

# T-FUZZY IDEALS IN BCI-ALGEBRAS

M. AKRAM AND K. H. DAR

*Received 22 September 2004 and in revised form 17 May 2005*

Using a  $t$ -norm  $T$ , we introduce the notions of  $T$ -fuzzy subalgebras and  $T$ -fuzzy  $H$ -ideals in BCI-algebras and investigate some of their properties.

## 1. Introduction

The notion of BCI-algebras was introduced by Iséki [3] which is a generalization of BCK-algebras [2]. This notion is originated from two different ways: one of the motivations is based on set theory, another motivation is from classical and nonclassical propositional calculi.

Zadeh [8] introduced the notion of fuzzy sets. Many researchers have applied this concept to mathematical branches, such as semigroup, loop, group, ring, semiring, field, near ring, vector spaces, topological spaces, functional analysis, automation. Jun et al. [4, 5] introduced the notions of fuzzy subalgebras and fuzzy ideals of BCK-algebras with respect to a  $t$ -norm  $T$ , and studied some of their properties. In this paper, we obtain some related results of  $T$ -fuzzy subalgebras and  $T$ -fuzzy  $H$ -ideals in BCI-algebras.

## 2. Preliminaries

In this section, we review some definitions that will be used in the sequel.

*Definition 2.1.* An algebra  $(X; *, 0)$  of type  $(2, 0)$  is called a BCI-algebra if, for all  $x, y, z \in X$ , the following axioms hold.

- (1)  $((x * y) * (x * z)) * (z * y) = 0$ .
- (2)  $(x * (x * y)) * y = 0$ .
- (3)  $x * x = 0$ .
- (4)  $x * y = 0$  and  $y * x = 0 \Rightarrow x = y$ .

In BCI-algebras, the following hold.

- (5)  $(x * 0) = x$ .
- (6)  $(x * y) * z = (x * z) * y$ .
- (7)  $0 * (y * x) = (0 * y) * (0 * x)$ .

**Definition 2.2.** Let  $S$  be a nonempty subset of a BCI-algebra  $X$ , then  $S$  is called a *subalgebra* of  $X$  if  $x * y \in S$  for all  $x, y \in S$ .

**Definition 2.3.** A subset  $A$  of a BCI-algebra  $(X; *, 0)$  is called an *ideal* of  $X$  if for any  $x, y \in X$ , the following conditions hold.

- (i)  $0 \in A$ .
- (ii)  $x * y$  and  $y \in A$  imply that  $x \in A$ .

**Definition 2.4** [6]. A subset  $A$  of a BCI-algebra  $(X; *, 0)$  is called an *H-ideal* of  $X$  if for any  $x, y, z \in X$ , the following conditions hold.

- (a)  $0 \in A$ .
- (b)  $(x * (y * z))$  and  $y \in A \Rightarrow x * z \in A$ .

**Definition 2.5.** A mapping  $f : X \rightarrow Y$  of BCI-algebras is called a *homomorphism* if  $f(x * y) = f(x) * f(y)$  for all  $x, y \in X$ .

Note that if  $f$  is a homomorphism of BCI-algebras, then  $f(0) = \bar{0}$ .

**Definition 2.6.** Let  $X$  be a nonempty set. A *fuzzy* (sub)set  $\mu$  of the set  $X$  is a mapping  $\mu : X \rightarrow [0, 1]$ . The *complement* of a fuzzy set  $\mu$  of a set  $X$  is denoted by  $\bar{\mu}$  and defined by  $\bar{\mu}(x) = 1 - \mu(x)$ , for all  $x \in X$ .

**Definition 2.7** [1]. A triangular norm (*t*-norm) is a function  $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$  that satisfies the following conditions.

- (T1)  $T(x, 1) = x$ .
- (T2)  $T(x, y) = T(y, x)$ .
- (T3)  $T(x, T(y, z)) = T(T(x, y), z)$ .
- (T4)  $T(x, y) \leq T(x, z)$  whenever  $y \leq z$ , for all  $x, y, z \in [0, 1]$ .

