

# DOUBLE-DUAL TYPES OVER THE BANACH SPACE $C(K)$

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Let  $K$  be a compact Hausdorff space and  $C(K)$  the Banach space of all real-valued continuous functions on  $K$ , with the sup-norm. Types over  $C(K)$  (in the sense of Krivine and Maurey) can be uniquely represented by pairs  $(\ell, u)$  of bounded real-valued functions on  $K$ , where  $\ell$  is lower semicontinuous,  $u$  is upper semicontinuous,  $\ell \leq u$ , and  $\ell(x) = u(x)$  for all isolated points  $x$  of  $K$ . A condition that characterizes the pairs  $(\ell, u)$  that represent double-dual types over  $C(K)$  is given.

## 1. Statement of the main theorem

The concept of *type over a Banach space  $E$*  was first introduced by Krivine and Maurey [7] in the context of separable Banach spaces. The reader is referred to Garling's monograph [4] for more details. We consider general, not necessarily separable Banach spaces. Let  $E$  be a Banach space. For every  $x \in E$ , we define a function  $\tau_x : E \rightarrow \mathbb{R}$  by letting  $\tau_x(y) = \|x + y\|$  for all  $y \in E$ .

*Definition 1.1.* A function  $\tau : E \rightarrow \mathbb{R}$  is a *type over  $E$*  if  $\tau$  is in the closure (with respect to the topology of pointwise convergence) of the set  $\{\tau_x : x \in E\}$ .

The definition given here is equivalent to the definition given in [1]. That is,  $\tau$  is a type over  $E$  if and only if there exists an ultrafilter  $\mathcal{U}$  over an infinite index set  $\lambda$  and a bounded family of elements  $(x_\alpha)_{\alpha \in \lambda}$  in  $E$  such that  $\tau(y) = \lim_{\alpha \in \mathcal{U}} \|x_\alpha + y\|$  for all  $y \in E$ . The reader is referred to [5] for more details regarding the choice of the ultrafilter.

Throughout, we let  $K$  be a compact Hausdorff topological space. The topology on  $K$  is denoted by  $\Omega$ . We let  $\ell_\infty(K)$  denote the Banach lattice of bounded real-valued functions on  $K$  equipped with the sup-norm. For  $f, g \in \ell_\infty(K)$ , the lattice ordering is defined pointwise.

An *sc pair (semicontinuous pair)* is a pair of functions  $(\ell, u)$  from  $\ell_\infty(K)$  such that  $\ell$  is lower semicontinuous (lsc),  $u$  is upper semicontinuous (usc),  $\ell \leq u$ , and  $\ell(x) = u(x)$  for all isolated points  $x \in K$ .

The Banach space of continuous real-valued functions on  $K$  with sup-norm is denoted by  $C(K)$ . The constant function with value 1 is denoted by  $\mathbf{1}$ .

The following theorem gives a concrete representation of types over  $C(K)$  in terms of sc pairs [9, 10].

**THEOREM 1.2.** *Let  $\tau : C(K) \rightarrow \mathbb{R}$  be a function. Then the following are equivalent:*

- (i)  $\tau$  is a type over  $C(K)$ ;
- (ii) there exists an sc pair  $(\ell, u)$  such that  $\tau(g) = \max\{\|\ell + g\|, \|u + g\|\}$  for all  $g \in C(K)$ .

The correspondence between types over  $C(K)$  and sc pairs  $(\ell, u)$  is one-to-one.

The following proposition is immediate from Definition 1.1; see [9] for more equivalent conditions and a detailed proof.

**PROPOSITION 1.3.** *Let  $E$  be a Banach space and  $\tau : E \rightarrow \mathbb{R}$  a function. Then the following are equivalent:*

- (i)  $\tau$  is a type over  $E$ ;
- (ii) for every finite subset  $\alpha \subseteq E$  and every  $\varepsilon > 0$ , there exists an element  $x = x(\alpha, \varepsilon) \in E$  such that  $|\tau(y) - \|x + y\|| < \varepsilon$  for all  $y \in \alpha$ ;
- (iii) there exists a bounded net  $(x_\alpha)_{\alpha \in I}$  in  $E$  such that

$$\lim_{\alpha, I} \|x_\alpha + y\| = \tau(y) \quad (1.1)$$

for all  $y \in E$ .

If  $\tau$  is a type over  $E$  and  $(x_\alpha)_{\alpha \in I}$  is as in (iii) above, we say that  $(x_\alpha)_{\alpha \in I}$  generates the type  $\tau$ . A net  $(x_\alpha)_{\alpha \in I}$  in  $E$  doubly generates  $\tau$  if for every  $\lambda \in [0, 1]$  and every  $y \in E$ ,

$$\lim_{\beta, I} \lim_{\alpha, I} \|y + \lambda x_\alpha + (1 - \lambda)x_\beta\| = \tau(y). \quad (1.2)$$

Let  $E$  be a Banach space and let  $E''$  be its second dual. Throughout, we consider  $E$  as a subspace of  $E''$ . For every fixed  $g'' \in E''$ , define the function  $\tau_{g''} : E \rightarrow \mathbb{R}$  by letting  $\tau_{g''}(x) = \|x + g''\|$  for all  $x \in E$ . It is immediate from the principle of local reflexivity that  $\tau_{g''}$  is a type over  $E$ .

If  $\tau$  is a type over  $E$  that can be represented in this way, we call  $\tau$  a *double-dual type over  $E$* .

Maurey [8] and Rosenthal [11] have given a characterization of double-dual types over separable Banach spaces. The author [9] has generalized this characterization to not necessarily separable Banach spaces as follows.

**THEOREM 1.4.** *Let  $E$  be a Banach space and  $\tau : E \rightarrow \mathbb{R}$  a type over  $E$ . Then the following are equivalent:*

- (i)  $\tau$  is a double-dual type over  $E$ ;
- (ii) there exists a net  $(x_\alpha)_{\alpha \in I}$  in  $E$  that doubly generates  $\tau$ .

This paper is devoted to proving the following characterization of double-dual types over  $C(K)$  in terms of the representation using the sc pairs.

