)-x(t)+\int_0^{x(t-\tau)}g'_{u_1}(t,x(t),\omega)\,d\omega+g(t,x(t+\tau),x(t))=0. \tag{2.9}$$

It is easy to see that (2.9) is equivalent to (1.7), so the system (1.7) is the Euler equation corresponding to the function $I(x)$. Then, we may get $2\gamma\tau$-periodic solutions of the system (1.7), by seeking critical points of the function $I(x)$.

3. Main results

Theorem 3.1. *Under the assumptions* $(A_1) \sim (A_2)$ *and the function* $g(t,u_1,u_2)$ *satisfying the following conditions:*

(C_1) *there exists a constant* $T > 0$ *such that*

$$\lim_{|u|\to 0} \frac{\int_0^{u_2} g(t,u_1,\omega)\,d\omega}{|u|^2} \le T, \tag{3.1}$$

where $|u| = \sqrt{u_1^2+u_2^2}$.

(C_2) *there exist constants* $\beta > 2$ *and* $\alpha > 0$ *such that*

$$\begin{aligned} 0 < G(t,u_1,u_2) &= \int_0^{u_2} g(t,u_1,\omega)\,d\omega \\ &\le \frac{1}{\beta}\left[\left(\int_0^{u_2} g'_{u_1}(t,u_1,\omega)\,d\omega\right)u_1 + g(t,u_1,u_2)u_2\right], \quad \forall |u| = \sqrt{u_1^2+u_2^2} \ge \alpha, \end{aligned} \tag{3.2}$$

then the problem (1.7) has an infinite number of nontrivial $2\gamma\tau$*-periodic solutions.*

It is not difficult to see that if $x(t)$ is a solution of the system (1.7), then $-x(t)$ is also a solution of the system (1.7) by the assumption (A_2). That is, the solution of the system (1.7) is a set that is symmetric with respect to the origin in $H^1_{2\gamma\tau}$. On the other hand, if we let $\eta(t,x) = G(t,x(t),x(t-\tau)) = \int_0^{x(t-\tau)} g(t,x(t),\omega)\,d\omega$, it is easy to see that $\eta(t,x)$ is an even function in x, so $I(x)$ is an even function in x and we may show that Theorem 3.1 holds by Lemma 1.4.

In order to exploit Lemma 1.4 to find the critical points of function $I(x)$ in (2.4), one needs to verify all the assumptions. First of all, we point out that the functional $I(\cdot)$, defined on $H^1_{2\gamma\tau}$, satisfies the Palais-Smale condition, that is, we have the following lemma.

Lemma 3.2. *Under the assumptions* $(A_1) \sim (A_2)$ *and the conditions* $(C_1) \sim (C_2)$, $I(u)$ *satisfies the P. S. condition.*

Proof. Let $\{u_n\} \subset H^1_{2\gamma\tau}$ and the constants c_1, c_2 satisfy

$$c_1 \le I(u_n) \le c_2, \tag{3.3}$$

$$I'(u_n) \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{3.4}$$

The above inequality (3.3) is equivalent to

$$c_1 < \int_0^{2\gamma\tau}\left[\frac{1}{2}\left(|u_n'(t)|^2 + |u_n(t)|^2\right) - \int_0^{u_n(t-\tau)} g(t,x(t),\omega)\,d\omega\right]dt < c_2. \tag{3.5}$$

Replacing x and y by u_n in (2.6), we have

$$\begin{aligned} \|u_n\|^2_{H^1_{2\gamma\tau}} = \int_0^{2\gamma\tau}\bigg[&\left(\int_0^{u_n(t-\tau)} g'_{\xi_1}(t,u_n(t),\omega)\,d\omega\right)u_n(t) \\ &+ g(t,u_n(t),u_n(t-\tau))u_n(t-\tau)\bigg]dt \\ &+ \langle I'(u_n),u_n\rangle. \end{aligned} \tag{3.6}$$

By (3.6), we have

$$\begin{aligned}&\int_0^{2\gamma\tau}\left[\left(\int_0^{u_n(t-\tau)} g'_{\xi_1}(t,u_n(t),\omega)\,d\omega\right)u_n(t)+g(t,u_n(t),u_n(t-\tau))u_n(t-\tau)\right]dt\\&\quad=\|u_n\|^2_{H^1_{2\gamma\tau}}-\langle I'_n(u_n),u_n\rangle.\end{aligned}\tag{3.7}$$