A simple example of such defined *t*-norm is a function  $T(x, y) = \min(x, y)$ . In a general case,  $T(x, y) \leq \min(x, y)$  and  $T(x, 0) = 0$  for all  $x, y \in [0, 1]$ .

### 3. $T$ -fuzzy subalgebras

In what follows, let  $X$  denote a BCI-algebra unless otherwise specified.

**Definition 3.1.** A fuzzy set  $\mu$  in  $X$  is called a subalgebra of  $X$  with respect to a *t*-norm  $T$  (briefly, a  $T$ -fuzzy subalgebra of  $X$ ) if  $\mu(x * y) \geq T(\mu(x), \mu(y))$  for all  $x, y \in X$ .

**Example 3.2.** Let  $X := \{0, 1, 2\}$  be a BCI-algebra with the following Cayley table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Define a fuzzy set  $\mu : X \rightarrow [0, 1]$  by  $\mu(0) = 0.81$  and  $\mu(x) = 0.25$  for all  $x \neq 0$  and let  $T_m : [0, 1] \times [0, 1] \rightarrow [0, 1]$  be a function defined by  $T_m(x, y) = \max(x + y - 1, 0)$  which is a *t*-norm for all  $x, y \in [0, 1]$ . Then  $T_m$  is a *t*-norm [7]. By routine calculations, it is easy

to check that  $\mu$  satisfies  $\mu(x * y) \geq T_m(\mu(x), \mu(y))$  for all  $x, y \in X$ . Hence,  $\mu$  is a  $T_m$ -fuzzy subalgebra of  $X$ .

**THEOREM 3.3.** *Let  $\mu$  be a  $T$ -fuzzy subalgebra of  $X$  and  $\alpha \in [0, 1]$ .*

- (i) *If  $\alpha = 1$ , then  $U(\mu; \alpha)$  is either empty or a subalgebra of  $X$ .*
- (ii) *If  $T = \min$ , then  $U(\mu; \alpha)$  is either empty or a subalgebra of  $X$ .*
- (iii)  *$\mu(0) \geq \mu(x)$  for all  $x \in X$ .*

*Proof.* (i) Assume that  $\alpha = 1$ . If  $x, y \in U(\mu; 1)$ , then  $\mu(x) \geq 1$  and  $\mu(y) \geq 1$ . It follows from Definitions 2.7 and 3.1 that  $\mu(x * y) \geq T(\mu(x), \mu(y)) \geq T(1, 1) = 1$  so that  $x * y \in U(\mu; 1)$ , that is,  $U(\mu; 1)$  is a subalgebra of  $X$ .

(ii) Assume that  $T = \min$  and let  $x, y \in U(\mu; \alpha)$ . Then,  $\mu(x * y) \geq T(\mu(x), \mu(y)) = \min(\mu(x), \mu(y)) \geq \min(\alpha, \alpha) = \alpha$ , and so  $x * y \in U(\mu; \alpha)$ .

Hence  $U(\mu; \alpha)$  is a subalgebra of  $X$ .

(iii) Since  $x * x = 0$  for all  $x \in X$ , we have  $\mu(0) = \mu(x * x) \geq T(\mu(x), \mu(x)) = \min(\mu(x), \mu(x)) = \mu(x)$ . This completes the proof.  $\square$

**THEOREM 3.4.** *Let  $\mu$  be a  $T$ -fuzzy subalgebra of  $X$ . If there is a sequence  $\{x_n\}$  in  $X$  such that  $\lim_{n \rightarrow \infty} T(\mu(x_n), \mu(x_n)) = 1$ , then  $\mu(0) = 1$ .*

*Proof.* Let  $x \in X$ , then  $\mu(0) = \mu(x * x) \geq T(\mu(x), \mu(x))$ . Hence  $\mu(0) \geq T(\mu(x_n), \mu(x_n))$  for any  $n \in \mathbb{N}$ . Since  $1 \geq \mu(0) \geq \lim_{n \rightarrow \infty} T(\mu(x_n), \mu(x_n)) = 1$ , it follows that  $\mu(0) = 1$ . This completes the proof.  $\square$

**Definition 3.5.** Let  $\lambda$  and  $\mu$  be  $T$ -fuzzy subalgebras of  $X$ . Then *direct product* of  $T$ -fuzzy subalgebras is defined by  $(\lambda \times \mu)(x, y) = T(\lambda(x), \mu(y))$ , for all  $x, y \in X$ .