THEOREM 1.5. Let  $\tau$  be a type over  $C(K)$ , represented by the sc pair  $(\ell, u)$  as in Theorem 1.2. Let

$$\begin{aligned} Y_\ell &= \left\{ x \in K : x \text{ is not isolated and } \ell(x) < \liminf_{y \rightarrow x} \ell(y) \right\}, \\ Y_u &= \left\{ x \in K : x \text{ is not isolated and } u(x) > \limsup_{y \rightarrow x} u(y) \right\}. \end{aligned} \quad (1.3)$$

The following are equivalent:

- (i)  $\tau$  is a double-dual type over  $C(K)$ ;
- (ii)  $Y_\ell \cap Y_u = \emptyset$ ;
- (iii) there exists a net  $(f_\alpha)_{\alpha \in I}$  which doubly generates  $\tau$ .

The next section will include a discussion of generating nets. In Section 3, several properties of singular points of sc pairs will be proved. The main Theorem 1.5 will then be proved in Section 4.

## 2. Generating nets in $C(K)$

In this section, we introduce concepts that are needed to prove the main theorem.

We use the standard notion for convergence of nets in topological spaces according to [3, Section 1.6]. We recall the basic definitions for the convenience of the reader.

*Definition 2.1.* (i) A partially ordered set  $(I, \leq)$  is a *directed set* if for any  $\alpha, \beta \in I$  there exists  $\gamma \in I$  such that  $\gamma \geq \alpha$  and  $\gamma \geq \beta$ . Such an element  $\gamma$  is called a *successor* of  $\alpha$  (and  $\beta$ ).

(ii) Let  $(I, \leq)$  be a directed set. For every element  $\alpha_0 \in I$ , define  $|\alpha_0| = \text{card}(\{\alpha \in I : \alpha \leq \alpha_0\})$ , the number of predecessors of  $\alpha_0$ .

(iii) Let  $(I, \leq)$  and  $(J, \leq)$  be directed sets. A function  $k : I \rightarrow J$  is *order-preserving* if  $\alpha \leq \beta \in I$  implies  $k(\alpha) \leq k(\beta)$ . A function  $k : I \rightarrow J$  is *cofinal* if for every  $\gamma \in J$  there exists  $\alpha \in I$  such that  $\gamma \leq k(\alpha)$ .

(iv) Let  $(I, \leq)$  be a directed set and  $K$  a topological space. We say that  $(x_\alpha)_{\alpha \in I}$  is a *net* in  $K$  indexed by  $I$  if  $x_\alpha \in K$  for all  $\alpha \in I$ . If  $K$  is a normed space, then  $(x_\alpha)_{\alpha \in I}$  is *bounded* if  $\{\|x_\alpha\| : \alpha \in I\}$  is bounded in  $\mathbb{R}$ .

(v) Let  $(I, \leq)$  be a directed set,  $K$  a topological space, and  $(x_\alpha)_{\alpha \in I}$  a net in  $K$  indexed by  $I$ . If  $j : I \rightarrow I$  is a cofinal order-preserving function, then  $(x_{j(\alpha)})_{\alpha \in I}$  is a *subnet* of  $(x_\alpha)_{\alpha \in I}$ .

(vi) Let  $(I, \leq)$  be a directed set,  $K$  a topological space, and  $(x_\alpha)_{\alpha \in I}$  a net in  $K$  indexed by  $I$ . Let  $x \in K$ . Then  $\lim_{\alpha, I} x_\alpha = x$  if and only if for every neighborhood  $U$  of  $x$  in  $K$  there exists  $\alpha \in I$  such that  $x_\beta \in U$  for all  $\beta \geq \alpha$ .

(vii) Let  $(I, \leq)$  be a directed set and  $(r_\alpha)_{\alpha \in I}$  a bounded net of real numbers. Then define

$$\begin{aligned} \limsup_{\alpha, I} r_\alpha &= \inf_{\alpha \in I} \sup_{\alpha \in I} \{r_\beta : \beta \in I \text{ and } \beta \geq \alpha\}, \\ \liminf_{\alpha, I} r_\alpha &= \sup_{\alpha \in I} \inf_{\alpha \in I} \{r_\beta : \beta \in I \text{ and } \beta \geq \alpha\}. \end{aligned} \quad (2.1)$$

Observe that  $\limsup_{\alpha, I} r_\alpha$  and  $\liminf_{\alpha, I} r_\alpha$  exist for every bounded net  $(r_\alpha)_{\alpha \in I}$  in  $\mathbb{R}$ .

We now consider the Banach lattice  $\ell_\infty(K)$  of bounded real-valued functions on  $K$ , equipped with the sup-norm.

A subset  $H \subseteq \ell_\infty$  is called *bounded* if  $\sup\{\|f\| : f \in H\} < \infty$ . Let  $H$  be such a set. The pointwise supremum of  $H$  is the real-valued function  $L$  defined by  $L(x) = \sup\{h(x) : h \in H\}$  for every  $x \in K$ . We write  $L = \bigvee H$  for this function. Similarly, the pointwise infimum of  $H$  is the real-valued function  $U$  defined by  $U(x) = \inf\{h(x) : h \in H\}$  for every  $x \in K$ . This function is denoted by  $\bigwedge H$ . Note that both  $\bigvee H$  and  $\bigwedge H$  are again in  $\ell_\infty(K)$ .

If  $H \subseteq \ell_\infty(K)$  is a bounded set of usc functions, then the pointwise infimum  $\bigwedge H$  is usc. Similarly, the pointwise supremum of a bounded set of lsc functions is lsc. Finally, it is clear that  $f \in C(K)$  is continuous if and only if  $f$  is usc and lsc. Therefore, if  $H$  is a bounded set of continuous functions on  $K$ , then  $\bigwedge H$  is usc and  $\bigvee H$  is lsc.

Let  $\tau$  be a type over  $C(K)$  and let  $(f_\alpha)_{\alpha \in I}$  generate  $\tau$  as in Proposition 1.3(iii) above. We construct the sc pair  $(\ell, u)$  of Theorem 1.2 as follows.