Next, we show that a sequence $\{u_n\}$ satisfying condition (3.3) and (3.4) is bounded. Denote $B_1=\{t\in[0,2\gamma\tau]\mid |u_n(t)|=\sqrt{u_n^2(t)+u_n^2(t-\tau)}\geq\alpha\}$, $B_2=[0,2\gamma\tau]\setminus B_1$. By the condition (C_2) and (3.6), we have

$$\begin{aligned}I(u_n)&=\frac{1}{2}\|u_n\|^2-\int_0^{2\gamma\tau}\left(\int_0^{u_n(t-\tau)} g(t,u_n(t),\omega)\,d\omega\right)dt\\&=\frac{1}{2}\|u_n\|^2-\int_{B_1}G(t,u_n(t),u_n(t-\tau))\,dt-\int_{B_2}G(t,u_n(t),u_n(t-\tau))\,dt\\&\geq\frac{1}{2}\|u_n\|^2-\int_{B_1}\frac{1}{\beta}\left[\left(\int_0^{u_n(t-\tau)} g'_{\xi_1}(t,u_n(t),\omega)\,d\omega\right)u_n(t)\right.\\&\qquad\left.+g(t,u_n(t),u_n(t-\tau))u_n(t-\tau)\right]dt-c_3\\&\geq\frac{1}{2}\|u_n\|^2-\frac{1}{\beta}\int_0^{2\gamma\tau}\left[\left(\int_0^{u_n(t-\tau)} g'_{\xi_1}(t,u_n(t),\omega)\,d\omega\right)u_n(t)\right.\\&\qquad\left.+g(t,u_n(t),u_n(t-\tau))u_n(t-\tau)\right]dt-c_4\\&=\frac{1}{2}\|u_n\|^2-\frac{1}{\beta}\left[\|u_n\|^2-\langle I'(u_n),u_n\rangle\right]-c_4\\&\geq\left(\frac{1}{2}-\frac{1}{\beta}\right)\|u_n\|^2-\frac{1}{\beta}\|I'(u_n)\|\|u_n\|-c_4.\end{aligned}\tag{3.8}$$

Remarks 3.3. In here and the following, $c_i>0$.

Then, by (3.3) and (3.4), it is easy to see

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{\beta}\right)\|u_n\|^2&\leq\frac{1}{\beta}\|I'(u_n)\|\,\|u_n\|+c_4\\&\leq c_5\|u_n\|+c_6.\end{aligned}\tag{3.9}$$

Since $\beta>2$, we know that $\{\|u_n\|\}$ is bounded.

Since $H^1_{2\gamma\tau}$ is a reflexive Banach space and the sequence $\{u_n\}$ is bounded, so $\{u_n\}$ has a weakly convergent subsequence. We still denote it by $\{u_n\}$ and suppose that $u_n\rightharpoonup u_0$ in $H^1_{2\gamma\tau}$ as $n\to\infty$.

So by (3.7) and the boundedness of $\|u_n\|$, we get that

$$\begin{aligned}&\|u_n\|^2_{H^1_{2\gamma\tau}}-\int_0^{2\gamma\tau}\left(\int_0^{u_n(t-\tau)} g'_{\xi_1}(t,u_n(t),\omega)\,d\omega\right)u_n(t)\,dt\\&\quad-\int_0^{2\gamma\tau}g(t,u_n(t),u_n(t-\tau))u_n(t-\tau)\,dt\longrightarrow 0,\quad(n\longrightarrow\infty).\end{aligned}\tag{3.10}$$

On the other hand, the weak convergence of $\{u_n\}$ of $H^1_{2\gamma\tau}$ implies the uniform convergence of $\{u_n\}$ in $C([0,1],R)$ (see [6]). Hence,

$$\begin{aligned}||u_n||^2_{H^1_{2\gamma\tau}} \longrightarrow \int_0^{2\gamma\tau}\left(\int_0^{u_0(t-\tau)} g'_{\xi_1}(t,u_0(t),\omega)\,d\omega\right)u_0(t)dt \\ +\int_0^{2\gamma\tau} g(t,u_0(t),u_0(t-\tau))u_0(t-\tau)dt, \quad (n\longrightarrow\infty).\end{aligned} \tag{3.11}$$

