

# ON WEAK CONVERGENCE OF ITERATES IN QUANTUM $L_p$ -SPACES ( $p \geq 1$ )

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Equivalent conditions are obtained for weak convergence of iterates of positive contractions in the  $L_1$ -spaces for general von Neumann algebra and general JBW algebras, as well as for Segal-Dixmier  $L_p$ -spaces ( $1 \leq p < \infty$ ) affiliated to semifinite von Neumann algebras and semifinite JBW algebras without direct summands of type  $I_2$ .

## 1. Introduction and preliminaries

This paper is devoted to a presentation of some results concerning ergodic-type properties of weak convergence of iterates of operators acting in  $L_1$ -space for general von Neumann algebras and JBW algebras, as well as Segal-Dixmier  $L_p$ -spaces ( $1 \leq p < \infty$ ) of operators affiliated with semifinite von Neumann algebras and semifinite JBW algebras.

The first results in the field of noncommutative ergodic theory were obtained independently by Sinaĭ and Ansĕlevič [21] and Lance [15]. Developments of the subject are reflected in the monographs of Jajte [13] and Krengel [14] (see also [8, 9, 10, 18]).

We will use facts and the terminology from the general theory of von Neumann algebras (see [5, 7, 17, 19, 22]), the general theory of Jordan and real operator algebras (see [2, 3, 11, 16]), and the theory of noncommutative integration (see [20, 23, 24]).

Let  $M$  be a von Neumann algebra, acting on a separable Hilbert space  $H$ ,  $M_*$  is a predual space of  $M$ , which always exists according to the Sakai theorem [19]. It is well known that  $M_*$  could be identified with  $L_1$ -space for  $M$ .

Spaces  $L_1$  and  $L_2$  of the operators affiliated with the semifinite von Neumann algebra  $M$  with semifinite faithful trace  $\tau$  were introduced by Segal (see [20]). This result was extended to  $L_p$ -space of operators affiliated with von Neumann algebras  $M$ ,  $\tau$ , and integrated with  $p$ th power by Dixmier (see [6]). For an alternative exposition of building  $L_p$  based on Grothendieck's idea of using rearrangements of functions, see also [24]. The theory of  $L_p$ -spaces was extended further to the von Neumann algebras with faithful normal weight  $\rho$ . However, these spaces lack some of the properties, for example, in general, these spaces do not intersect.

Recall some standard terminology (see [8, 9, 10, 14]).

**Definition 1.1.** A linear mapping  $T$  from  $M_*$  in itself is called a *contraction* if its norm is not greater than one.

**Definition 1.2.** A contraction  $T$  is said to be *positive* if

$$TM_{*+} \subset M_{*+}. \quad (1.1)$$

We will consider the two topologies on the space  $M_*$ : the *weak topology*, or the  $\sigma(M_*, M)$  topology, and the *strong topology* of the  $M_*$ -space norm convergence.

**Definition 1.3.** A matrix  $(a_{n,i})$ ,  $i, n = 1, 2, \dots$ , of real numbers is called *uniformly regular* if

$$\sup_n \sum_{i=1}^{\infty} |a_{n,i}| \leq C < \infty; \quad \lim_{n \rightarrow \infty} \sup_i |a_{n,i}| = 0; \quad \lim_{n \rightarrow \infty} \sum_i a_{n,i} = 1. \quad (1.2)$$

## 2. Main result: the case of quantum $L_1$ -spaces

**2.1. The case of noncommutative  $L_1$ -spaces.** The following theorem is valid.

**THEOREM 2.1.** *The following conditions for a positive contraction  $T$  in the predual space of a complex von Neumann algebras  $M$  are equivalent.*