For every  $\alpha \in I$ , define a lower semicontinuous functions  $\ell_\alpha$  and an upper semicontinuous function  $u_\alpha$  on  $K$  by setting

$$\begin{aligned}\ell_\alpha &= \bigvee \{f \in C(K) : f \leq f_\beta \ \forall \beta \geq \alpha\}, \\ u_\alpha &= \bigwedge \{f \in C(K) : f \geq f_\beta \ \forall \beta \geq \alpha\}.\end{aligned}\tag{2.2}$$

Then set

$$u = \bigwedge_\alpha u_\alpha, \quad \ell = \bigvee_\alpha \ell_\alpha.\tag{2.3}$$

Here are some basic properties of the functions  $\ell$  and  $u$  defined in (2.3). See [10] for details.

*Remark 2.2.* Let  $(f_\alpha)_{\alpha \in I}$  be a bounded net of functions and let  $\ell_\alpha$ ,  $\ell$ ,  $u_\alpha$ , and  $u$  be as in (2.3) above.

- (i) If  $\alpha_1, \alpha_2 \in I$  and  $\alpha_1 \leq \alpha_2$ , then  $\ell_{\alpha_1} \leq \ell_{\alpha_2} \leq \ell$  and  $u_{\alpha_1} \geq u_{\alpha_2} \geq u$ .
- (ii) If  $x \in K$  and  $\varepsilon > 0$ , then there exists an  $\alpha_0 = \alpha(x, \varepsilon) \in I$  such that for all indices  $\alpha > \alpha_0$ ,

$$\ell_\alpha(x) \geq \ell(x) - \varepsilon, \quad u_\alpha(x) \leq u(x) + \varepsilon.\tag{2.4}$$

- (iii) For every  $\beta \in I$ , every  $x \in K$ , every  $\delta > 0$ , and every neighborhood  $U$  of  $x$ , there exists  $y \in U$  and  $\gamma \geq \beta$  such that  $f_\gamma(y) \leq \ell_\beta(x) + \delta$ .
- (iv) For every  $\beta \in I$ , every  $x \in K$ , every  $\delta > 0$ , and every neighborhood  $U$  of  $x$ , there exists  $y \in U$  and  $\gamma \geq \beta$  such that  $f_\gamma(y) \geq u_\beta(x) - \delta$ .

*Proof.* (i) and (ii) are trivial. To prove (iii) let  $\beta \in I$ , let  $x \in K$ ,  $\delta > 0$ , and  $U$  a neighborhood of  $x$ . Suppose that for every  $y \in U$  and all  $\gamma \geq \beta$  we have  $f_\gamma(y) > \ell_\beta(x) + \delta$ . Then we may choose a function  $g \in C(K)$  such that  $g \leq f_\gamma$  for all  $\gamma \geq \beta$  and  $g(x) = \ell_\beta(x) + \delta$ . This would imply that  $\ell_\beta(x) = \bigvee\{f \in C(K) : f \leq f_\gamma \text{ for all } \gamma \geq \beta\} \geq g(x) = \ell_\beta(x) + \delta$ . This is a contradiction. The proof of (iv) is dual to the proof of (iii).  $\square$

Let  $(f_\alpha)_{\alpha \in I}$  be a bounded net of functions in  $C(K)$  that generates a type  $\tau$  over  $C(K)$ . Choose  $\ell$  and  $u$  as in (2.3) and assume  $x \in K$  and  $u(x) = r$ . It can be shown that for every neighborhood  $U$  of  $x$  and for every  $\varepsilon > 0$ , there exists an index  $\alpha_0$  such that for every

$\alpha \geq \alpha_0$ , there exists  $y \in U$  such that  $f_\alpha(y) > r - \varepsilon$ . If  $U$ ,  $\varepsilon$ , and  $r$  are fixed, then we define for every  $\alpha \in I$ ,

$$V_\alpha := \{y \in U : f_\alpha(y) > r - \varepsilon\}. \quad (2.5)$$

Hence, for every  $x \in K$ , every neighborhood  $U$  of  $x$ , and every  $\varepsilon > 0$ , there exists an index  $\alpha_0$  such that  $V_\alpha \neq \emptyset$  for all  $\alpha > \alpha_0$ .

The following definition introduces stronger conditions.

*Definition 2.3.* Let  $(f_\alpha)_{\alpha \in I}$  be a bounded net of functions in  $C(K)$ . Let  $\ell$  and  $u$  be as in (2.3).

- (i)  $(f_\alpha)_{\alpha \in I}$  generates  $u$  at  $x$  within  $\Omega$  if for every  $\varepsilon > 0$  and every neighborhood  $U$  of  $x$ , there exists an index  $\alpha_0$  such that for all  $\alpha > \alpha_0$ , there exists  $\beta_0$  such that  $V_\alpha \cap V_\beta \neq \emptyset$  for all  $\beta > \beta_0$ .
- (ii) The net  $(f_\alpha)_{\alpha \in I}$  generates  $u$  within  $\Omega$  if it generates  $u$  at  $x$  within  $\Omega$  for every  $x \in K$ .
- (iii)  $(f_\alpha)_{\alpha \in I}$  generates  $\ell$  at  $x$  within  $\Omega$  if  $(-f_\alpha)_{\alpha \in I}$  generates  $-\ell$  at  $x$  within  $\Omega$ .
- (iv) The net  $(f_\alpha)_{\alpha \in I}$  generates  $\ell$  within  $\Omega$  if it generates  $\ell$  at  $x$  within  $\Omega$  for every  $x \in K$ .

*PROPOSITION 2.4.* Let  $(f_\alpha)_{\alpha \in I}$  be a bounded net of functions in  $C(K)$  that generates a type  $\tau$ . Let  $u$  be as in (2.3).

- (i) If  $u$  is continuous at  $x$ , then  $(f_\alpha)_{\alpha \in I}$  generates  $u$  at  $x$  within  $\Omega$ .
- (ii) If  $\lim_{\alpha, I} f_\alpha(x) = u(x)$ , then  $(f_\alpha)_{\alpha \in I}$  generates  $u$  at  $x$  within  $\Omega$ .
- (iii) If  $(x_\beta)_{\beta \in I}$  is a net in  $K$  that converges to  $x$  and if  $\lim_{\beta, I} u(x_\beta) = u(x)$  and  $\lim_{\alpha, I} f_\alpha(x_\beta) = u(x_\beta)$  for all  $\beta$ , then  $(f_\alpha)_{\alpha \in I}$  generates  $u$  at  $x$  within  $\Omega$ .