This means that $\|u_n\|$ is convergent in $H^1_{2\gamma\tau}$, then, by Lemma 1.6, we get that the function I satisfies the P. S. condition. □

Lemma 3.4. *Under the assumptions $(A_1)\sim(A_2)$ and the conditions $(C_1)\sim(C_2)$, then there exist $\rho,\alpha>0$ and finite dimensional subspace E of $H^1_{2\gamma\tau}$, such that*

$$I(x)_{E^\perp\cap S_\rho}\geq\alpha. \tag{3.12}$$

Proof. Let $v_j(t)=(\gamma\tau/k\pi)\sin(k\pi/\gamma\tau)t$, $j=1,2,\ldots$, then

$$\begin{aligned}\int_0^{2\gamma\tau}|v_j(t)|^2dt=\frac{\gamma^2\tau^2}{k^2\pi^2}\gamma\tau,\\ \int_0^{2\gamma\tau}|v'_j(t)|^2dt=\gamma\tau.\end{aligned} \tag{3.13}$$

Define an n-dimensional linear space as follows:

$$E=\operatorname{span}\{v_1,v_2,\ldots,v_n\},\qquad X=E^\perp. \tag{3.14}$$

For all $x(t)\in S_\rho\bigcap X$, we get that

$$\begin{aligned}|x(t)| &\leq \left|\int_0^{2\gamma\tau}x'(t)dt\right| \leq \int_0^{2\gamma\tau}|x'(t)|dt\\ &\leq\sqrt{2\gamma\tau}\left(\int_0^{2\gamma\tau}|x'(t)|^2dt\right)^{1/2}\leq\sqrt{2\gamma\tau}\|x\|_{H^1_{2\gamma\tau}}.\end{aligned} \tag{3.15}$$

By the periodicity of $x(t)$, it is easy to see that $|x(t-\tau)|\leq\sqrt{2\gamma\tau}\|x\|_{H^1_{2\tau}}$. On the other hand, by the condition (C_2), for all $\varepsilon_0>0$, there exists $\delta>0$, such that when $u=(x(t),x(t-\tau))$ satisfies

$$x^2(t)+x^2(t-\tau)\leq 2\left(\sqrt{2\gamma\tau}\|x\|_{H^1_{2\gamma\tau}}\right)^2=4\gamma\tau\|x\|^2_{H^1_{2\tau}}\leq\delta^2, \tag{3.16}$$

we have

$$|G(t,x(t),x(t-\tau))|=\int_0^{x(t-\tau)}g(t,x(t),\omega)\,d\omega\leq(T+\varepsilon_0)\left[x^2(t)+x^2(t-\tau)\right]. \tag{3.17}$$

At the same time, it is not difficult to see that

$$\int_0^{2\gamma\tau} |x(t)|^2 dt \le \frac{\gamma^2\tau^2}{k^2\pi^2}\int_0^{2\gamma\tau} |x'(t)|^2 dt \tag{3.18}$$

holds when $x(t) \in S_\rho \bigcap X$. So one gets

$$\int_0^{2\gamma\tau} |x(t)|^2 dt \le \frac{\gamma^2\tau^2}{k^2\pi^2+\gamma^2\tau^2}\rho^2. \tag{3.19}$$

By the above equality and (3.17) as well as the periodicity of $x(t-\tau)$, we have

$$\begin{aligned} I(x) &= \int_0^{2\gamma\tau}\left[\frac{1}{2}\left(|x'(t)|^2+|x(t)|^2\right)-\int_0^{x(t-\tau)} g(t,x(t),x(t-\tau))\right]dt \\ &\ge \frac{1}{2}\|x\|^2_{H^1_{2\gamma\tau}} - 2(T+\varepsilon_0)\int_0^{2\gamma\tau}|x(t)|^2 dt \\ &\ge \frac{1}{2}\rho^2 - \frac{2(T+\varepsilon_0)\gamma^2\tau^2}{k^2\pi^2+\gamma^2\tau^2}\rho^2 = \frac{1}{2}\left(1-\frac{4(T+\varepsilon_0)\gamma^2\tau^2}{k^2\pi^2+\gamma^2\tau^2}\right)\rho^2 > 0. \end{aligned} \tag{3.20}$$

Remark 3.5. We may choose $T > 0$ and $\varepsilon_0 > 0$ such that the above equality holds.