**THEOREM 3.6.** *If  $\mu_1$  and  $\mu_2$  are  $T$ -fuzzy subalgebras of  $X$ , then  $\mu = \mu_1 \times \mu_2$  is a  $T$ -fuzzy subalgebra of  $X \times X$ .*

*Proof.* For any  $(x_1, x_2)$  and  $(y_1, y_2) \in X \times X$ , we have

$$\begin{aligned}
 \mu((x_1, x_2) * (y_1, y_2)) &= \mu(x_1 * y_1, x_2 * y_2) \\
 &= (\mu_1 \times \mu_2)(x_1 * y_1, x_2 * y_2) \\
 &= T(\mu_1(x_1 * y_1), \mu_2(x_2 * y_2)) \\
 &\geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2))) \\
 &= T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2))) \\
 &= T((\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2)) \\
 &= T(\mu(x_1, x_2), \mu(y_1, y_2)).
 \end{aligned} \tag{3.1}$$

Hence,  $\mu = \mu_1 \times \mu_2$  is a  $T$ -fuzzy subalgebra of  $X \times X$ .  $\square$

**Definition 3.7.** Let  $f$  be a mapping on  $X$ . If  $v$  is a fuzzy set in  $f(X)$ , then fuzzy set  $\mu = v \circ f$  (i.e.,  $(v \circ f)(x) = v(f(x))$ ) in  $X$  is called *preimage* of  $v$  under  $f$ .

**THEOREM 3.8.** *An epimorphism preimage of a  $T$ -fuzzy subalgebra of  $X$  is a  $T$ -fuzzy subalgebra.*

*Proof.* Let  $f : X \rightarrow Y$  be an epimorphism of BCI-algebras, let  $\nu$  be a  $T$ -fuzzy subalgebra of  $Y$ , and let  $\mu$  be the preimage of  $\nu$  under  $f$ . Then for any  $x, y \in X$ , we have

$$\begin{aligned}
 \mu(x * y) &= (\nu \circ f)(x * y) \\
 &= \nu(f(x * y)) = \nu(f(x) * f(y)) \\
 &\geq T(\nu(f(x)), \nu(f(y))) \\
 &= T((\nu \circ f)(x), (\nu \circ f)(y)) \\
 &= T(\mu(x), \mu(y)).
 \end{aligned} \tag{3.2}$$

Hence,  $\mu$  is a fuzzy subalgebra of  $X$  with respect to a  $t$ -norm  $T$ .  $\square$

*Definition 3.9.* If  $\mu$  is a fuzzy set in a subalgebra  $X$  and  $f$  is a mapping defined on  $X$ , the fuzzy set  $\mu^f$  in  $f(X)$  defined by

$$\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x) \quad \forall y \in f(X) \tag{3.3}$$

is called the *image* of  $\mu$  under  $f$ .

*Definition 3.10.* A fuzzy set  $\mu$  in  $X$  has the *Sup property* if for any subset  $A \subseteq X$ , there exists  $a_0 \in A$  such that  $\mu(a_0) = \sup_{a \in A} \mu(a)$ .