The statement is also true if  $u$  is replaced with  $\ell$ .

*Proof.* To show (i) let  $\varepsilon > 0$  and  $U$  a neighborhood of  $x$ . We may assume that  $|u(y) - u(x)| < \varepsilon/2$  for all  $y \in U$ . By Remark 2.2(ii) there exists  $\alpha_0 \in I$  such that for all  $\alpha > \alpha_0$ ,

$$V_\alpha = \{y \in U : f_\alpha(y) > u(x) - \varepsilon\} \neq \emptyset. \quad (2.6)$$

Now fix such an  $\alpha$  and choose  $y \in V_\alpha$ . Then (using Remark 2.2(iii)) there exists  $\beta_0$  such that for every  $\beta \geq \beta_0$ ,

$$f_\beta(z_\beta) > u(y) - \frac{\varepsilon}{2} \quad (2.7)$$

for some  $z_\beta \in V_\alpha$ . Therefore,  $z_\beta \in V_\alpha \cap V_\beta$ , which shows that the net  $(f_\alpha)_{\alpha \in I}$  generates  $u$  at  $x$  within  $\Omega$ . Statement (ii) is immediate from the definition.

To show (iii) let  $U$  be a neighborhood of  $x$  and  $\varepsilon > 0$ . There exists  $\beta \in I$  such that  $x_\beta \in U$  and  $|u(x) - u(x_\beta)| < \varepsilon/2$ . Fix such a  $\beta \in I$  and choose  $\alpha_0 \in I$  such that  $|f_\alpha(x_\beta) - u(x_\beta)| < \varepsilon/2$  for all  $\alpha > \alpha_0$ . Then  $x_\beta \in V_\alpha = \{y \in U : f_\alpha(y) > u(x) - \varepsilon\}$  for all  $\alpha > \alpha_0$ ; that is,  $V_\alpha \cap V_{\alpha'} \neq \emptyset$  for all  $\alpha, \alpha' > \alpha_0$ .  $\square$

### 3. Singular points of semicontinuous pairs

Our next goal is to find necessary and sufficient conditions on  $\ell$  and  $u$  for the existence of a single net that generates both  $\ell$  and  $u$  within  $\Omega$ .

*Definition 3.1.* Let  $u$  be a usc function and  $x \in K$ . We call  $x$  a *singular* point of  $u$ , if  $x$  is not an isolated point of  $K$  and

$$u(x) > \limsup_{y \rightarrow x} u(y). \quad (3.1)$$

Similarly, we call  $x$  a *singular* point of an lsc function  $\ell$  if  $x$  is not isolated and

$$\ell(x) < \liminf_{y \rightarrow x} \ell(y). \quad (3.2)$$

We call  $x$  a *regular point* of  $u$  (resp.,  $\ell$ ) if  $x$  is not isolated and not a singular point of  $u$  (resp.,  $\ell$ ).

It is immediate from the definition that  $x$  is a singular point of  $u$  if and only if there exists an open neighborhood  $U$  of  $x$  such that

$$u(x) > \sup \{u(y) : y \in U \setminus \{x\}\}. \quad (3.3)$$

If  $U$  is such a neighborhood and  $V \subseteq U$  is another neighborhood of  $x$ , then

$$u(x) > \sup \{u(y) : y \in V \setminus \{x\}\}. \quad (3.4)$$

If  $x$  is a regular point of  $u$ , then there exists a net  $(x_\beta)_{\beta \in I}$  in  $K$  which converges to  $x$  such that  $u(x) = \lim_{\beta, I} u(x_\beta)$  and  $x_\beta \neq x$  for all  $\beta \in I$ .

**PROPOSITION 3.2.** Let  $(\ell, u)$  be an sc pair in  $\ell_\infty(K)$ . Let  $x \in K$  be a nonisolated point and  $(f_\alpha)_{\alpha \in I}$  a net which generates both  $\ell$  and  $u$  within  $\Omega$  at  $x$ .

- (i) If  $x$  is a singular point of  $u$ , then  $x$  is a regular point of  $\ell$  and  $\lim_{\alpha, I} f_\alpha(x) = u(x)$ .
- (ii) If  $x$  is a singular point of  $\ell$ , then  $x$  is a regular point of  $u$  and  $\lim_{\alpha, I} f_\alpha(x) = \ell(x)$ .

*Proof.* First we prove the following claim, which is the second statement of (i).

$$\text{If } x \text{ is a singular point of } u, \text{ then } \lim_{\alpha, I} f_\alpha(x) = u(x). \quad (3.5)$$

*Proof of the claim.* Let  $x$  be a singular point of  $u$  and suppose  $\lim_{\alpha, I} f_\alpha(x) \neq u(x)$ . Choose  $\varepsilon > 0$  and an open neighborhood  $U'$  of  $x$  such that

$$u(x) - \varepsilon > \sup \{u(y) : y \in U' \setminus \{x\}\} \quad (3.6)$$

and such that

$$\limsup_{\alpha, I} f_\alpha(x) < u(x) - 2\varepsilon. \quad (3.7)$$

There exists a further open neighborhood  $U$  of  $x$  such that  $x \in U \subseteq \overline{U} \subseteq U'$  and  $\overline{U}$  is compact. We may fix  $\alpha_0$  such that for all  $\alpha > \alpha_0$ ,

$$f_\alpha(x) < u(x) - \varepsilon, \quad V_\alpha = \left\{ y \in U : f_\alpha(y) > u(x) - \frac{\varepsilon}{3} \right\} \neq \emptyset. \quad (3.8)$$

Let  $\alpha > \alpha_0$ . Then

$$W_\alpha = \left\{ y \in U : f_\alpha(y) < u(x) - \frac{2\varepsilon}{3} \right\} \quad (3.9)$$

is an open neighborhood of  $x$  which is disjoint from  $V_\alpha$ . Since  $(f_\alpha)_{\alpha \in I}$  generates  $u$  at  $x$  within  $\Omega$ , there exists  $\beta_0$  such that for all  $\beta > \beta_0$ , we may choose

$$y_\beta \in \left\{ y \in V_\alpha : f_\beta(y) > u(x) - \frac{\varepsilon}{3} \right\}. \quad (3.10)$$

By passing to a subnet if necessary, we may assume that  $\lim_{\beta \in I} y_\beta = y$  for some  $y \in \overline{U}$ . We obtain  $u_\beta(y) \geq u(x) - \varepsilon/3$  for all  $\beta \in I$  with  $\beta \geq \beta_0$  and hence

$$u(y) \geq u(x) - \frac{\varepsilon}{3}, \quad (3.11)$$

which contradicts (3.6). So  $\lim_{\alpha \in I} f_\alpha(x) = u(x)$  and the claim is established.  $\square$

The dual statement of claim (3.5) reads as follows:

$$\text{if } x \text{ is a singular point of } \ell, \text{ then } \lim_{\alpha \in I} f_\alpha(x) = \ell(x). \quad (3.12)$$

It is proved using an argument dual to the proof of claim (3.5). This shows the second part of (ii).

