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PROPERTIES OF SOME $*$-DENSE-IN-ITSELF SUBSETS

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$\mathcal{I}$-open sets were introduced and studied by Janković and Hamlett (1990) to generalize the well-known Banach category theorem. Quasi-$\mathcal{I}$-openness was introduced and studied by Abd El-Monsef et al. (2000). These are $*$-dense-in-itself sets of the ideal spaces. In this note, properties of these sets are further investigated and characterizations of these sets are given. Also, their relation with $\mathcal{I}$-dense sets and $\mathcal{I}$-locally closed sets is discussed. Characterizations of completely codense ideals are given in terms of semi-preopen sets.

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1. Introduction and preliminaries. The subject of ideals in topological spaces has been studied by Kuratowski [12] and Vaidyanathaswamy [20]. An *ideal* $\mathcal{I}$ on a topological space (X,τ) is a collection of subsets of X which satisfies that (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : \wp(X) \to \wp(X)$, called a *local function* [12] of A with respect to $\mathcal{I}$ and τ, is defined as follows: for $A \subset X$, $A^*(\mathcal{I},\tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau \mid x \in U\}$. We will make use of the basic facts concerning the local functions [10, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $\text{cl}^*(\cdot)$ for a topology $\tau^*(\mathcal{I},\tau)$, called the $*$-*topology*, finer than τ, is defined by $\text{cl}^*(A) = A \cup A^*(\mathcal{I},\tau)$ [19]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* or $\tau^*(\mathcal{I})$ for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then $(X,\tau,\mathcal{I})$ is called an ideal space. By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subset X$, $\text{cl}(A)$ and $\text{int}(A)$ will denote the closure and interior of A in (X,τ), respectively, and $\text{cl}^*(A)$ and $\text{int}^*(A)$ will denote the closure and interior of A in (X,τ^*), respectively. A subset A of a space (X,τ) is *semiopen* [13] if there exists an open set G such that $G \subset A \subset \text{cl}(G)$ or, equivalently, $A \subset \text{cl}(\text{int}(A))$. The complement of a semiopen set is semiclosed. The smallest semiclosed set containing A is called the *semiclosure* of A and is denoted by $\text{scl}(A)$. Also, $\text{scl}(A) = A \cup \text{int}(\text{cl}(A))$ [4, Theorem 1.5(a)]. The largest semiopen set contained in A is called the *semi-interior* of A and is denoted by $\text{sint}(A)$. A subset A of a space (X,τ) is an α-set [15] if $A \subset \text{int}(\text{cl}(\text{int}(A)))$. The family of all α-sets in (X,τ) is denoted by τ^α. τ^α is a topology on X which is finer than τ. The complement of an α-set is called an α-*closed* set. The closure and interior of A in (X,τ^α) are denoted by $\text{cl}_\alpha(A)$ and $\text{int}_\alpha(A)$, respectively. If $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X,τ), then $\tau^*(\mathcal{N},\tau) = \tau^\alpha$ and $\text{cl}_\alpha(A) = A \cup A^*(\mathcal{N})$ [10]. An open subset A of a space (X,τ) is said to be *regular open* if $A = \text{int}(\text{cl}(A))$. The complement of a regular open set is *regular closed*. A subset A of a space (X,τ) is said to be *preopen* [14] if $A \subset \text{int}(\text{cl}(A))$. The family of all preopen sets is denoted by $\text{PO}(X,\tau)$ or simply $\text{PO}(X)$. The largest preopen set contained in A is called the *preinterior* of A and is denoted by $\text{pint}(A)$ and $\text{pint}(A) = A \cap \text{int}(\text{cl}(A))$ [4]. A is preopen if and only if there is a regular open set G such that $A \subset G$ and $\text{cl}(A) = \text{cl}(G)$ [7, Proposition 2.1]. A subset A of a space (X,τ) is *semi-preopen* [4] if there exists a preopen set G such that $G \subset A \subset \text{cl}(G)$. The family of all semi-preopen sets in (X,τ) is denoted by $\text{SPO}(X,\tau)$ or simply $\text{SPO}(X)$. The complement of a semi-preopen set is called *semi-preclosed*. The largest semi-preopen set contained in A is called the *semi-preinterior* of A and is denoted by $\text{spint}(A)$. Also, $\text{spint}(A) = A \cap \text{cl}(\text{int}(\text{cl}(A)))$ for every A of X [4]. Given a space (X,τ) and ideals $\mathcal{I}$ and $\mathfrak{I}$ on X, the *extension* of $\mathcal{I}$ via $\mathfrak{I}$ [11], denoted by $\mathcal{I} * \mathfrak{I}$, is the ideal given by $\mathcal{I} * \mathfrak{I} = \{A \subset X \mid A^*(\mathcal{I}) \in \mathfrak{I}\}$. In particular, $\mathcal{I} * \mathcal{N} = \{A \subset X \mid \text{int}(A^*(\mathcal{I})) = \phi\}$ is an ideal containing both $\mathcal{I}$ and $\mathcal{N}$ and $\mathcal{I} * \mathcal{N}$ is usually denoted by $\tilde{\mathcal{I}}$. The following lemmas will be useful in the sequel.