- (i) *The sequence  $\{T^i\}_{i=1,2,\dots}$  converges weakly.*
- (ii) *For each strictly increasing sequence of natural numbers  $\{k_i\}_{i=1,2,\dots}$ ,*

$$n^{-1} \sum_{i < n} T^{k_i} \quad (2.1)$$

*converges strongly.*

- (iii) *For any uniformly regular matrix  $(a_{n,i})$ , the sequence  $\{A_n(T)\}_{n=1,2,\dots}$ ,*

$$A_n(T) = \sum_i a_{n,i} T^i, \quad (2.2)$$

*converges strongly.*

**Proof of Theorem 2.1.** We first prove the following lemma.

**LEMMA 2.2.** *Let there exist a uniformly regular matrix  $(a_{n,i})$  such that for each strictly increasing sequence  $\{k_i\}_{i=1,2,\dots}$  of natural numbers,*

$$B_n = \sum_i a_{n,i} T^{k_i} \quad (2.3)$$

*converges strongly. Then the sequence  $\{T^i\}_{i=1,2,\dots}$  converges weakly.*

**Proof.** Let  $(a_{n,i})$  be a matrix with the aforementioned properties. Then the limit  $B_n$  is not dependant upon the choice of the sequence  $\{k_i\}_{i=1,2,\dots}$ . In fact, let  $\{k_i\}_{i=1,2,\dots}$  and  $\{l_i\}_{i=1,2,\dots}$  be the sequences for which the limits  $B_n$  are different. This means that for some  $x \in M_*$ ,

$$\sum_i a_{n,i} T^{k_i} x \rightarrow x_1, \quad \sum_i a_{n,i} T^{l_i} x \rightarrow x_2, \quad (2.4)$$

for  $n \rightarrow \infty$ . For a matrix  $(a_{n,i})$ , we build increasing sequences  $\{i_j\}_{j=1,2,\dots}$  and  $\{n_j\}_{j=1,2,\dots}$ , such that

$$\lim_{j \rightarrow \infty} \left( \sum_{i < i_{j-1}} |a_{n_j,i}| + \sum_{i > i_j} |a_{n_j,i}| \right) = 0. \quad (2.5)$$

Let

$$m_i = k_i \quad \text{for } i \in [i_{2j-1}, i_{2j}), \quad m_i = l_i \quad \text{for } i \in [i_{2j}, i_{2j+1}), \quad j = 1, 2, \dots \quad (2.6)$$

Then

$$\lim_j \left\| \sum_i a_{n_{2j+1},i} T^{m_i} x - x_1 \right\| = 0; \quad \lim_j \left\| \sum_i a_{n_{2j},i} T^{m_i} x - x_2 \right\| = 0, \quad (2.7)$$

which contradict (2.3), and therefore  $x_1 = x_2$ . Let now  $y \in M$  such that

$$(T^n x - x_1, y) \rightarrow 0, \quad (2.8)$$

when  $n \rightarrow \infty$ . We choose a subsequence  $\{k_i\}$  such that

$$(T^{k_i} x - x_1, y) \rightarrow \gamma \neq 0, \quad (2.9)$$

where  $\gamma$  is a real number. Then, from the uniform regularity of the matrix  $(a_{n,i})$ , it follows that

$$\lim_n \left( \sum_i a_{n,i} T^{k_i} x - x_1, y \right) = \gamma, \quad (2.10)$$

which contradicts the choice of the matrix  $(a_{n,i})$ . □

The implication (iii)  $\Rightarrow$  (ii) is trivial because the matrix  $(a_{n,i})$ ,

$$a_{n,i} = \frac{1}{n} \sum_{i < n} \delta_{j,k_i}, \quad (2.11)$$

is uniformly regular. Applying Lemma 2.2 to the matrix

$$a_{n,i} = \frac{1}{n}, \quad (2.12)$$

$i \leq n$  and  $a_{n,i} = 0$  for  $i > n$ , we get the implication (ii)  $\Rightarrow$  (i).

To prove the implication (i)  $\Rightarrow$  (iii), we would need the following lemma.