To prove the first part of (i) observe that  $x$  singular for  $u$  implies  $\ell(x) < u(x)$ , and therefore  $\lim_{\alpha \in I} f_\alpha(x) = u(x) \neq \ell(x)$ . Using the contrapositive of statement (3.12) above shows that  $x$  is not a singular point of  $\ell$ ; that is,  $x$  is a regular point of  $\ell$ .

Likewise, (3.5) can be used to show that if  $x$  a singular point of  $\ell$ , then  $x$  is a regular point of  $u$ .  $\square$

Let  $(\ell, u)$  be an sc pair and  $Y_\ell$  and  $Y_u$  the sets of singular points of  $\ell$  and  $u$ , respectively. If  $(f_\alpha)_{\alpha \in I}$  is a net that generates both  $\ell$  and  $u$  within  $\Omega$ , then  $Y_\ell$  and  $Y_u$  are disjoint by Proposition 3.2. The following proposition proves the existence of such a net, provided that  $Y_\ell$  and  $Y_u$  are disjoint.

**PROPOSITION 3.3.** *Let  $K$  be a compact Hausdorff space and  $(\ell, u)$  an sc pair in  $\ell_\infty(K)$ . Consider the sets  $Y_\ell, Y_u$  of singular points of  $\ell, u$ , respectively. Suppose that  $Y_\ell \cap Y_u = \emptyset$ . Then there exists a net  $(f_\alpha)_{\alpha \in I}$  of continuous functions which generates  $\ell$  and  $u$  within  $\Omega$ .*

The proof of this proposition requires the following theorem.

**THEOREM 3.4** (Edwards [2]). *Let  $U$  be a usc function and  $L$  an lsc function on a compact Hausdorff space  $K$  such that  $U \leq L$ . Then there exists a continuous function  $F$  such that  $U \leq F \leq L$ .*

A proof of this theorem can be found in Kaplan [6, (48.5)].

*Proof of Proposition 3.3.* Let  $\mathcal{U}$  be a base for the topology  $\Omega$  such that  $\mathcal{U}$  does not contain the empty set and the only finite sets in  $\mathcal{U}$  are singletons. Let  $I = \mathfrak{P}_{<\infty}(\mathcal{U}) \setminus \{\emptyset\}$ , the set of finite subsets of  $\mathcal{U}$ , be partially ordered by inclusion.

By induction on  $|\alpha|$  construct an increasing net of integers  $(k_\alpha)_{\alpha \in I}$  and for every  $1 \leq k \leq k_\alpha$  construct functions  $g_\alpha^{(1)}, g_\alpha^{(2)}$  and  $f_\alpha \in C(K)$  and finite collections of nonempty open sets  $\mathfrak{B}_\alpha = \{V_{\alpha,1}, \dots, V_{\alpha,k_\alpha}\} \supseteq \alpha$  and elements  $z_{i,\alpha,k} \in V_{\alpha,k}$  for  $i = 1, 2$  and all  $1 \leq k \leq k_\alpha$ , such that the following conditions hold for every  $\alpha \in I$  and every  $k = 1, \dots, k_\alpha$ :

$$u(z_{1,\alpha,k}) \geq \sup \{u(y) : y \in V_{\alpha,k}\} - \frac{1}{|\alpha|}, \quad (3.13)$$

$$\ell(z_{2,\alpha,k}) \leq \inf \{\ell(y) : y \in V_{\alpha,k}\} + \frac{1}{|\alpha|}, \quad (3.14)$$

$$g_\alpha^{(1)}(z_{j,\alpha,k}) = u(z_{j,\alpha,k}) \quad \text{for } j = 1, 2, \quad (3.15)$$

$$g_\alpha^{(2)}(z_{j,\alpha,k}) = \ell(z_{j,\alpha,k}) \quad \text{for } j = 1, 2, \quad (3.16)$$

$$u \leq g_\alpha^{(1)} \leq \bigwedge_{\beta < \alpha} g_\beta^{(1)} \leq \|u\| \mathbf{1}, \quad (3.17)$$

$$\ell \geq g_\alpha^{(2)} \geq \bigvee_{\beta < \alpha} g_\beta^{(2)} \geq -\|\ell\| \mathbf{1}, \quad (3.18)$$

$$f_\alpha(z_{1,\alpha,k}) = u(z_{1,\alpha,k}), \quad f_\alpha(z_{2,\alpha,k}) = \ell(z_{2,\alpha,k}). \quad (3.19)$$

Furthermore, for every  $\beta < \alpha$  and every  $1 \leq k \leq k_\beta$ , the following nonempty open sets are required to be among the elements of  $\mathfrak{B}_\alpha$ :

$$V_{1,\beta,k}^{(\alpha)} = \left\{ y \in V_{\beta,k} : f_\beta(y) > u(z_{1,\beta,k}) - \frac{1}{|\alpha|} \right\}, \quad (3.20)$$

$$V_{2,\beta,k}^{(\alpha)} = \left\{ y \in V_{\beta,k} : f_\beta(y) < \ell(z_{2,\beta,k}) + \frac{1}{|\alpha|} \right\}. \quad (3.21)$$

We use induction on  $|\alpha|$ . If  $\alpha = \emptyset$ , let  $f_\emptyset = g_\emptyset^{(1)} = \|u\| \mathbf{1}$  and  $g_\emptyset^{(2)} = -\|\ell\| \mathbf{1}$  and set  $\mathfrak{B}_\emptyset = \emptyset$ . With this choice, conditions (3.13)–(3.21) are either trivial or vacuously true.