Lemma 1.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. If $A \subset A^*$, then*
(a) $A^* = \text{cl}(A) = \text{cl}^*(A)$,
(b) $A^*(\tilde{\mathcal{I}}) = A^*(\mathcal{N})$.

Proof. Clearly, for every subset A of X, $\text{cl}^*(A) \subset \text{cl}(A)$. Let $x \notin \text{cl}^*(A)$. Then there exists a τ^*-open set G containing x such that $G \cap A = \phi$. There exists $V \in \tau$ and $I \in \mathcal{I}$ such that $x \in V - I \subset G$. $G \cap A = \phi \Rightarrow (V - I) \cap A = \phi \Rightarrow (V \cap A) - I = \phi \Rightarrow ((V \cap A) - I)^* = \phi \Rightarrow (V \cap A)^* - I^* = \phi \Rightarrow (V \cap A)^* = \phi \Rightarrow V \cap A^* = \phi \Rightarrow V \cap A = \phi$. Since V is an open set containing x, $x \notin \text{cl}(A)$ and so $\text{cl}(A) \subset \text{cl}^*(A)$. Hence $\text{cl}(A) = \text{cl}^*(A)$. Since $A \subset A^* \subset \text{cl}(A)$, $\text{cl}(A) = A^*$. This proves (a).

(b) By [11, Theorem 3.2], $A^*(\tilde{\mathcal{I}}) = \text{cl}(\text{int}(A^*))$ and so by (a), $A^*(\tilde{\mathcal{I}}) = \text{cl}(\text{int}(\text{cl}(A))) = A^*(\mathcal{N})$. □

Lemma 1.2. *Let (X,τ) be a space and $A \subset X$. If A is semiopen, then* $\text{cl}(A) = \text{cl}_\alpha(A)$ *and if A is semiclosed, then* $\text{int}(A) = \text{int}_\alpha(A)$ [18, Lemma 2.1].

Lemma 1.3. *If $(X,\tau,\mathcal{I})$ is an ideal space, then the following are equivalent.*
(a) *For every $A \in \tau$, $A \subset A^*$.*
(b) *For every $A \in \text{SO}(X,\tau)$, $A \subset A^*$.*

Proof. Since $\tau \subset \text{SO}(X,\tau)$, it is enough to prove that (a)⇒(b). Suppose $A \in \text{SO}(X,\tau)$. Then there exists an open set H such that $H \subset A \subset \text{cl}(H)$. Since H is open, $H \subset H^*$ and so, by Lemma 1.1, $A \subset \text{cl}(H) = H^* \subset A^*$. Hence $A \subset A^*$. □

2. Completely codense ideal. An ideal $\mathcal{I}$ on a space (X,τ) is said to be *codense* [6] if $\tau \cap \mathcal{I} = \{\phi\}$ or, equivalently, $X = X^*$ [10]. $\mathcal{I}$ is said to be *completely codense* [6] if $\text{PO}(X) \cap \mathcal{I} = \{\phi\}$ or, equivalently, $\mathcal{I} \subset \mathcal{N}$ [6, Theorem 4.13]. Every completely codense ideal is codense. The converse implication is not true, since in $\mathbb{R}$, the set of all real numbers with the usual topology, the ideal $\mathcal{C}$ of all countable subsets is codense but not completely codense [6]. The following theorem characterizes completely codense ideals.

Theorem 2.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then the following are equivalent.*
(a) *$\mathcal{I}$ is completely codense.*
(b) $\mathrm{SPO}(X)\cap\mathcal{I}=\{\phi\}$.
(c) $A\subset A^*$ *for every* $A\in\mathrm{SPO}(X)$.
(d) $\mathrm{spint}(A)=\phi$ *for every* $A\in\mathcal{I}$.

Proof. (a)⇒(b). Suppose $A\in\mathrm{SPO}(X)\cap\mathcal{I}$. $A\in\mathcal{I}\Rightarrow A\in\mathcal{N}$ and so $\mathrm{int}(\mathrm{cl}(A))=\phi$. $A\in\mathrm{SPO}(X)\Rightarrow A\subset\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))\Rightarrow A=\phi$. Therefore, $\mathrm{SPO}(X)\cap\mathcal{I}=\{\phi\}$.