That is, Lemma 3.4 holds. □

Lemma 3.6. *Under the assumptions* $(A_1) \sim (A_2)$ *and the conditions* $(C_1) \sim (C_2)$, *for all finite dimensional subspace* $\widetilde{E}$ *of* $H^1_{2\gamma\tau}$, *there is an* $R = R(\widetilde{E}) > 0$, *such that*

$$I(x) \le 0, \quad \forall x \in \widetilde{E}\backslash B_R. \tag{3.21}$$

Proof. For an arbitrary finite dimensional subspace $E_1 \subset H^1_{2\gamma\tau}$, by (C_2), we know that there exist constant $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\int_0^{x(t-\tau)} g(t,x(t),\omega)d\omega \ge \alpha_1\left|\sqrt{u_1^2+u_2^2}\right|^\beta - \alpha_2. \tag{3.22}$$

So, for any given $\varphi \in E_1$, $\|\varphi\|_{H^1_{2\gamma\tau}} = 1$ and $\mu > 0$, we have

$$\begin{aligned} I(\mu\varphi) &= \frac{1}{2}\mu^2\|\varphi\|^2_{H^1_{2\gamma\tau}} - \int_0^{2\gamma\tau}\left(\int_0^{\mu\varphi(t-\tau)} g(t,\mu\varphi(t),\omega)d\omega\right)dt \\ &\le \frac{1}{2}\mu^2\|\varphi\|^2_{H^1_{2\gamma\tau}} - \alpha_1\mu^\beta\int_0^{2\gamma\tau}\left|\sqrt{\varphi^2(t)+\varphi^2(t-\tau)}\right|^\beta dt + 2\tau\alpha_2 \longrightarrow -\infty, \quad \mu \longrightarrow +\infty. \end{aligned} \tag{3.23}$$

Then there exists $\mu_0 > 0$.

Remark 3.7. In fact, μ_0 is a minimum value of μ that the above inequality holds on the unit ball of the finite dimensional subspace E_1.

For any given $\varphi \in E_1$, $\|\varphi\|_{H^1_{2\gamma\tau}} = 1$, when $\mu \geq \mu_0$, such that $I(\mu\varphi) < 0$. So choosing $R = \mu_0$, we get that

$$I(x) \leq 0, \quad \forall x \in \widetilde{E_1} \backslash B_R. \tag{3.24}$$

Since E_1 was arbitrary, we know Lemma 3.6 holds. □

By (A_2), we get that $I(\theta) = 0$. So, by Lemmas 3.2, 3.4, and 3.6, we know that I has infinite nontrivial critical points, that is, the problem (1.7) has infinite nontrivial $2\gamma\tau$-periodic solutions.

We next consider the nonlinear mixed-type delay equations

$$x''(t-\tau) + \lambda(t) f_1(t, x(t), x(t-\tau), x(t-2\tau)) = x(t-\tau), \quad \lambda(t) > 0. \tag{3.25}$$

Our basic assumptions is that

(A_1') $f_1(t, u_1, u_2, u_3) \in C(R^4, R)$, and $\partial f_1(t, u_1, u_2, u_3)/\partial t \neq 0$, and there exists a continuous function $g_1(t, u, v) \in C(R^2, R)$ such that

$$f_1(t, u_1, u_2, u_3) = g_1(t, u_1, u_2) + \int_0^{u_3} g'_{1u_2}(t, u_2, \omega)\, d\omega; \tag{3.26}$$

(A_2') $f(t+\tau, u_1, u_2, u_3) = f(t, u_1, u_2, u_3)$, and $\lambda(t) \in C(R, R)$ satisfies $\lambda(t+\tau) = \lambda(t)$ as well as

$$f_1(t, -u_1, -u_2, -u_3) = -f_1(t, u_1, u_2, u_3). \tag{3.27}$$

Under the assumptions (A_1') ~ (A_2'), similar to Theorem 3.1, it is easy to see that the corresponding energy functional of the system (3.25) is

$$I(x) = \int_0^{2\gamma\tau} \left[\frac{1}{2}\left(|x'(t)|^2 + |x(t)|^2\right) - \lambda(t) \int_0^{x(t-\tau)} g_1(t, x(t), \omega)\, d\omega\right] dt. \tag{3.28}$$