**LEMMA 2.3.** *Let  $Q$  be a contraction in the Hilbert space  $H$ . Then the weak convergence of  $Q^n x$  in  $H$ , where  $x \in H$ , implies the strong convergence of*

$$\sum_i a_{n,i} Q^i x \quad (2.13)$$

for any uniformly regular matrix  $(a_{n,i})$ .

*Proof.* If the weak limit  $Q^n x$  exists and is equal to  $x_1$ , then

$$Qx_1 = Q\left(\lim_{n \rightarrow \infty} Q^n x\right) = x_1, \quad (2.14)$$

where the limit is considered in the weak topology, that is,  $x_1$  is  $Q$ -invariant. Replacing  $x$  on  $x - x_1$  (if necessary), we may suppose that  $Q^n x$  converges weakly to  $\mathbf{0}$ , and hence

$$(Q^n x, x) \longrightarrow 0. \quad (2.15)$$

We are going to show that

$$\sum_n a_{i,n} Q^n x \xrightarrow{\|\cdot\|} \mathbf{0}, \quad (2.16)$$

where  $(a_{i,n})$  is uniformly regular matrix. One can see that

$$\left\| \sum_i a_{N,i} Q^i x \right\|^2 \leq \sum_i \sum_j a_{N,i} a_{N,j} (Q^i x, Q^j x) \leq \sum_i \sum_j |a_{N,i} a_{N,j} (Q^i x, Q^j x)|. \quad (2.17)$$

We fix  $\varepsilon > 0$ . Because  $Q$  is a contraction, the limit  $\|Q^n x\|$  does exist. Now, we can find  $K > 0$ , such that for  $k > K$  and  $j \geq 0$ ,

$$\|Q^k x\| - \|Q^{k+j} x\| \leq \varepsilon^2, \quad |(Q^k x, x)| \leq \varepsilon. \quad (2.18)$$

Then,

$$\begin{aligned} & |(Q^k x, x) - (Q^{k+j} x, Q^j x)| \\ &= |(Q^k x, x) - (Q^{*j} Q^{k+j} x, x)| \\ &\leq \|Q^k x - Q^{*j} Q^{k+j} x\| \cdot \|x\| = (\|Q^k x - Q^{*j} Q^{k+j} x\|^2)^{1/2} \cdot \|x\| \\ &= (\|Q^k x\|^2 - 2\|Q^{k+j} x\|^2 + \|Q^{*j} Q^{k+j} x\|^2)^{1/2} \cdot \|x\| \\ &\leq (\|Q^k x\|^2 - \|Q^{k+j} x\|^2) \cdot \|x\| \leq \varepsilon \cdot \|x\|, \end{aligned} \quad (2.19)$$

and therefore

$$|(Q^{k+j} x, Q^j x)| \leq \varepsilon \cdot (1 + \|x\|) \quad (2.20)$$

for all  $k > K$  and  $j \geq 0$ , or for  $|i - j| \geq k$ , the inequality

$$|(Q^i x, Q^j x)| \leq \varepsilon \cdot (1 + \|x\|) \quad (2.21)$$

is valid. We will fix  $\eta > 0$ , and let  $N$  be a natural number such that

$$\max_i |a_{n,i}| < \eta, \quad (2.22)$$

for  $n \geq N$ . Then the expression (1) for  $n \geq N$  could be estimated in the following way:

$$\begin{aligned}
 & \sum_i \sum_j |a_{N,i} a_{N,j} (Q^i x, Q^j x)| \\
 &= \sum_{|i-j| \leq k} |a_{n,i} a_{n,j} (Q^i x, Q^j x)| + \sum_{|i-j| > k} |a_{n,i} a_{n,j} (Q^i x, Q^j x)| \\
 &\leq \sum_i |a_{n,i}| \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + \sum_i \sum_j |a_{n,i} a_{n,j}| \cdot \varepsilon \cdot (1 + \|x\|) \\
 &\leq C \cdot \eta \cdot \|x\|^2 \cdot (2k-1) + C^2 \cdot \varepsilon \cdot (1 + \|x\|).
 \end{aligned} \tag{2.23}$$