(b)⇒(c). Suppose $A\in\mathrm{SPO}(X)$ and $x\notin A^*$. Then there exists an open set G containing x such that $G\cap A\in\mathcal{I}$. Since $A\in\mathrm{SPO}(X)$, $G\cap A\in\mathrm{SPO}(X)$, by [4, Theorem 2.7] and so by hypothesis, $G\cap A=\phi$ which implies that $x\notin A$.

(c)⇒(d). Let $A\in\mathcal{I}$ such that $\mathrm{spint}(A)\neq\phi$. Then there exists a nonempty semi-preopen set G such that $G\subset A$ and so $G^*\subset A^*=\phi$. Since $G\subset G^*$, $G=\phi$ which is a contradiction. Therefore, $\mathrm{spint}(A)=\phi$.

(d)⇒(a). Let $A\in\mathrm{PO}(X)\cap\mathcal{I}$. Then $A\in\mathrm{PO}(X)\Rightarrow A\subset\mathrm{int}(\mathrm{cl}(A))\subset\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$. $A\in\mathcal{I}\Rightarrow\mathrm{spint}(A)=\phi\Rightarrow A\cap\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))=\phi\Rightarrow A=\phi$. □

Corollary 2.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space with a completely codense ideal $\mathcal{I}$.*
(a) *If* $A\in\mathrm{SPO}(X)$, *then* $A^*(\mathcal{I})$ *is regular closed,* $A^*(\mathcal{I})=A^*(\mathcal{N})$, *and* $\mathrm{cl}(A)=\mathrm{cl}^*(A)=\mathrm{cl}_\alpha(A)$.
(b) *If A is semi-preclosed, then* $\mathrm{int}(A)=\mathrm{int}^*(A)=\mathrm{int}_\alpha(A)$.

Proof. (a) If $A\in\mathrm{SPO}(X)$, by Theorem 2.1(c), $A\subset A^*\subset\mathrm{cl}(A)$ and so $A^*=\mathrm{cl}(A)$ which implies that A^* is regular closed, since the closure of a semi-preopen set is regular closed [4, Theorem 2.4]. Therefore, $A^*=\mathrm{cl}(\mathrm{int}(A^*))=\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))=A^*(\mathcal{N})$. $\mathrm{cl}(A)=\mathrm{cl}^*(A)$ by Theorem 2.1(c) and Lemma 1.1. Also, $\mathrm{cl}^*(A)=A\cup A^*(\mathcal{I})=A\cup A^*(\mathcal{N})=\mathrm{cl}_\alpha(A)$. This proves (a).

(b) The proof follows from (a). □

3. $\mathcal{I}$-open sets. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is τ^*-*closed* [10] (resp., $*$-*dense in itself* [9], $*$-*perfect* [9]) if $A^*\subset A$ (resp., $A\subset A^*$, $A=A^*$). Clearly, A is $*$-perfect if and only if A is τ^*-closed and $*$-dense in itself. The following Theorem 3.1 is useful in the sequel.

Theorem 3.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and let U and A be subsets of X such that $A\subset U\subset A^*$. Then U is $*$-dense in itself, and U^* and A^* are $*$-perfect.*

Proof. $A\subset U\subset A^*$ implies that $U^*=A^*$ and so U is $*$-dense in itself. Since $(A^*)^*\subset A^*$, $A\subset A^*$ implies that A^* is $*$- perfect and so U^* is $*$-perfect. □

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-locally closed, [5] if $A=G\cap V$, where G is open and V is $*$-perfect. Clearly, every $*$-perfect set is $\mathcal{I}$-locally closed. The following theorem gives a characterization of $\mathcal{I}$-locally closed sets.

Theorem 3.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space. A subset A of X is $\mathcal{I}$-locally closed if and only if $A=G\cap A^*$ for some open set G.*

Proof. Suppose A is $\mathcal{I}$-locally closed. Then $A = G\cap V$ where G is open and V is $*$-perfect. Now $A^* = (G\cap V)^* \supset G\cap V^* = G\cap V = A$. Also, $A\subset V$ implies that $A^*\subset V^* = V$. Therefore, $G\cap A^* = G\cap(A^*\cap V) = (G\cap V)\cap A^* = A\cap A^* = A$. Conversely, if $A = G\cap A^*$ where G is open, then $A\subset A^*$ and so by Theorem 3.1, A^* is $*$-perfect and so A is $\mathcal{I}$-locally closed. □

The following corollary follows from [10, Theorems 2.1 and 2.2 and Theorem 6.1(d)].