If  $\alpha \in I$  and  $\alpha \neq \emptyset$ , suppose as inductive hypothesis that the construction has been completed for every  $\beta \in I$  with  $\beta < \alpha$ . Let

$$\mathfrak{B}_\alpha = \left\{ V_{i,\beta,k}^{(\alpha)} : i = 1, 2; \beta < \alpha; 1 \leq k \leq k_\beta \right\} \cup \alpha \cup \bigcup_{\beta < \alpha} \mathfrak{B}_\beta, \quad (3.22)$$

where  $V_{1,\beta,k}$  and  $V_{2,\beta,k}$  are as in (3.20) for all  $1 \leq k \leq k_\alpha$ . Observe that  $\mathfrak{B}_\alpha$  is a finite collection of nonempty open sets. Say

$$\mathfrak{B}_\alpha = \{V_{\alpha,1}, \dots, V_{\alpha,k_\alpha}\}, \quad (3.23)$$

where  $(V_{\alpha,k})_{k=1}^{k_\alpha}$  are pairwise distinct. For  $i = 1, 2$  and  $1 \leq k \leq k_\alpha$ , we choose  $z_{i,\alpha,k} \in V_{\alpha,k}$  satisfying (3.13) and (3.14), and such that for all  $1 \leq k, j \leq k_\alpha$ , and  $i_1, i_2 \in \{1, 2\}$ , we have  $z_{i_1,\alpha,k} = z_{i_2,\alpha,j}$  if and only if either  $j = k$  and  $i_1 = i_2$  or  $j = k$  and  $V_{\alpha,k}$  is a singleton.

Note that such a choice is possible, since the singular points of  $\ell$  and  $u$  are disjoint and the only finite sets in  $\mathcal{U}$  are singletons.

We now construct  $g_\alpha^{(1)}$  and  $g_\alpha^{(2)}$  satisfying (3.15) through (3.18).

By inductive hypothesis in (3.17),  $u \leq \bigwedge_{\beta < \alpha} g_{\beta}^{(1)} \leq \|u\| \mathbf{1}$ . We define an lsc function  $L$  on  $K$  by setting

$$L(x) = \begin{cases} u(x) & \text{if } x = z_{j,\alpha,k} \quad \text{for some } j = 1, 2; 1 \leq k \leq k_{\alpha}, \\ \bigwedge_{\beta < \alpha} g_{\beta}^{(1)}(x) & \text{otherwise.} \end{cases} \quad (3.24)$$

Because  $u \leq L$ , we may apply Theorem 3.4 and obtain  $g_{\alpha}^{(1)} \in C(K)$  with  $u \leq g_{\alpha}^{(1)} \leq L$ . This choice of  $g_{\alpha}^{(1)}$  satisfies (3.15) and (3.17). We use a dual construction to define  $g_{\alpha}^{(2)}$  satisfying conditions (3.16) and (3.18).

In order to construct  $f_{\alpha}$  define a usc function  $U$  and an lsc function  $L$  on  $K$  by setting for every  $x \in K$ ,

$$\begin{aligned} U(x) &= \begin{cases} g_{\alpha}^{(1)}(x) & \text{if } x = z_{1,\alpha,k} \quad \text{for some } 1 \leq k \leq k(\alpha), \\ g_{\alpha}^{(2)}(x) & \text{otherwise,} \end{cases} \\ L(x) &= \begin{cases} g_{\alpha}^{(2)}(x) & \text{if } x = z_{2,\alpha,k} \quad \text{for some } 1 \leq k \leq k(\alpha), \\ g_{\alpha}^{(1)}(x) & \text{otherwise.} \end{cases} \end{aligned} \quad (3.25)$$

Observe that  $U \leq L$ ; by Theorem 3.4 there exists a continuous function  $f_{\alpha}$  with  $U \leq f_{\alpha} \leq L$ . By construction of  $U$  and  $L$  and (3.15), we have

$$U(x_{1,\alpha,k}) = L(x_{1,\alpha,k}) = g_{\alpha}^{(1)}(z_{1,\alpha,k}) = u(z_{1,\alpha,k}) \quad (3.26)$$

for all  $1 \leq k \leq k_{\alpha}$ . Hence,  $f_{\alpha}(z_{1,\alpha,k}) = u(z_{1,\alpha,k})$ . Furthermore,

$$U(x_{2,\alpha,k}) = L(x_{2,\alpha,k}) = g_{\alpha}^{(2)}(z_{2,\alpha,k}) = \ell(z_{2,\alpha,k}) \quad (3.27)$$

for all  $1 \leq k \leq k_{\alpha}$ . Thus,  $f_{\alpha}(z_{2,\alpha,k}) = \ell(z_{2,\alpha,k})$ . Condition (3.19) follows from these last two observations.

This completes the construction and we now proceed to show that the net  $(f_{\alpha})_{\alpha \in I}$  generates  $u$  and  $\ell$  within  $\Omega$ .

Fix  $x \in K$ ,  $\varepsilon > 0$ , and  $U \in \Omega$ . Choose  $n \in \mathbb{N}$  such that  $1/n < \varepsilon/2$ . Fix  $\beta \in I$  with  $|\beta| > n$ , such that for some  $V \in \beta$  we have  $x \in V \subseteq U$ . Choose  $1 \leq k \leq k_{\beta}$  such that  $V = V_{\beta,k} \in \mathfrak{B}_{\beta}$ . Applying (3.13) yields

$$u(z_{1,\beta,k}) \geq \sup \{u(y) : y \in V_{\beta,k}\} - \frac{1}{|\beta|} \geq u(x) - \frac{1}{|\beta|}. \quad (3.28)$$

So by (3.19),  $f_{\beta}(z_{1,\beta,k}) = u(z_{1,\beta,k})$ . Now let  $\alpha > \beta$ . By (3.20) there exists  $1 \leq j \leq k_{\alpha}$  such that

$$V_{\alpha,j} = V_{1,\beta,k}^{(\alpha)} = \left\{ y \in V_{\beta,k} : f_{\beta}(y) > u(z_{1,\beta,k}) - \frac{1}{|\alpha|} \right\}. \quad (3.29)$$