**THEOREM 3.11.** *An epimorphism image of a fuzzy subalgebra with Sup property is a fuzzy subalgebra.*

*Proof.* Let  $f : X \rightarrow Y$  be an epimorphism of  $X$  and let  $\mu$  be a fuzzy subalgebra of  $X$  with Sup property. Let  $f(x), f(y) \in f(X)$  and let  $x_0, y_0 \in f^{-1}(f(x))$  be such that

$$\begin{aligned}
 \mu(x_0) &= \sup_{t \in f^{-1}(f(x))} \mu(t), \\
 \mu(y_0) &= \sup_{t \in f^{-1}(f(y))} \mu(t),
 \end{aligned} \tag{3.4}$$

respectively. Then,

$$\begin{aligned}
 \mu^f(f(x) * f(y)) &= \sup_{z \in f^{-1}(f(x) * f(y))} \mu(z) \\
 &\geq \min \{ \mu(x_0), \mu(y_0) \} \\
 &= \min \left\{ \sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t) \right\} \\
 &= \min \{ \mu^f(f(x)), \mu^f(f(y)) \}.
 \end{aligned} \tag{3.5}$$

Hence  $\mu^f$  is a fuzzy subalgebra of  $Y$ .  $\square$

**Definition 3.12** [7]. A  $t$ -norm  $T$  on  $[0,1]$  is called a *continuous  $t$ -norm* if  $T$  is a continuous function from  $[0,1] \times [0,1]$  to  $[0, 1]$  with respect to the usual topology. Note that the function  $\min$  is a continuous  $t$ -norm.

**THEOREM 3.13.** Let  $T$  be a continuous  $t$ -norm and let  $f$  be a homomorphism on  $X$ . If  $\mu$  is a  $T$ -fuzzy subalgebra of  $X$ , then  $\mu^f$  is a  $T$ -fuzzy subalgebra of  $f(X)$ .

*Proof.* Let  $Z_1 = f^{-1}(y_1)$ ,  $Z_2 = f^{-1}(y_2)$ , and  $Z_{12} = f^{-1}(y_1 * y_2)$ , where  $y_1, y_2 \in f(X)$ .

Consider the set  $Z_1 * Z_2 = \{x \in X \mid x = z_1 * z_2 \text{ for some } z_1 \in Z_1 \text{ and } z_2 \in Z_2\}$ . If  $x \in Z_1 * Z_2$ , then  $x = x_1 * x_2$  for some  $x_1 \in Z_1$  and  $x_2 \in Z_2$ .

Thus  $f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2$ , that is,  $x \in f^{-1}(y_1 * y_2) = Z_{12}$ . Hence  $Z_1 * Z_2 \subseteq Z_{12}$ . It follows that

$$\begin{aligned} \mu^f(y_1 * y_2) &= \sup_{x \in f^{-1}(y_1 * y_2)} \mu(x) \\ &= \sup_{Z_{12}} \mu(x) \geq \sup_{x \in Z_1 * Z_2} \mu(x) \\ &\geq \sup_{x_1 \in Z_1, x_2 \in Z_2} \mu(x_1 * x_2) \\ &\geq \sup_{x_1 \in Z_1, x_2 \in Z_2} T(\mu(x_1), \mu(x_2)). \end{aligned} \quad (3.6)$$

Since  $T$  is continuous, for every  $\varepsilon > 0$ , there exists a number  $\delta > 0$  such that if  $\sup_{x_1 \in Z_1} \mu(x_1) - x_1^* \leq \delta$  and  $\sup_{x_2 \in Z_2} \mu(x_2) - x_2^* \leq \delta$ , then

$$T\left(\sup_{x_1 \in Z_1} \mu(x_1), \sup_{x_2 \in Z_2} \mu(x_2)\right) - T(x_1^*, x_2^*) \leq \varepsilon. \quad (3.7)$$

Choose  $z_1 \in Z_1$  and  $z_2 \in Z_2$  such that  $\sup_{x_1 \in Z_1} \mu(x_1) - \mu(z_1) \leq \delta$  and  $\sup_{x_2 \in Z_2} \mu(x_2) - \mu(z_2) \leq \delta$ , then  $T(\sup_{x_1 \in Z_1} \mu(x_1), \sup_{x_2 \in Z_2} \mu(x_2)) - T(\mu(z_1), \mu(z_2)) \leq \varepsilon$ . Consequently,  $\mu^f(y_1 * y_2) \geq \sup_{x_1 \in Z_1, x_2 \in Z_2} T(\mu(x_1), \mu(x_2)) \geq T(\sup_{x_1 \in Z_1} \mu(x_1), \sup_{x_2 \in Z_2} \mu(x_2)) = T(\mu^f(y_1), \mu^f(y_2))$ , which shows that  $\mu^f$  is a  $T$ -fuzzy subalgebra of  $f(X)$ .  $\square$