Corollary 3.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space.*
(a) *Every $\mathcal{I}$-locally closed set is $*$-dense in itself.*
(b) *Every open, $*$-dense-in-itself subset of X is $\mathcal{I}$-locally closed.*
(c) *If $\mathcal{I}$ is codense, then every open set is $\mathcal{I}$-locally closed.*

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-open [11] if $A\subset \operatorname{int}(A^*)$. The family of all $\mathcal{I}$-open sets is denoted by $\mathrm{IO}(X,\tau,\mathcal{I})$, $\mathrm{IO}(X,\tau)$, or $\mathrm{IO}(X)$. The complement of an $\mathcal{I}$-open set is said to be $\mathcal{I}$-closed. The largest $\mathcal{I}$-open set contained in A is called the $\mathcal{I}$-interior of A and is denoted by $\mathrm{Iint}(A)$ and $\mathrm{Iint}(A) = A\cap\operatorname{int}(A^*)$ [11, Theorem 4.1(3)]. The following theorem gives some properties of $\mathcal{I}$-open sets.

Theorem 3.4. *If A is an $\mathcal{I}$-open subset of an ideal space $(X,\tau,\mathcal{I})$, then*
(a) *A is $*$-dense in itself,*
(b) *$A^* = \operatorname{cl}(A) = \operatorname{cl}^*(A)$ and $\operatorname{cl}(A)$ and A^* are regular closed,*
(c) *A^* is $*$-perfect and $\mathcal{I}$-locally closed,*
(d) *$\operatorname{int}(A^*)$ is $*$-dense in itself and $\mathcal{I}$-locally closed,*
(e) *$\operatorname{cl}(\operatorname{int}(A^*)) = A^*(\tilde{\mathcal{I}})$ is $*$-dense in itself,*
(f) *$A^* = (\operatorname{int}(A^*))^* = (\operatorname{cl}(\operatorname{int}(A^*)))^* = (A^*(\tilde{\mathcal{I}}))^*(\mathcal{I})$,*
(g) *$(\operatorname{int}(A^*))^*$ and $(\operatorname{cl}(\operatorname{int}(A^*)))^*$ are $\mathcal{I}$-locally closed,*
(h) *$\operatorname{int}(A^*)$ is $\mathcal{I}$-open.*

Proof. (a) follows from the definition. (b) follows from (a), Lemma 1.1, and the fact that every $\mathcal{I}$-open set is preopen [1] and the closure of a preopen set is regular closed [7, Proposition 2.1(ii)]. (c) follows from Theorem 3.1 and from the fact that every $*$-perfect set is $\mathcal{I}$-locally closed. (d) follows from Theorem 3.1 and Corollary 3.3(b). (e) $\operatorname{cl}(\operatorname{int}(A^*)) = A^*(\tilde{\mathcal{I}})$ by [11, Theorem 3.2] and since $A\subset\operatorname{int}(A^*)\subset\operatorname{cl}(\operatorname{int}(A^*))\subset A^*$, by Theorem 3.1, $\operatorname{cl}(\operatorname{int}(A^*))$ is $*$-dense in itself. (f) From the inequality in the proof of (e), we have $A^* = (\operatorname{int}(A^*))^* = (\operatorname{cl}(\operatorname{int}(A^*)))^*$. Each is equal to $(A^*(\tilde{\mathcal{I}}))^*(\mathcal{I})$ by (e). (g) and (h) follow from (c) and (f), respectively. □

Theorem 3.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is $\mathcal{I}$-open and V is semiopen, then*
(a) *$V\cap A$ is $*$-dense in itself,*
(b) *$(V\cap A)^*$ is $*$-perfect and $\mathcal{I}$-locally closed,*
(c) *$\operatorname{cl}(V)\cap A$ is $*$-dense in itself,*
(d) *$(\operatorname{cl}(V)\cap A)^*$ is $*$-perfect and $\mathcal{I}$-locally closed.*

Proof. Since $V\cap A\subset\operatorname{cl}(V)\cap A\subset(V\cap A)^*$ by [1, Theorem 2.10], $V\cap A$ is $*$-dense in itself and by Theorem 3.1, $\operatorname{cl}(V)\cap A$ is $*$-dense in itself and so by Theorem 3.1, $(V\cap A)^*$ and $(\operatorname{cl}(V)\cap A)^*$ are $*$-perfect and so are $\mathcal{I}$-locally closed. □

The following theorem shows that (X,τ) and (X,τ^α) have the same collection of $\mathcal{I}$-open sets.

Theorem 3.6. *If $(X,\tau,\mathcal{I})$ is an ideal space, then $\mathrm{IO}(X,\tau,\mathcal{I}) = \mathrm{IO}(X,\tau^\alpha,\mathcal{I})$.*

Proof. $A \in \mathrm{IO}(X,\tau)$ if and only if $A \subset \mathrm{int}(A^*)$ if and only if $A \subset \mathrm{int}_\alpha(A^*)$, by Lemma 1.2 if and only if $A \in \mathrm{IO}(X,\tau^\alpha)$. □

Corollary 3.7. *If $(X,\tau,\mathcal{I})$ is an ideal space where $\mathcal{I}$ is completely codense, then $\mathrm{IO}(X,\tau) = \mathrm{IO}(X,\tau^*) = \mathrm{IO}(X,\tau^\alpha)$.*

Proof. Follows from Corollary 2.2(b). □

The following theorem and corollary are generalizations of [1, Theorem 2.6(iii) and Corollary 2.1(ii)], respectively.