THEOREM 3.8. *Under the assumptions* (A_1') ~ (A_2'), *and the function* $g_1(t, u_1, u_2)$ *satisfying the following conditions:*

(F_1) $\lim_{|u|\to 0}(\int_0^{x(t-\tau)} g_1(t, x(t), \omega) d\omega / |u|^2) = 1$, *where* $|u| = \sqrt{u_1^2 + u_2^2}$;

(F_2) *there exists an* $\alpha > 0$ *such that* $g_1(t, u_1, \alpha) \leq 0$, *for all* $(t, u_1) \in [0, \tau] \times (R \setminus [-\alpha, \alpha])$. *Denote* $\kappa = \min_{t \in [0,\tau]} \lambda(t)$, *then when*

$$\kappa > \frac{n^2(\pi^2 + \gamma^2\tau^2)}{4\gamma\tau^2}, \tag{3.29}$$

the problem (3.25) has at least $2n$ *nontrivial* $2\gamma\tau$*-periodic solutions.*

Similar to Theorem 3.1, we may show that Theorem 3.8 holds by mean of Z_2 group index theory, that is, Lemma 1.5.

Proof. Let

$$h(t,u_1,u_2) = \begin{cases} g_1(t,u_1,\alpha), & u_2 > \alpha, \\ g_1(t,u_1,u_2), & |u| \le \alpha, \\ g_1(t,u_1,-\alpha), & u_2 < -\alpha, \end{cases} \tag{3.30}$$

so $h(t,-u_1,-u_2) = -h(t,u_1,u_2)$ is obvious. Let us consider the functional defined on $H^1_{2\gamma\tau}$,

$$I(x) = \int_0^{2\gamma\tau} \left[\frac{1}{2}\left(|x'(t)|^2 + |x(t)|^2\right) - \lambda(t)\int_0^{x(t-\tau)} h(t,x(t),\omega)\,d\omega\right]dt. \tag{3.31}$$

First, we show that $I(x)$ has a lower bound.

By the periodicity $x(t)$, $x(t-\tau)$, one gets $\max_{t\in[0,2\gamma\tau]}|x(t)| = \max_{t\in[0,2\gamma\tau]}|x(t-\tau)|$. Then, we have $\max_{t\in[0,2\gamma\tau]}|x(t)| < \alpha$ when $\max_{t\in[0,2\gamma\tau]}|x(t-\tau)| < \alpha$. On the other hand, by (F_2), we get $x(t-\tau)h(t,x(t),x(t-\tau)) \le 0$ when $|x(t-\tau)| \ge \alpha$. So, $\int_0^{2\gamma\tau}(\int_0^{\alpha}|h(t,x(t),\omega)d\omega|)dt$ is bounded. Denote $M = \int_0^{2\gamma\tau}(\int_0^{\alpha}|h(t,x(t),\omega)d\omega|)dt$, and $L = \max_{t\in[0,\tau]}\lambda(t)$, then we get

$$I(x) = \frac{1}{2}\|x\|^2 - \int_0^{2\gamma\tau}\lambda(t)\int_0^{x(t-\tau)} h(t,x(t),\omega)\,d\omega \ge \frac{1}{2}\|x\|^2 - LM. \tag{3.32}$$

So, $I(x)$ has a lower bound, by the condition (P_1) of Lemma 1.5, we get $i_2(I) = 0$.

Secondly, we will show that $I(x)$ satisfies the P. S. condition. Let $\{x_n\} \subset H^1_{2\gamma\tau}$, and the constants c_1, c_2 satisfy

$$\begin{gathered} c_1 \le I(x_n) \le c_2, \\ I'(x_n) \longrightarrow 0, \quad (n \longrightarrow \infty). \end{gathered} \tag{3.33}$$

By (3.32), we know

$$\|x\|_{H^1_{2\gamma\tau}} \le \sqrt{2LM + 2c_2}. \tag{3.34}$$

So, $\|x_n\|_{H^1_{2\gamma\tau}}$ is bounded. Similarly to the proof of Lemma 3.2, it is easy to see $I(x)$ satisfies the P. S. condition.