From the arbitrariness of the values of  $\varepsilon$  and  $\eta$ , it follows that the strong convergence is present and the lemma is proven.  $\square$

We prove the implication (i)  $\Rightarrow$  (iii). Let  $x \in M_{*+}$  and the sequence  $\{T^i x\}_{i=1,2,\dots}$  converges weakly. Without the loss of generality, we can consider  $\|x\| \leq 1$ , and let

$$\bar{x} = \lim_{n \rightarrow \infty} T^n x, \tag{2.24}$$

where the limit is understood in the weak sense. We consider

$$y = \sum_{n=0}^{\infty} 2^{-n} T^n x. \tag{2.25}$$

The series that defines  $y$  is convergent in the norm of the space  $M_*$ . From the positivity of  $x$  and the properties of the operator  $T$ , it follows that

$$Ty \leq 2y, \tag{2.26}$$

and, therefore, for all  $k = 1, 2, \dots$ ,

$$s(T^k y) \leq s(y), \tag{2.27}$$

where we denote by  $s(z)$  the support of the normal functional  $z$ .

LEMMA 2.4. Let  $u \in M_{*+}$  and  $s(u) \leq s(y)$ . Then  $s(\bar{u}) \leq s(\bar{x})$ , where

$$\bar{u} = \lim_{n \rightarrow \infty} T^n u. \tag{2.28}$$

*Proof.* In fact, We fix  $\varepsilon > 0$ . From the density of the set

$$\mathfrak{L}_y = \{w \in M_{*+}, w \leq \lambda y, \text{ for some } \lambda > 0\} \tag{2.29}$$

in the set

$$\mathfrak{S} = \{w \in M_{*+}, s(w) \leq s(y)\} \tag{2.30}$$

in the norm of the space  $M_*$ , it follows that there are  $\lambda > 0$  and  $w \in \mathfrak{L}_y$  such that

$$\|w - u\| \leq \varepsilon, \quad w \leq \lambda y. \tag{2.31}$$

Let

$$\bar{w} = \lim_{n \rightarrow \infty} T^n w. \quad (2.32)$$

Then

$$\begin{aligned} \bar{w}(1-s(\bar{x})) &= \lim_{n \rightarrow \infty} (T^n(w))(1-s(\bar{x})) \\ &\leq \lambda \cdot \lim_{n \rightarrow \infty} (T^n y)(1-s(\bar{x})) \\ &\leq \lambda \cdot \lim_{n \rightarrow \infty} \left( \sum_{k=0}^{\infty} 2^{-k} \cdot (T^{n+k} x)(1-s(\bar{x})) \right) \\ &= \lambda \cdot \sum_{k=0}^{\infty} 2^{-k} \lim_{n \rightarrow \infty} (T^{n+k} x)(1-s(\bar{x})) = 0. \end{aligned} \quad (2.33)$$

Because the operator  $T$  does not increase the norm of the functionals from  $M_*$ , we get that

$$\bar{u}(1-s(\bar{x})) = \lim_{n \rightarrow \infty} (T^n u)(1-s(\bar{x})) \leq \lim_{n \rightarrow \infty} (T^n w)(1-s(\bar{x})) + \lim_{n \rightarrow \infty} \|T^n(w - u)\| \leq \varepsilon. \quad (2.34)$$

The needed inequality follows from the arbitrariness of  $\varepsilon$ .  $\square$

We introduce the following notion. For  $\mu \in M_*$ , we will denote by  $\mu \cdot E$ , where  $E$  is a projection from the algebra  $M$ , the functional

$$(\mu \cdot E)(A) = \mu(EAE), \quad (2.35)$$

where  $A \in M$ .