Theorem 3.8. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $A \in \mathrm{IO}(X)$ and $B \in \tau^\alpha$, then $A \cap B \in \mathrm{IO}(X)$.*

Proof. $A \in \mathrm{IO}(X,\tau) \Rightarrow A \in \mathrm{IO}(X,\tau^\alpha)$ and so by [1, Theorem 2.6(ii)], $A\cap B \in \mathrm{IO}(X,\tau^\alpha)$ which implies that $A \cap B \in \mathrm{IO}(X,\tau)$. □

Corollary 3.9. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is $\mathcal{I}$-closed and B is α-closed, then $A\cup B$ is $\mathcal{I}$-closed.*

Every $\mathcal{I}$-open set is preopen but the converse need not be true [1, Example 2.3]. The following theorem characterizes $\mathcal{I}$-open sets in terms of preopen sets.

Theorem 3.10. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. Then the following are equivalent.*

(a) *A is $\mathcal{I}$-open.*

(b) *$A \subset A^*$ and $\mathrm{scl}(A) = \mathrm{int}(\mathrm{cl}(A))$.*

(c) *$A \subset A^*$ and A is preopen.*

Proof. $A \in \mathrm{IO}(X)$ if and only if $A \subset A^*$ and $A \subset \mathrm{int}(A^*)$ if and only if $A \subset A^*$ and $A \subset \mathrm{int}(\mathrm{cl}(A))$, since $\mathrm{cl}(A) = A^*$ if and only if $A \subset A^*$ and $A \cup \mathrm{int}(\mathrm{cl}(A)) = \mathrm{int}(\mathrm{cl}(A))$ if and only if $A \subset A^*$ and $\mathrm{scl}(A) = \mathrm{int}(\mathrm{cl}(A))$. Therefore, (a) and (b) are equivalent. It is clear that (a) and (c) are equivalent. □

Corollary 3.11. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$.*

(a) *If A is semiclosed and $\mathcal{I}$-open, then A is regular open.*

(b) *If A is semiopen and $\mathcal{I}$-closed, then A is regular closed.*

(c) *If A is $\mathcal{I}$-open, then $\mathrm{sint}(\mathrm{scl}(A)) = \mathrm{int}(\mathrm{scl}(A)) = \mathrm{int}(\mathrm{cl}(A))$.*

For subsets of any ideal space $(X,\tau,\mathcal{I})$, openness and $\mathcal{I}$-openness are independent concepts [1, Examples 2.1 and 2.2]. The following Theorem 3.12 shows that the two concepts coincide for $*$-perfect sets. Corollary 3.13 follows from the fact that every τ^*-closed, $\mathcal{I}$-open set is $*$-perfect.

Theorem 3.12. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$.*

(a) *If A is $*$-dense in itself, then $\mathrm{Iint}(A^*) = \mathrm{int}(A^*)$.*

(b) *If A is $*$-perfect, then* $\mathrm{Iint}(A) = \mathrm{int}(A)$ *and so, for $*$-perfect sets, the concepts open and $\mathcal{I}$-open coincide.*

Proof. Since A is $*$-dense in itself, A^* is $*$-perfect, by Theorem 3.1. Now $\mathrm{Iint}(A^*) = A^* \cap \mathrm{int}((A^*)^*) = A^* \cap \mathrm{int}(A^*) = \mathrm{int}(A^*)$. This proves (a). (b) follows from (a). □

Corollary 3.13. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. If A is τ^*-closed and $\mathcal{I}$-open, then A is open.*

In [17, Remark 4], it was stated that $\mathcal{I}$ is codense if and only if $\tau \subset \mathrm{IO}(X)$. The following Theorem 3.14(a) follows from the above result. Theorem 3.14(b) follows from Theorem 3.6 and the fact that $\mathrm{SO}(X) \cap \mathcal{I} = \{\phi\}$ if and only if $\tau \cap \mathcal{I} = \{\phi\}$. Theorem 3.15 is a characterization of completely codense ideals.

Theorem 3.14. *Let $(X,\tau,\mathcal{I})$ be an ideal space.*
(a) *If* $\mathrm{SO}(X) \subset \mathrm{IO}(X)$, *then $\mathcal{I}$ is codense.*
(b) *$\mathcal{I}$ is codense if and only if $\tau^\alpha \subset \mathrm{IO}(X)$.*

Theorem 3.15. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is completely codense if and only if* $\mathrm{PO}(X) = \mathrm{IO}(X)$.