#### 4. $T$ -fuzzy $H$ -ideals

**Definition 4.1.** A fuzzy set  $\mu$  in  $X$  is called  $T$ -fuzzy ideals of  $X$  if

- (1)  $\mu(0) \geq \mu(x)$  for all  $x \in X$ ,
- (2)  $\mu(x) \geq T(\mu(x * y), \mu(y))$  for all  $x, y \in X$ .

**Definition 4.2.** A fuzzy set  $\mu$  in  $X$  is called  $T$ -fuzzy  $H$ -ideals of  $X$  if

- (TF1)  $\mu(0) \geq \mu(x)$  for all  $x \in X$ ,
- (TF2)  $\mu(x * z) \geq T(\mu(x * (y * z)), \mu(y))$  for all  $x, y, z \in X$ .

*Example 4.3.* Let  $X := \{0, a, b, c\}$  be a BCI-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	a	c	0

Define a fuzzy set  $\mu : X \rightarrow [0, 1]$  by  $\mu(0) = t_1$  and  $\mu(x) = t_2$  for all  $x \neq 0$ , where  $t_1 > t_2$  and let  $T_m : [0, 1] \times [0, 1] \rightarrow [0, 1]$  be a function defined by  $T_m(x, y) = \max(x + y - 1, 0)$  which is a  $t$ -norm for all  $x, y \in [0, 1]$ . By routine calculations, it is easy to check that  $\mu$  is a  $T_m$ -fuzzy  $H$ -ideal of  $X$ .

**PROPOSITION 4.4.** Every  $T$ -fuzzy  $H$ -ideal in a BCI-algebra  $X$  is a  $T$ -fuzzy ideal of  $X$ .

*Proof.* For  $x, y, z \in X$ , we have

$$\begin{aligned}
 \mu_A(x * z) &\geq T(\mu_A(x * (y * z)), \mu_A(y)), \\
 \mu_A(x * 0) &\geq T(\mu_A(x * (y * 0)), \mu_A(y)) \quad (\text{putting } z = 0) \\
 \mu_A(x) &\geq T(\mu_A(x * y), \mu_A(y)) \quad (\text{using (5)}),
 \end{aligned} \tag{4.1}$$

which completes the proof.  $\square$

**PROPOSITION 4.5.** Every  $T$ -fuzzy  $H$ -ideal of a BCI-algebra  $X$  is a  $T$ -fuzzy subalgebra of  $X$ .

*Proof.* For  $x, y, z \in X$ , we have

$$\begin{aligned}
 \mu_A(x * z) &\geq T(\mu_A(x * (y * z)), \mu_A(y)), \\
 \mu_A(x * y) &\geq T(\mu_A(x * (y * y)), \mu_A(y)) \quad (\text{replacing } z \text{ by } y) \\
 \mu_A(x * y) &\geq T(\mu_A(x * 0), \mu_A(y)) \quad (\text{using (3)}) \\
 \mu_A(x * y) &\geq T(\mu_A(x), \mu_A(y)) \quad (\text{using (5)}).
 \end{aligned} \tag{4.2}$$

This ends the proof.  $\square$

**THEOREM 4.6.** If  $\mu$  is a  $T$ -fuzzy  $H$ -ideal of  $X$ , then each nonempty level subset  $U(\mu; 1)$  is  $H$ -ideal of  $X$ .