We fix  $\varepsilon > 0$ . We will find a number  $N$ , such that

$$(T^n x)(1-s(\bar{x})) < \varepsilon^2 \quad (2.36)$$

for  $n > N$ .

Then,

$$\begin{aligned} &\|T^N x \cdot s(\bar{x}) - T^N x\| \\ &= \sup_{\substack{A \in M \\ \|A\|_\infty \leq 1}} |(T^N x)((1-s(\bar{x}))A(1-s(\bar{x}))) \\ &\quad + (T^N x)((s(\bar{x}))A(1-s(\bar{x}))) + (T^N x)((1-s(\bar{x}))A(s(\bar{x})))| \\ &\leq \varepsilon \cdot (\varepsilon + 2\|x\|^{1/2}), \end{aligned} \quad (2.37)$$

because

$$|\mu(AB)|^2 \leq \mu(A^*A) \cdot \mu(B^*B), \quad (2.38)$$

where  $\mu \in M_{*+}$  and  $A, B \in M$ .

Let  $w \in \mathfrak{L}_{\overline{y}}$  be such that

$$w \leq \lambda \overline{x} \quad (2.39)$$

for some  $\lambda > 0$  and

$$\|T^N x \cdot s(\overline{x}) - w\| \leq \varepsilon. \quad (2.40)$$

Then, for  $n > N$ , the following is valid:

$$\|T^n x - T^{n-N} w\| \leq \|T^{n-N}(T^N x - T^N x \cdot s(\overline{x}))\| + \|T^{n-N}(T^N x \cdot s(\overline{x}) - w)\| \leq 4 \cdot \varepsilon. \quad (2.41)$$

By taking the weak limit in the inequality (2.37) and because the unit ball of  $M_*$  is closed weakly, we will get

$$\|\overline{x} - \overline{w}\| \leq 4 \cdot \varepsilon, \quad (2.42)$$

where

$$\overline{w} = \lim_{n \rightarrow \infty} T^n w. \quad (2.43)$$

We now consider the algebra  $M_{s(x)}$ . The functional  $\overline{x}$  is faithful on the algebra  $M_{s(x)}$ . We will consider the representation  $\pi_{\overline{x}}$  of the algebra  $M_{s(x)}$  constructed using the functional  $x$  [7]. Because the functional  $\overline{x}$  is faithful, we can conclude that the representation  $\pi_{\overline{x}}$  is faithful on the algebra  $M_{s(\overline{x})}$ , and therefore  $\pi_{\overline{x}}$  is an isomorphism of the algebra  $M_{s(\overline{x})}$  and some algebra  $\mathfrak{A}$ . The algebra  $\mathfrak{A}$  is a von Neumann algebra, and its preconjugate space  $\mathfrak{A}_*$  is isomorphic to the space  $M_* \cdot s(\overline{x})$  ([19]). We note now that

$$TM_* \cdot s(\overline{x}) \subset M_* \cdot s(\overline{x}). \quad (2.44)$$

In fact,

$$T\mathfrak{L}_y \subset \mathfrak{L}_y, \quad (2.45)$$

and therefore, by taking the norm closure, we get

$$TS \subset S; \quad (2.46)$$

by taking now the linear span, we get

$$TM_* \cdot s(\overline{x}) \subset M_* \cdot s(\overline{x}). \quad (2.47)$$

We denote by  $\overline{T}$  the isomorphic image of the operator  $T$ , acting on the space  $\mathfrak{A}_*$ . Let

$$u \in \mathfrak{A}_{*+}, \quad u \leq \lambda \overline{x}, \quad (2.48)$$

for some  $\lambda > 0$ . Then there exists the operator  $B \in \mathfrak{A}'$ , where  $\mathfrak{A}'$  is a commutant of  $\mathfrak{A}$ , such that

$$(AB\Omega, \Omega) = u(A) \quad (2.49)$$

for all  $A \in \mathfrak{A}$ . Note, that from Lemma 2.3,

$$(\overline{T}u)(A) = u((\overline{T})^* A) = ((\overline{T})^* A)B\Omega, \Omega = (A((\overline{T}^*)' B)\Omega, \Omega). \quad (2.50)$$