Proof. Suppose $\mathcal{I}$ is completely codense and $G \in \mathrm{PO}(X)$. Then $G \subset G^*$, by Theorem 2.1(c) and so $\mathrm{cl}(G) = G^*$. $G \in \mathrm{PO}(X)$ implies $G \subset \mathrm{int}(\mathrm{cl}(G)) = \mathrm{int}(G^*)$ and so $G \in \mathrm{IO}(X)$. Therefore, $\mathrm{PO}(X) \subset \mathrm{IO}(X)$. Clearly, $\mathrm{IO}(X) \subset \mathrm{PO}(X)$. Conversely, if $G \in \mathrm{SPO}(X)$, then there exists $V \in \mathrm{PO}(X)$ such that $V \subset G \subset \mathrm{cl}(V)$ and by hypothesis, $V \subset V^*$ and so by Lemma 1.1, $\mathrm{cl}(V) = V^*$. Hence by Theorem 3.1, G is $*$-dense in itself and so by Theorem 2.1, $\mathcal{I}$ is completely codense. □

In the following Theorem 3.16, we show that if A is $\mathcal{I}$-open, then $\mathrm{sint}(A^*)$ is regular closed.

Theorem 3.16. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$.*
(a) *For every subset A of X,* $\mathrm{cl}(\mathrm{Iint}(A)) = \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{sint}(A^*)$.
(b) *If A is $\mathcal{I}$-open, then* $A^* = \mathrm{cl}(A) = \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{sint}(A^*)$ *and so* $\mathrm{sint}(A^*)$ *is regular closed.*

Proof. If A is a subset of X, then $\mathrm{sint}(A^*) = A^* \cap \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(A^*))$. To prove the other equality, since $\mathrm{Iint}(A) = A \cap \mathrm{int}(A^*)$, $\mathrm{cl}(\mathrm{Iint}(A)) = \mathrm{cl}(A \cap \mathrm{int}(A^*)) \supset \mathrm{cl}(A) \cap \mathrm{int}(A^*) = \mathrm{int}(A^*)$ and so $\mathrm{cl}(\mathrm{Iint}(A)) \supset \mathrm{cl}(\mathrm{int}(A^*))$. To prove the reverse direction, note that $\mathrm{Iint}(A) \subset \mathrm{int}(A^*)$ and so $\mathrm{cl}(\mathrm{Iint}(A)) \subset \mathrm{cl}(\mathrm{int}(A^*))$. This completes the proof of (a). (b) follows from (a) and Theorem 3.4(b). □

A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-dense [6] if $A^* = X$. Clearly, every $\mathcal{I}$-dense set is dense but the converse is not true. If G is any proper dense subset of an ideal space $(X,\tau,\mathcal{I})$ where $\mathcal{I}$ is the maximal ideal $\wp(X)$, then G is not $\mathcal{I}$-dense. In particular, if $\mathcal{I}$ is not codense, then X is not $\mathcal{I}$-dense and hence no subset of X is $\mathcal{I}$-dense [6]. Therefore, the existence of an $\mathcal{I}$-dense set implies that the ideal is codense. The following theorem characterizes $\mathcal{I}$-open sets in terms of $\mathcal{I}$-dense sets.

Theorem 3.17. *Let $(X,\tau,\mathcal{I})$ be an ideal space with a codense ideal $\mathcal{I}$ and $A \subset X$. Then the following are equivalent.*
(a) *A is $\mathcal{I}$-open.*
(b) *There is a regular open set G such that $A \subset G$ and $A^* = G^*$.*
(c) *$A = G \cap D$ where G is regular open and D is $\mathcal{I}$-dense.*
(d) *$A = G \cap D$ where G is open and D is $\mathcal{I}$-dense.*

Proof. (a)⇒(b). That A is $\mathcal{I}$-open implies $A \subset \operatorname{int}(A^*) \subset A^*$. Let $G = \operatorname{int}(A^*)$. Then $A \subset G$ and $\operatorname{int}(\operatorname{cl}(G)) = \operatorname{int}(\operatorname{cl}(\operatorname{int}(A^*))) = \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))) = \operatorname{int}(\operatorname{cl}(A)) = \operatorname{int}(A^*) = G$ and so G is regular open. $G^* = (\operatorname{int}(A^*))^* = A^*$, by Theorem 3.4(f).