In particular,

$$z_{1,\beta,k} \in V_{\alpha,j}. \quad (3.30)$$

Observe that by (3.13)

$$u(z_{1,\alpha,j}) \geq \sup \{u(y) : y \in V_{\alpha,j}\} - \frac{1}{|\alpha|} \quad (3.31)$$

and  $z_{1,\alpha,j} \in V_{\alpha,j}$ . Thus,

$$\begin{aligned} f_\beta(z_{1,\alpha,j}) &> u(z_{1,\beta,k}) - \frac{1}{|\alpha|} \quad \text{by (3.29)} \\ &\geq u(x) - \frac{1}{|\beta|} - \frac{1}{|\alpha|} \quad \text{by (3.28)} \\ &> u(x) - \varepsilon. \end{aligned} \quad (3.32)$$

On the other hand,

$$\begin{aligned} f_\alpha(z_{1,\alpha,j}) &= u(z_{1,\alpha,j}) \quad \text{by (3.19)} \\ &\geq \sup \{u(y) : y \in V_{\alpha,j}\} - \frac{1}{|\alpha|} \quad \text{by (3.31)} \\ &\geq u(z_{1,\beta,k}) - \frac{1}{|\alpha|} \quad \text{by (3.30)} \\ &\geq u(x) - \frac{1}{|\beta|} - \frac{1}{|\alpha|} \quad \text{by (3.28)} \\ &> u(x) - \varepsilon. \end{aligned} \quad (3.33)$$

Therefore,

$$z_{1,\alpha,j} \in \{y \in U : f_\alpha(y) > u(x) - \varepsilon\} \cap \{y \in U : f_\beta(y) > u(x) - \varepsilon\} \neq \emptyset \quad (3.34)$$

for all  $\alpha \geq \beta$ . This shows that the net  $(f_\alpha)_{\alpha \in I}$  generates  $u$  within  $\Omega$ . The proof that it generates  $\ell$  within  $\Omega$  follows from a similar argument.  $\square$

Let  $\tau$  be a type over  $C(K)$  that is represented by the sc pair  $(\ell, u)$  as in Theorem 1.2. Propositions 3.2 and 3.3 prove that the sets  $Y_\ell$  and  $Y_u$  of singular points of  $\ell$  and  $u$  are disjoint if and only if there exists a net  $(f_\alpha)_{\alpha \in I}$  that generates both  $\ell$  and  $u$  within  $\Omega$ .

#### 4. Proof of the main theorem

We now consider a net  $(f_\alpha)_{\alpha \in I}$  that generates a type  $\tau$  over  $C(K)$ . As before, let this type be represented by the sc pair  $(\ell, u)$ .

To establish the main theorem, we will now prove that the net doubly generates the type  $\tau$  if and only if the net generates both  $\ell$  and  $u$  within  $\Omega$ . This is accomplished in the following two lemmas.

**LEMMA 4.1.** *Let  $K$  be a compact Hausdorff space and  $\tau$  a type over  $C(K)$ . Let  $(f_\alpha)_{\alpha \in I}$  be a net that doubly generates  $\tau$ . Let  $(\ell, u)$  be the sc pair such that  $\tau(g) = \max\{\|\ell + g\|, \|u + g\|\}$  for all  $g \in C(K)$ . Then  $(f_\alpha)_{\alpha \in I}$  generates  $\ell$  and  $u$  within  $\Omega$ .*

*Proof.* Assume the conclusion does not hold. Then either  $(f_\alpha)_{\alpha \in I}$  does not generate  $u$  within  $\Omega$  at some  $x \in K$ , or it does not generate  $\ell$  within  $\Omega$  at some  $x \in K$ . We distinguish between these two cases.

*Case 1.*  $(f_\alpha)_{\alpha \in I}$  does not generate  $u$  at  $x$  within  $\Omega$ . Let  $\lambda = 1/2$ . There exists an open neighborhood  $U$  of  $x$  and  $\varepsilon > 0$  such that for all  $\alpha_0 \in I$  and  $\beta_0 \in I$  there exist  $\alpha > \alpha_0$  and  $\beta > \beta_0$ , for which

$$\{y \in U : f_\alpha(y) > u(x) - \varepsilon\} \cap \{y \in U : f_\beta(y) > u(x) - \varepsilon\} = \emptyset. \quad (4.1)$$

Let  $U_0 = \{y \in U : u(y) < u(x) + \varepsilon/2\}$  and choose an open neighborhood  $U_1$  of  $x$  such that  $U_1 \subseteq \overline{U}_1 \subseteq U_0 \subseteq U$ . We claim that there exists  $\alpha_0 \in I$  such that  $\|f_\alpha\| \leq \|\tau\| + \varepsilon/2$  and  $f_\alpha|_{\overline{U}_1} \leq u(x) + \varepsilon/2$  for all  $\alpha \geq \alpha_0$ . (Here,  $\|\tau\| = \tau(0)$ .)

First observe that there exists  $\alpha_1$  such that for all  $\alpha \geq \alpha_1$  we have  $\|f_\alpha\| \leq \|\tau\| + \varepsilon/2$ . Suppose there does not exist  $\alpha_0 \geq \alpha_1$  such that  $f_\alpha|_{\overline{U}_1} \leq u(x) + \varepsilon/2$  for all  $\alpha \geq \alpha_0$ . Then there exist a cofinal order-preserving map  $i : I \rightarrow I$  such that  $f_{i(\alpha)}(y_{i(\alpha)}) > u(x) + \varepsilon/2$ , where  $y_{i(\alpha)} \in \overline{U}_1$  for all  $\alpha \in I$ . We may assume that  $(y_{i(\alpha)})_{\alpha \in I}$  converges to  $y_0 \in \overline{U}_1$ . Thus,  $u(y_0) \geq u(x) + \varepsilon/2$ , which contradicts the choice of  $U_0$  and establishes the claim.

Fix a function  $g \in C(K)$  such that  $g|_{K \setminus U_1} = 0$  and  $g(x) = 3\|\tau\|$  and  $0 \leq g \leq 3\|\tau\|$ . Observe that  $\|u + g\| \geq g(x) + u(x) = 3\|\tau\| + u(x)$ .