Also, from

$$\overline{T}\mathfrak{A}_{*+} \subset \mathfrak{A}_{*+}, \quad \|\overline{T}u\| \leq \|u\|, \quad \overline{T}\bar{x} = \bar{x}, \quad (2.51)$$

it follows that

$$(\overline{T})^* \mathfrak{A}_+; \quad (\overline{T}^*) \mathbf{1} \leq \mathbf{1}, \quad \|(\overline{T})^* A\|_\infty \leq \|A\|_\infty, \quad (2.52)$$

for all  $A \in \mathfrak{A}$ . Based on the lemma, we now conclude that

$$\|(\overline{T}^* B)\|_\infty \leq \|B\|_\infty; \quad \overline{T}^{*'} \mathfrak{A}'_+ \subset \mathfrak{A}'_+; \quad \overline{T}^{*'} \mathbf{1} \leq \mathbf{1}, \quad (2.53)$$

for all  $B \in \mathfrak{A}'$ .

The space  $\mathfrak{A}'_{\text{sa}}$  is a pre-Hilbert space of the selfadjoint operators from  $\mathfrak{A}'$  with the scalar product

$$(B, C)_{\bar{x}} = (CB\Omega, \Omega), \quad (2.54)$$

and using the Kadison inequality [5], we have

$$((\overline{T}^{*'} B)(\overline{T}^{*'} B)\Omega, \Omega) \leq (\overline{T}^{*'} (B^2)\Omega, \Omega) \leq (B\Omega, B\Omega), \quad (2.55)$$

that is, the operator  $\overline{T}^{*'}$  is a contraction in the pre-Hilbert space  $(\mathfrak{A}'_{\text{sa}}, (\cdot, \cdot)_{\bar{x}})$ .

We will identify  $M_* \cdot s(\bar{x})$  and  $\mathfrak{A}_*$ . Because  $w \in \mathfrak{L}$ , that is,

$$w \leq \lambda \bar{x} \quad (2.56)$$

for some  $\lambda > 0$ , then

$$\overline{w} \leq \lambda \bar{x} \quad (2.57)$$

as well. Let

$$w(A) = (BA\Omega, \Omega), \quad \overline{w}(A) = (\overline{B}A\Omega, \Omega), \quad (2.58)$$

for all  $A \in \mathfrak{A}$ , where  $B, \overline{B} \in \mathfrak{A}'$ .



Let now  $(a_{n,i})$  be a uniformly regular matrix. Using Lemma 2.3, we will find  $k \in \mathbb{N}$  so that

$$\begin{aligned}
 & \left\| \sum_i a'_{k,i} T^i w - \bar{w} \right\| \\
 &= \sup_{\substack{A \in \mathfrak{A} \\ \|A\|_\infty = 1}} \left| \left( \sum_{i=1}^\infty a'_{k,i} (\bar{T}^{*'})^i (B - \bar{B}) A \Omega, \Omega \right) \right| \\
 &\leq \left( \sum_{i=1}^\infty a'_{k,i} (\bar{T}^{*'})^i (B - \bar{B}) \Omega, \sum_{i=1}^\infty a'_{k,i} (\bar{T}^{*'})^i (B - \bar{B}) \Omega \right)^{1/2} \cdot \sup_{\substack{A \in \mathfrak{A} \\ \|A\|_\infty \leq 1}} (A \Omega, A \Omega)^{1/2} \\
 &\leq (\bar{x}(1))^{1/2} \cdot \left\| \sum_{i=1}^\infty a'_{k,i} (\bar{T}^{*'})^i (B - \bar{B}) \right\|_{(\cdot, \cdot)_{\bar{x}}} < \varepsilon
 \end{aligned} \tag{2.59}$$

for  $k > K$ , where by  $(a'_{n,i})$ , we will denote a matrix with the elements

$$a'_{n,i} = \left( \sum_{j>N} a_{n,j} \right)^{-1} a_{n,j+N}. \tag{2.60}$$

It is easy to see that the matrix  $(a'_{n,i})$  will be uniformly regular as well.