Finally, we show that Theorem 3.8 holds by Lemma 1.5.

Denote $\beta_k(t) = (\gamma\tau/k\pi)\cos(k\pi/\gamma\tau)t$, $k = 1,2,3,\dots,n$, then

$$\begin{gathered} \int_0^{2\gamma\tau} |\beta_k(t)|^2 dt = \frac{\gamma^2\tau^2}{k^2\pi^2}\gamma\tau, \\ \int_0^{2\gamma\tau} |\beta_k'(t)|^2 dt = \gamma\tau. \end{gathered} \tag{3.35}$$

Define the n-dimensional space

$$E_n = \operatorname{span}\{\beta_1(t),\beta_2(t),\ldots,\beta_n(t)\}. \tag{3.36}$$

It is obvious that E_n is a symmetric set. Suppose $\rho > 0$, then

$$E_n \bigcap S_\rho = \left\{ \sum_{k=0}^{n} b_k\beta_k \,\middle|\, \sum_{k=0}^{n} b_k^2 \gamma\tau\left(1+\frac{\gamma^2\tau^2}{k^2\pi^2}\right) = \rho^2 \right\}. \tag{3.37}$$

On the other hand, we may choose ε such that $0 < \varepsilon < \kappa n^2\pi^2/\gamma^2\tau^2(2\gamma^2\tau^2/n^2 - (\pi^2 + \gamma^2\tau^2)/\kappa)$. By ($F_2$), we know that there exists $\delta > 0$, when $\|x_n(t)\|_C^2 + \|x(t-\tau)\|_C^2 \le \delta$ (where $\|x_n(t)\|_C^2 = \max_{0\le t\le 2\gamma\tau}|x(t)|$) such that

$$\begin{aligned}
\lambda(t)\int_0^{x(t-\tau)} & h(t,x(t),\omega)\,d\omega \\
&\ge (\lambda(t)-\varepsilon)\Big[|x(t)|^2+|x(t-\tau)|^2\Big] \\
&\ge (\kappa-\varepsilon)\Big[|x(t)|^2+|x(t-\tau)|^2\Big], \quad \forall t\in[0,2\gamma\tau].
\end{aligned} \tag{3.38}$$

So, choose $\rho = \delta$, when $x \in E_n \bigcap S_\rho$, we have

$$\begin{aligned}
I(x) &= \int_0^{2\gamma\tau}\left[\frac{1}{2}\left(|x'(t)|^2+|x(t)|^2\right) - \lambda(t)\int_0^{x(t-\tau)} h(t,x(t),\omega)\,d\omega\right]dt \\
&\le \frac{1}{2}\sum_{k=0}^{n}\gamma\tau b_k^2\left(1+\frac{\gamma^2\tau^2}{k^2\pi^2}\right) - (\kappa-\varepsilon)\int_0^{2\gamma\tau}\Big[|x(t)|^2+|x(t-\tau)|^2\Big]dt \\
&\le \frac{1}{2}\sum_{k=0}^{n}\gamma\tau b_k^2\left(1+\frac{\gamma^2\tau^2}{k^2\pi^2}\right) - 2(\kappa-\varepsilon)\int_0^{2\gamma\tau}|x(t)|^2dt \\
&\le \frac{1}{2}\sum_{k=0}^{n}\gamma\tau b_k^2\left(1+\frac{\gamma^2\tau^2}{k^2\pi^2}\right) - 2(\kappa-\varepsilon)\sum_{k=0}^{n}\gamma\tau b_k^2\frac{\gamma^2\tau^2}{k^2\pi^2} \\
&\le \frac{1}{2}\sum_{k=0}^{n}\gamma\tau b_k^2\left(1+\frac{\gamma^2\tau^2}{\pi^2}\right) - 2(\kappa-\varepsilon)\sum_{k=0}^{n}\gamma\tau b_k^2\frac{\gamma^2\tau^2}{n^2\pi^2} \\
&\le \frac{\kappa\gamma\tau}{2\pi^2}\left(\frac{\pi^2+\gamma^2\tau^2}{\kappa} - \frac{4\gamma^2\tau^2}{n^2} + \varepsilon\frac{\gamma^2\tau^2}{\kappa n^2\pi^2}\right) < 0.
\end{aligned} \tag{3.39}$$

(The above equality makes use of $\kappa > n^2(\pi^2+\gamma^2\tau^2)/4\gamma\tau^2$ and $0<\varepsilon<\kappa n^2\pi^2/\gamma^2\tau^2(2\gamma^2\tau^2/n^2 - (\pi^2+\gamma^2\tau^2)/\kappa)$.)