*Proof.* Suppose that  $\mu$  is a  $T$ -fuzzy  $H$ -ideal of  $X$ . Since  $U(\mu, 1)$  is nonempty, there exists  $x \in U(\mu; 1)$ . It follows from (TF1) that  $\mu(0) \geq \mu(x) \geq 1$ , that is,  $0 \in U(\mu; 1)$ . Let  $x, y, z \in X$  be such that  $x * (y * z) \in U(\mu; 1)$  and  $y \in U(\mu; 1)$ . Then  $\mu(x * z) \geq T(\mu(x * (y * z)), \mu(y)) \geq T(1, 1) = 1$  so that  $x * z \in U(\mu; 1)$ .

Hence  $U(\mu; 1)$  is a  $H$ -ideal of  $X$ .  $\square$

**THEOREM 4.7.** *If  $\lambda$  and  $\mu$  are  $T$ -fuzzy  $H$ -ideals of a BCI-algebra  $X$ , then  $\lambda \times \mu$  is a  $T$ -fuzzy  $H$ -ideal of  $X \times X$ .*

*Proof.* For any  $(x, y) \in X \times X$ , we have  $(\lambda \times \mu)(0, 0) = T(\lambda(0), \mu(0)) \geq T(\lambda(x), \mu(y)) = (\lambda \times \mu)(x, y)$ . Let  $x = (x_1, x_2)$ ,  $y = (y_1, y_2)$ , and  $z = (z_1, z_2) \in X \times X$ . Then

$$\begin{aligned}
 (\lambda \times \mu)(x * z) &= (\lambda \times \mu)((x_1, x_2) * (z_1, z_2)) \\
 &= (\lambda \times \mu)(x_1 * z_1, x_2 * z_2) \\
 &= T(\lambda(x_1 * z_1), \mu(x_2 * z_2)) \\
 &\geq T(T(\lambda(x_1 * (y_1 * z_1)), \lambda(y_1)), T(\mu(x_2 * (y_2 * z_2)), \mu(y_2))) \\
 &= T(T(\lambda(x_1 * (y_1 * z_1)), \mu(x_2 * (y_2 * z_2))), T(\lambda(y_1), \mu(y_2))) \\
 &= T((\lambda \times \mu)((x_1 * (y_1 * z_1), x_2 * (y_2 * z_2))), (\lambda \times \mu)((y_1, y_2))) \\
 &= T((\lambda \times \mu)((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))), (\lambda \times \mu)((y_1, y_2))) \\
 &= T((\lambda \times \mu)(x * (y * z)), (\lambda \times \mu)(y)).
 \end{aligned} \tag{4.3}$$

Hence  $\lambda \times \mu$  is a  $T$ -fuzzy  $H$ -ideal of  $X \times X$ . □

**THEOREM 4.8.** *Let  $f : X \rightarrow Y$  be a homomorphism of BCI-algebras. If  $\mu$  is a  $T$ -fuzzy  $H$ -ideal of  $Y$ , then  $\mu^f$  is a  $T$ -fuzzy  $H$ -ideal of  $X$ .*

*Proof.* For any  $x \in X$ , we have  $\mu^f(x) = \mu(f(x)) \leq \mu(\acute{0}) = \mu(f(0)) = \mu^f(0)$ . Thus,  $\mu^f(x) \leq \mu^f(0)$ , for all  $x \in X$ . Let  $x, y, z \in X$ . Then

$$\begin{aligned}
 T(\mu^f(x * (y * z)), \mu^f(y)) &= T(\mu(f(x * (y * z))), \mu(f(y))) \\
 &= T(\mu(f(x) * (f(y) * f(z))), \mu(f(y))) \\
 &\leq \mu(f(x) * f(z)) = \mu(f(x * z)) = \mu^f(x * z).
 \end{aligned} \tag{4.4}$$

Hence  $\mu^f$  is a  $T$ -fuzzy  $H$ -ideal of  $X$ . □

**THEOREM 4.9.** *Let  $f : X \rightarrow Y$  be an epimorphism of BCI-algebras. If  $\mu^f$  is a  $T$ -fuzzy  $H$ -ideal of  $X$ , then  $\mu$  is a  $T$ -fuzzy  $H$ -ideal of  $Y$ .*