Then, for a big enough  $k > K$ , we will have

$$\begin{aligned}
 & \left\| \sum_i a_{k,i} T^i x - \bar{x} \right\| \leq \sum_{i \leq N} |a_{k,i}| \|T^i x - \bar{x}\| + \sum_{i > N} |a_{k,i}| \|T^i x - T^{i-N} w\| \\
 & \quad + \sum_{i > N} |a_{k,i}| \left| 1 - \left( \sum_{j > N} a_{k,j} \right)^{-1} \right| \|T^{i-N} w\| \\
 & \quad + \left\| \sum_{j=1}^\infty a_{k,j+N} \cdot \left( \sum_{i > N} a_{k,i} \right)^{-1} T^j w - \bar{w} \right\| \\
 & \quad + \left\| \left( \sum_{i \leq N} a_{k,i} \right) \cdot \bar{w} \right\| + \left\| \sum_{i > N} a_{k,i} \right\| \|\bar{w} - \bar{x}\| \\
 & \leq \sum_{i \leq N} 2 \cdot \frac{\varepsilon}{N} + \sum_{i > N} |a_{k,i}| \cdot 4\varepsilon + \sum_{i > N} |a_{k,i}| (1 - (1 + \varepsilon)^{-1}) \cdot 2 \\
 & \quad + \sum_{i \leq N} 2 \cdot \frac{\varepsilon}{N} + (1 + \varepsilon) \cdot 4\varepsilon \\
 & \leq 2\varepsilon + (1 + \varepsilon) \cdot 4\varepsilon + \varepsilon \cdot 2 \cdot (1 + \varepsilon) + \varepsilon + 2\varepsilon + (1 + \varepsilon) \cdot 4\varepsilon \leq 25\varepsilon.
 \end{aligned} \tag{2.61}$$

The arbitrariness of  $\varepsilon$  proves the needed statement. The proof of the theorem is now completed.  $\square$

**2.2. The case of  $L_1$ -spaces for JBW algebras.** The  $L_1$ -spaces for semifinite JBW algebras were considered by [4] (see also [1, 12]), where it has been proven that they do coincide

with predual spaces. A semifinite JBW algebra  $A$  is always represented as

$$A = A_{\text{sp}} \dot{+} A_{\text{ex}}, \quad (2.62)$$

where  $A_{\text{sp}}$  is isometrically isomorphic to operator JW algebra, and  $A_{\text{ex}}$  is isometrically isomorphic to the space  $C(X, M_3^8)$  of all continuous mappings from a Hyperstonean compact topological space  $X$  onto the exceptional Jordan algebra  $M_3^8$  (see [11]). In this case, when  $A$  does not have direct summands of type  $I_2$ , it is going to be a selfadjoint part of a real von Neumann algebra  $R(A_{\text{sp}})$ , whose complexification

$$R(A_{\text{sp}}) \dot{+} iR(A_{\text{sp}}) = M, \quad (2.63)$$

where  $M$  is the enveloping von Neumann algebra of  $A_{\text{sp}}$ , and the predual space of  $A$ , and the space