(b)⇒(c). Let G be a regular open set such that $A \subset G$ and $A^* = G^*$. Let $D = A \cup (X - G)$. Then $A = G \cap D$ where G is regular open. Now $D^* = (A \cup (X-G))^* = A^* \cup (X-G)^* = G^* \cup (X-G)^* = (G \cup (X-G))^* = X^* = X$, since $\mathcal{I}$ is codense. Therefore, D is $\mathcal{I}$-dense which proves (c).

(c)⇒(d) is clear.

(d)⇒(a). Suppose $A = G \cap D$ where G is open and D is $\mathcal{I}$-dense. Now $G = G \cap X = G \cap D^* \subset (G \cap D)^*$ and so $G \subset \operatorname{int}((G \cap D)^*) = \operatorname{int}(A^*)$. Therefore, $A \subset G \subset \operatorname{int}(A^*)$ which implies that A is $\mathcal{I}$-open. □

The following theorem is a generalization of [1, Theorem 2.14(ii)].

Theorem 3.18. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. If A is $\mathcal{I}$-closed and α-open, then $A = \operatorname{cl}(A) = \operatorname{int}(\operatorname{cl}(A)) = \operatorname{cl}(\operatorname{int}(A))$ and so A is both regular open and regular closed.*

Proof. A is $\mathcal{I}$-closed ⇒ $X - A$ is $\mathcal{I}$-open ⇒ $X - A \subset \operatorname{int}(X-A)^*$ ⇒ $X - A \subset \operatorname{int}(\operatorname{cl}(X - A))$ ⇒ $X - A \subset X - \operatorname{cl}(\operatorname{int}(A))$ ⇒ $\operatorname{cl}(\operatorname{int}(A)) \subset A$. A is α-open ⇒ A is semiopen and preopen [16] ⇒ $\operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(A))$ and $A \subset \operatorname{int}(\operatorname{cl}(A))$. Therefore, $\operatorname{int}(\operatorname{cl}(A)) \subset \operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(A)) \subset A \subset \operatorname{int}(\operatorname{cl}(A))$ and so $A = \operatorname{cl}(A) = \operatorname{cl}(\operatorname{int}(A)) = \operatorname{int}(\operatorname{cl}(A))$. □

4. Quasi-$\mathcal{I}$-open sets. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is quasi-$\mathcal{I}$-open [2] if $A \subset \operatorname{cl}(\operatorname{int}(A^*))$. Every $\mathcal{I}$-open set is quasi-$\mathcal{I}$-open and every quasi-$\mathcal{I}$-open set is semi-preopen but the converse implications need not be true [2, Examples 1 and 2]. Also, quasi-$\mathcal{I}$-openness and semiopenness (resp., preopenness) are independent concepts [2, Examples 1 and 2]. The family of all quasi-$\mathcal{I}$-open sets is denoted by $Q\mathcal{I}O(X,\tau)$. The following theorem gives some of the properties of quasi-$\mathcal{I}$-open sets, the proof of which is similar to the proof of Theorem 3.4.

Theorem 4.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and A a quasi-$\mathcal{I}$-open subset of X. Then*
(a) *A is $*$-dense in itself,*
(b) *$A^* = \operatorname{cl}(A) = \operatorname{cl}^*(A)$,*
(c) *A^* is $*$-perfect, regular closed, and $\mathcal{I}$-locally closed,*
(d) *$\operatorname{cl}(\operatorname{int}(A^*)) = A^*(\tilde{\mathcal{I}})$ is $*$-dense in itself,*
(e) *$A^* = (\operatorname{cl}(\operatorname{int}(A^*)))^* = (A^*(\tilde{\mathcal{I}}))^*(\mathcal{I})$,*
(f) *$(\operatorname{cl}(\operatorname{int}(A^*)))^*$ is $*$-perfect and $\mathcal{I}$-locally closed.*

Corollary 4.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space. A subset A of X is quasi-$\mathcal{I}$-open if and only if* $A \subset A^*(\tilde{\mathcal{I}})$ [2, Theorem 3].

Theorem 4.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space and let U and A be subsets of X such that $A \subset U \subset A^*$. Then U^* is $*$-perfect, and if A is quasi-$\mathcal{I}$-open, then U is quasi-$\mathcal{I}$-open and so* $\text{cl}(\text{int}(A^*))$ *is quasi-$\mathcal{I}$-open.*

Proof. By Theorem 3.1, $U^* = A^*$ and U^* is $*$-perfect. A is quasi-$\mathcal{I}$-open $\Rightarrow A \subset \text{cl}(\text{int}(A^*)) = \text{cl}(\text{int}(U^*))$. Now $U \subset A^* \Rightarrow U \subset (\text{cl}(\text{int}(U^*)))^* \Rightarrow U \subset \text{cl}(\text{cl}(\text{int}(U^*))) = \text{cl}(\text{int}(U^*))$. Therefore, U is quasi-$\mathcal{I}$-open. Since $A \subset \text{cl}(\text{int}(A^*)) \subset A^*$, $\text{cl}(\text{int}(A^*))$ is quasi-$\mathcal{I}$-open. □

Every quasi-$\mathcal{I}$-open set is semi-preopen but the converse is not true [2]. [2, Proposition 3(iii)] says that every semiopen set which is $*$-dense in itself is quasi-$\mathcal{I}$-open. The following Theorem 4.4 is a generalization of this result and shows that for $*$-dense in itself, the concepts quasi-$\mathcal{I}$-open and semi-preopen are equivalent. Theorem 4.5(a) gives a characterization of codense ideals and Theorem 4.5(b) gives a characterization of completely codense ideals.