*Proof.* Let  $y \in Y$ , there exists  $x \in X$  such that  $f(x) = y$ . Then  $\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(0) = \mu(f(0)) = \mu(\acute{0})$ . Let  $x, y, z \in Y$ . Then there exist  $a, b, c \in X$  such that  $f(a) = x$ ,  $f(b) = y$ , and  $f(c) = z$ . It follows that

$$\begin{aligned}
 \mu(x * z) &= \mu(f(a) * f(c)) = \mu(f(a * c)) = \mu^f(a * c) \\
 &\geq T(\mu^f(a * (b * c)), \mu^f(b)) \\
 &= T(\mu(f(a * (b * c))), \mu(f(b))) \\
 &= T(\mu(f(a) * (f(b) * f(c))), \mu(f(b))) \\
 &= T(\mu(x * (y * z)), \mu(y)).
 \end{aligned} \tag{4.5}$$

Hence  $\mu$  is a  $T$ -fuzzy  $H$ -ideal of  $Y$ . □

**Definition 4.10.** Let  $T$  be a  $t$ -norm and let  $\lambda$  and  $\mu$  be two fuzzy sets in  $X$ . Then the  $T$ -product of  $\lambda$  and  $\mu$  is denoted by  $[\lambda \cdot \mu]_T$  and defined by  $[\lambda \cdot \mu]_T(x) = T(\lambda(x), \mu(x))$ , for all  $x \in X$ .

Note that

- (i)  $[\lambda \cdot \mu]_T = [\mu \cdot \lambda]_T$ ,
- (ii)  $[\lambda \cdot \mu]_T$  is a fuzzy set in  $X$ .

**THEOREM 4.11.** Let  $\lambda$  and  $\mu$  be two  $T$ -fuzzy  $H$ -ideals of  $X$ . If a  $t$ -norm  $T^*$  dominates  $T$ , that is, if  $T^*(T(\alpha, \gamma), T(\beta, \delta)) \geq T(T^*(\alpha, \beta), T^*(\gamma, \delta))$  for all  $\alpha, \beta, \gamma, \delta \in [0, 1]$ , then  $T^*$ -product  $[\lambda \cdot \mu]_T^*$  is a  $T$ -fuzzy  $H$ -ideal of  $X$ .

*Proof.* For any  $x \in X$ , we have  $[\lambda \cdot \mu]_T^*(0) = T^*(\lambda(0), \mu(0)) \geq T^*(\lambda(x), \mu(x)) = [\lambda \cdot \mu]_T^*(x)$ . Let  $x, y, z \in X$ . Then

$$\begin{aligned}
 [\lambda \cdot \mu]_T^*(x * z) &= T^*(\lambda(x * z), \mu(x * z)) \\
 &\geq T^*(T(\lambda(x * (y * z)), \lambda(y)), T(\mu(x * (y * z)), \mu(y))) \\
 &\geq T(T^*(\lambda(x * (y * z)), \mu(x * (y * z))), T^*(\lambda(y), \mu(y))) \\
 &= T([\lambda \cdot \mu]_T^*(x * (y * z)), [\lambda \cdot \mu]_T^*(y)).
 \end{aligned} \tag{4.6}$$

This proves that  $[\lambda \cdot \mu]_T^*$  is a  $T$ -fuzzy  $H$ -ideal of  $X$ . □

**COROLLARY 4.12.** The  $T$ -product of two  $T$ -fuzzy  $H$ -ideals of  $X$  is a  $T$ -fuzzy ideal of the same BCI-algebra  $X$ .

**THEOREM 4.13.** Let  $T$  and  $T^*$  be  $T$ -norms in which  $T^*$  dominates  $T$ . Let  $f : X \rightarrow Y$  be an epimorphism of BCI-algebras. If  $\lambda$  and  $\mu$  are  $T$ -fuzzy  $H$ -ideals of  $Y$ , then  $f^{-1}([\lambda \cdot \mu]_T^*) = [f^{-1}(\lambda), f^{-1}(\mu)]_T^*$ .