$$A_* = (A_{\text{sp}})_* \dot{+} (A_{\text{ex}})_*, \quad (2.64)$$

where  $(A_{\text{sp}})_*$  is the predual space of  $A_{\text{sp}}$ , and  $(A_{\text{ex}})_*$  is the predual space of  $A_{\text{ex}}$  (see, e.g., [2, 11]). The main result for the summand  $A_{\text{ex}}$  follows immediately from the result for  $C(X)$ , and the fact that the algebra  $M_3^8$  is finite dimensional. So, without the loss of generality, we are interested in the operator case only. But in the operator case, the space  $(A_{\text{sp}})_*$  is a selfadjoint part of  $R_* = (R(A_{\text{sp}}))_*$ , and

$$M_* = R_* \dot{+} iR_* \quad (2.65)$$

(see [2, 16] for details). So, the main result for  $R_*$  thus follows from the complex case by restriction of scalars, and we obtain the main result for  $L_1$ -spaces affiliated to semifinite JBW algebras without direct type  $I_2$  summand.

### 3. Main result: the case of quantum $L_p$ -spaces ( $1 < p < \infty$ )

In the case of a noncommutative  $L_p$ -space for a semifinite von Neumann algebra, the main result is discussed in [25].

We will discuss here the nonassociative case.

In this section,  $A$  denotes a semifinite JBW algebra without direct summands of type  $I_2$ , with a faithful normal trace  $\tau$ . By  $L_p$ , we denote the space of operators affiliated to  $A$ , and integrated with  $p$ th power ( $p > 1$ , see, e.g., [1, 2, 12]). Space  $L_q$  (here  $q = p/(p-1)$ ) is a dual as Banach space to  $L_p$  (see [1, 12]). The following theorem is valid.

**THEOREM 3.1.** *The following conditions for a positive contraction  $T$  in the  $L_p$  are equivalent.*

- (i) *The sequence  $\{T^i x\}_{i=1,2,\dots}$  converges in  $\sigma(L_p, L_q)$  topology for  $x \in L_p$ .*
- (ii) *For each strictly increasing sequence of natural numbers  $\{k_i\}_{i=1,2,\dots}$ ,*

$$n^{-1} \sum_{i < n} T^{k_i} x \quad (3.1)$$

*converges in norm of  $L_p$  for all  $x \in L_p$ .*

(iii) For any uniformly regular matrix  $(a_{n,i})$ , the sequence  $\{A_n(T)x\}_{n=1,2,\dots}$ ,

$$A_n(T)x = \sum_i a_{n,i} T^i x, \quad (3.2)$$

converges in norm of  $L_p$  for all  $x \in L_p$ .

For the sake of completeness, we give the following definitions (see, e.g., [25]) and sketch of the proof. Let  $\phi$  be a gauge function

$$\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \quad (3.3)$$

with

$$\phi(0) = 0, \quad \lim_{t \rightarrow \infty} \phi(t) = \infty. \quad (3.4)$$

Hahn-Banach theorem implies for strictly convex Banach spaces  $E$  with conjugate  $E'$  that there exists a duality map

$$\Phi : E \longrightarrow E', \quad (3.5)$$

associated with  $\phi$  such that

$$\langle x, \Phi(x) \rangle = \|x\| \|\Phi(x)\|, \quad \|\Phi(x)\| = \phi(x). \quad (3.6)$$

*Definition 3.2.* Map  $\Phi$  is said to satisfy property (S) uniformly if for every  $\epsilon > 0$ , there exists  $\delta(\epsilon) > 0$ , such that for any  $x, y \in E$ ,

$$|\langle x, \Phi(y) \rangle| < \delta(\epsilon) \quad (3.7)$$

implies that

$$|\langle y, \Phi(x) \rangle| < \epsilon. \quad (3.8)$$

*Proof.* From [12, Section 4], it follows that the duality map defined as

$$\Phi(a) = s|a|^{p-1}, \quad (3.9)$$

for

$$a = s|a| \in A \quad (3.10)$$

(where  $a = s|a|$  is a polar decomposition of element  $a$ ) satisfies the property (S) uniformly. Hence, the statement of the theorem follows from [25, Theorem 3.1].  $\square$

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