So, $i_1(I) \ge n$. On the other hand, by (A_2'), we know $I(\theta) = 0$. By Lemma 1.5, we have that the problem (3.25) has at least $2n$ nontrivial $2\gamma\tau$-periodic solutions. □

Example 3.9. We consider periodic solutions of the following mixed-type differential equations:

$$x''(t-\tau)+4\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+2x^2(t-\tau)+x^2(t-2\tau))x(t-\tau)-x(t-\tau)=0. \tag{3.40}$$

By

$$\begin{aligned} f(t,x(t),x(t-\tau),x(t-2\tau)) &= 4\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+2x^2(t-\tau)+x^2(t-2\tau))x(t-\tau)-x(t-\tau)\\ &= 4\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+x^2(t-\tau))x(t-\tau)\\ &\quad +4\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t-\tau)+x^2(t-2\tau))x(t-\tau)-x(t-\tau), \end{aligned} \tag{3.41}$$

we have

$$g(t,x(t),x(t-\tau)) = 4\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+x^2(t-\tau))x(t-\tau), \tag{3.42}$$

that is,

$$g(t,u_1,u_2) = 4\left[1+\sin^2\frac{\pi t}{\tau}\right](u_1^2+u_2^2)u_2. \tag{3.43}$$

So, we get

$$\begin{aligned} \int_0^{x(t-\tau)} g(t,x(t),\omega)\,d\omega &= \int_0^{x(t-\tau)} 4\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+\omega)\,\omega\,d\omega\\ &= \left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+x^2(t-\tau))^2. \end{aligned} \tag{3.44}$$

It is easy to verify $g(t,u_1,u_2)$ satisfies $(A_1)\sim(A_2)$ and $(C_1)\sim(C_2)$. So, by Theorem 3.1, we know the mixed-type differential (3.40) has an infinite number nontrivial $2\gamma\tau$-periodic solution.

Example 3.10. Let us consider the system

$$x''(t-\tau)+\lambda(t)f_1(t,x(t),x(t-\tau)) = x(t-\tau), \tag{3.45}$$

where

$$\begin{aligned} f_1(t,x(t),x(t-\tau),x(t-2\tau)) &\\ = 4x(t-\tau)-4\left[1+\sin^2\frac{\pi t}{\tau}\right]&(x^2(t)+2x^2(t-\tau)+x^2(t-2\tau))x(t-\tau), \end{aligned} \tag{3.46}$$

then we get

$$g_1(t,u_1,u_2) = 2u_2 - 4\left[1+\sin^2\frac{\pi t}{\tau}\right](u_1^2+u_2^2)u_2. \tag{3.47}$$

So, we have

$$\begin{aligned}\int_0^{x(t-\tau)}&\left(2u_2 - 4\left[1+\sin^2\frac{\pi t}{\tau}\right](x(t)^2+u_2^2)u_2\right)du_2\\&= x^2(t)+x^2(t-\tau)-\left[1+\sin^2\frac{\pi t}{\tau}\right](x^2(t)+x^2(t-\tau))^2,\end{aligned} \tag{3.48}$$

$$\lim_{|u|\to 0}\frac{\int_0^{u_2} g_1(t,u_1,\omega)\,d\omega}{|u|^2} = \lim_{|u|\to 0}\frac{u_1^2+u_2^2-[1+\sin^2(\pi t/\tau)](u_1^2+u_2^2)^2}{u_1^2+u_2^2} = 1.$$

By Theorem 3.8, when (3.29) the problem (3.45) has at least $2n$ nontrivial $2\gamma\tau$-periodic solutions.

Acknowledgments

The authors would like to thank the referee for useful remarks on the preliminary version of this paper. The project is supported by NNSF of China (10471155), the Foundation of the Guangdong province, Natural Science Committee (031608), and a specific Foundation for Ph.D. Specialities of Educational Department of China (20020558092).

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