*Proof.* For any  $x \in X$ , we have

$$\begin{aligned}
 [f^{-1}([\lambda \cdot \mu]_T^*)](x) &= [\lambda \cdot \mu]_T^*(f(x)) \\
 &= T^*(\lambda(f(x)), \mu(f(x))) \\
 &= T^*([f^{-1}(\lambda)](x), [f^{-1}(\mu)](x)) \\
 &= [f^{-1}(\lambda), f^{-1}(\mu)]_T^*(x).
 \end{aligned} \tag{4.7}$$

This completes the proof. □

**COROLLARY 4.14.** If  $f : X \rightarrow Y$  is an epimorphism of BCI-algebras, then  $f^{-1}([\lambda \cdot \mu]_T) = [f^{-1}(\lambda), f^{-1}(\mu)]_T$  for any  $T$ -fuzzy  $H$ -ideals  $\lambda$  and  $\mu$  of  $Y$ .

## Acknowledgments

The first author was supported by PUCIT. The authors are highly grateful to the referee for valuable comments and suggestions.



## References

- [1] M. T. Abu Osman, *On some product of fuzzy subgroups*, Fuzzy Sets and Systems **24** (1987), no. 1, 79–86.
- [2] Y. Imai and K. Iséki, *On axiom systems of propositional calculi. XIV*, Proc. Japan Acad. **42** (1966), 19–22.
- [3] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. **42** (1966), 26–29.
- [4] Y. B. Jun and K. H. Kim, *Imaginable fuzzy ideals of BCK-algebras with respect to a  $t$ -norm*, J. Fuzzy Math. **8** (2000), no. 3, 737–744.
- [5] Y. B. Jun and Q. Zhang, *Fuzzy subalgebras of BCK-algebras with respect to a  $t$ -norm*, Far East J. Math. Sci. (FJMS) **2** (2000), no. 3, 489–495.
- [6] H. M. Khalid and B. Ahmad, *Fuzzy  $H$ -ideals in BCI-algebras*, Fuzzy Sets and Systems **101** (1999), no. 1, 153–158.
- [7] Y. Yu, J. N. Mordeson, and S. N. Cheng, *Elements of L-Algebra*, Lecture Notes in Fuzzy Math. and Computer Sci., Creighton University, Nebraska, 1994.
- [8] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

M. Akram: Punjab University College of Information Technology, University of the Punjab, P.O. Box 54000, Lahore, Pakistan

*E-mail address:* m.akram@pucit.edu.pk

K. H. Dar: School of Mathematical Sciences, National College of Business Administration & Economics, 40/E-1, Gulberg III, Lahore, Pakistan

*E-mail address:* prof\_khdar@yahoo.com

## Special Issue on Space Dynamics

### Call for Papers

Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

The main objective of this Special Issue is to publish topics that are under study in one of those lines. The idea is to get the most recent researches and published them in a very short time, so we can give a step in order to help scientists and engineers that work in this field to be aware of actual research. All the published papers have to be peer reviewed, but in a fast and accurate way so that the topics are not outdated by the large speed that the information flows nowadays.

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/mpe/guidelines.html>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	July 1, 2009
First Round of Reviews	October 1, 2009
Publication Date	January 1, 2010

### Lead Guest Editor

**Antonio F. Bertachini A. Prado**, Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; [prado@dem.inpe.br](mailto:prado@dem.inpe.br)

### Guest Editors

**Maria Cecilia Zanardi**, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; [cecilia@feg.unesp.br](mailto:cecilia@feg.unesp.br)

**Tadashi Yokoyama**, Universidade Estadual Paulista (UNESP), Rio Claro, 13506-900 São Paulo, Brazil; [tadashi@rc.unesp.br](mailto:tadashi@rc.unesp.br)

**Silvia Maria Giuliatti Winter**, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; [silvia@feg.unesp.br](mailto:silvia@feg.unesp.br)