Further, for each  $\alpha \geq \alpha_0$ , there exists  $\alpha_2 \geq \alpha$  and a cofinal order-preserving function  $j = j_{\alpha_2} : I \rightarrow I$  such that

$$\left\{ y \in U_1 : \frac{1}{2}f_{\alpha_2}(y) > \frac{1}{2}u(x) - \frac{1}{2}\varepsilon \right\} \cap \left\{ y \in U_1 : \frac{1}{2}f_{j(\beta)}(y) > \frac{1}{2}u(x) - \frac{1}{2}\varepsilon \right\} = \emptyset. \quad (4.2)$$

Fix such  $\alpha_2$  and  $j = j_{\alpha_2}$ . If  $y \in \overline{U}_1$ ,

$$g(y) + \frac{1}{2}f_{\alpha_2}(y) + \frac{1}{2}f_{j(\beta)}(y) \leq 3\|\tau\| + u(x) - \frac{\varepsilon}{4} \quad (4.3)$$

for all  $\beta \in I$ . If  $y \in K \setminus U_1$ , we have

$$g(y) + \frac{1}{2}f_{\alpha_2}(y) + \frac{1}{2}f_{j(\beta)}(y) \leq \|\tau\| + \frac{\varepsilon}{2}. \quad (4.4)$$

Observe that  $\lim_{\beta, I} \|g + 1/2f_{\alpha_2} + 1/2f_\beta\|$  exists. Thus,

$$\lim_{\beta, I} \left\| g + \frac{1}{2}f_{\alpha_2} + \frac{1}{2}f_\beta \right\| = \lim_{\beta, I} \left\| g + \frac{1}{2}f_{\alpha_2} + \frac{1}{2}f_{j(\beta)} \right\| \leq 3\|\tau\| + u(x) - \frac{\varepsilon}{4}. \quad (4.5)$$

Hence,

$$\liminf_{\alpha, I} \lim_{\beta, I} \left\| g + \frac{1}{2}f_\alpha + \frac{1}{2}f_\beta \right\| \leq 3\|\tau\| + u(x) - \frac{\varepsilon}{4} < \lim_{\alpha, I} \|g + f_\alpha\|. \quad (4.6)$$

This contradicts the assumption that  $(f_\alpha)_{\alpha \in I}$  doubly generates  $\tau$ .

*Case 2.*  $(f_\alpha)_{\alpha \in I}$  does not generate  $\ell$  at  $x$  within  $\Omega$ . This case is handled with an argument dual to the one in Case 1.  $\square$

LEMMA 4.2. Let  $K$  be a compact Hausdorff space and  $\tau$  a type over  $C(K)$ . Let  $(\ell, u)$  be the sc pair such that  $\tau(g) = \max\{\|\ell + g\|, \|u + g\|\}$  for all  $g \in C(K)$ . Assume that  $(f_\alpha)_{\alpha \in I}$  generates  $\ell$  and  $u$  within  $\Omega$ . Then  $(f_\alpha)_{\alpha \in I}$  doubly generates  $\tau$ .

*Proof.* Fix  $g \in C(K)$ . Because  $\tau(g) = \max\{\|\ell + g\|, \|u + g\|\}$ , we distinguish between two cases.

*Case 1.* Suppose that  $\tau(g) = \|u + g\|$ . Choose  $x \in K$  such that  $\|u + g\| = u(x) + g(x)$ . Let  $\varepsilon > 0$  and choose a neighborhood  $U$  of  $x$  such that  $|g(y) - g(x)| < \varepsilon/2$  for all  $y \in U$ . Choose  $\alpha_0 \in I$  such that for all  $\alpha > \alpha_0$ , there exists  $\beta_0 \in I$  such that for all  $\beta > \beta_0$ , we have  $f_\alpha(z) > u(x) - \varepsilon/2$  and  $f_\beta(z) > u(x) - \varepsilon/2$  for some  $z \in U$ . Then

$$\|g + \lambda f_\alpha + (1 - \lambda) f_\beta\| \geq |g(z) + \lambda f_\alpha(z) + (1 - \lambda) f_\beta(z)| > u(x) + g(x) - \varepsilon = \|u + g\| - \varepsilon. \quad (4.7)$$

Therefore,

$$\liminf_{\alpha, I} \lim_{\beta, I} \|g + \lambda f_\beta + (1 - \lambda) f_\alpha\| \geq \|u + g\| - \varepsilon. \quad (4.8)$$

On the other hand,

$$\begin{aligned} & \limsup_{\alpha, I} \lim_{\beta, I} \|g + \lambda f_\alpha + (1 - \lambda) f_\beta\| \\ & \leq \limsup_{\alpha, I} \lambda \|g + f_\alpha\| + \lim_{\beta, I} (1 - \lambda) \|f_\beta + g\| \\ & \leq \|u + g\|. \end{aligned} \quad (4.9)$$

Because  $\varepsilon$  was arbitrary, this shows that

$$\lim_{\alpha, I} \lim_{\beta, I} \|g + \lambda f_\alpha + (1 - \lambda) f_\beta\| \quad (4.10)$$

exists and equals  $\tau(g)$ .

*Case 2.* If  $\tau(g) = \|\ell + g\|$ , consider the net  $(-f_\alpha)_{\alpha \in I}$ , which generates  $-u$  and  $-\ell$  within  $\Omega$  and the function  $-g \in C(K)$ . We infer from Case 1 that

$$\lim_{\beta, I} \lim_{\alpha, I} \|g + \lambda f_\alpha + (1 - \lambda) f_\beta\| = \lim_{\beta, I} \lim_{\alpha, I} \| -g + \lambda(-f_\alpha) + (1 - \lambda)(-f_\beta)\| = \|\ell + g\|. \quad (4.11)$$

Therefore,  $\lim_{\beta, I} \lim_{\alpha, I} \|g + \lambda f_\alpha + (1 - \lambda) f_\beta\| = \tau(g)$  for all  $g \in C(K)$ .  $\square$

*Proof of Theorem 1.5.* The equivalence between (i) and (iii) is Theorem 1.4 above. The implication (ii)  $\Rightarrow$  (iii) follows from Proposition 3.2 and Lemma 4.1 and (iii)  $\Rightarrow$  (ii) follows from Proposition 3.3 and Lemma 4.2.  $\square$

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