Theorem 4.4. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A is semi-preopen and $*$-dense in itself, then A is quasi-$\mathcal{I}$-open.*

Proof. $A \subset A^* \Rightarrow \text{cl}(A) = A^*$, by Lemma 1.1. A is semi-preopen $\Rightarrow A \subset \text{cl}(\text{int}(\text{cl}(A))) = \text{cl}(\text{int}(A^*))$ and so A is quasi-$\mathcal{I}$-open. □

Theorem 4.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then*

(a) *$\mathcal{I}$ is codense if and only if* $\text{SO}(X) \subset Q\mathcal{I}O(X)$,

(b) *$\mathcal{I}$ is completely codense if and only if* $\text{SPO}(X) = Q\mathcal{I}O(X)$.

Proof. (a) Suppose $\mathcal{I}$ is codense. Let $G \in \text{SO}(X)$. By [10, Theorem 6.1] and Lemma 1.3, G is $*$-dense in itself and so by [2, Proposition 3(iii)], $G \in Q\mathcal{I}O(X)$. Conversely, suppose that $\text{SO}(X) \subset Q\mathcal{I}O(X)$. If $G \in \text{SO}(X)$, then $G \in Q\mathcal{I}O(X)$ and so $G \subset G^*$. Therefore, $\mathcal{I}$ is codense by [10, Theorem 6.1] and Lemma 1.3.

(b) Suppose $\mathcal{I}$ is completely codense and $G \in \text{SPO}(X)$. Then $G \subset G^*$, by Theorem 2.1(c) and so $\text{cl}(G) = G^*$. $G \in \text{SPO}(X) \Rightarrow G \subset \text{cl}(\text{int}(\text{cl}(G))) = \text{cl}(\text{int}(G^*))$ and so $G \in Q\mathcal{I}O(X)$. Therefore, $\text{SPO}(X) \subset Q\mathcal{I}O(X)$. Clearly, $Q\mathcal{I}O(X) \subset \text{SPO}(X)$. Conversely, if $G \in \text{SPO}(X)$, then $G \in Q\mathcal{I}O(X)$, by hypothesis, and so $G \subset G^*$, and so by Theorem 2.1(c), $\mathcal{I}$ is completely codense. □

In [2], it was established that the intersection of a quasi-$\mathcal{I}$-open set with an α-set is semi-preopen. The following theorem is a generalization of the above result.

Theorem 4.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then (a) $Q\mathcal{I}O(X,\tau) = Q\mathcal{I}O(X,\tau^\alpha)$ and (b) $A \in Q\mathcal{I}O(X,\tau)$ and $B \in \tau^\alpha$ implies $A \cap B \in Q\mathcal{I}O(X,\tau)$.*

Proof. $A \in Q\mathcal{I}O(X,\tau)$ if and only if $A \subset \text{cl}(\text{int}(A^*))$ if and only if $A \subset \text{cl}_\alpha(\text{int}_\alpha(A^*))$ [3] if and only if $A \in Q\mathcal{I}O(X,\tau^\alpha)$ which proves (a). $A \in Q\mathcal{I}O(X,\tau)$ and $B \in \tau^\alpha \Rightarrow A \in Q\mathcal{I}O(X,\tau^\alpha)$ and $B \in \tau^\alpha \Rightarrow A \cap B \in Q\mathcal{I}O(X,\tau^\alpha)$; by [2, Proposition 2] implies $A \cap B \in Q\mathcal{I}O(X,\tau)$. □

[2, Lemma 2] states that $W^*(\mathcal{N}) \subset W$ for every subset W of X in the ideal space $(X,\tau,\mathcal{N})$. That is, every subset of X is τ^*-closed and so τ^* is the discrete topology. This is not always the case. For example, if we consider $\mathbb{R}$ with the usual topology τ and the ideal $\mathcal{N}$ of nowhere dense subsets of $\mathbb{R}$, then $Q^* = \mathbb{R}$ and so Q is not τ^*-closed. Therefore, [2, Proposition 4] is no longer valid. Also, it was established that every τ^*-closed, quasi-$\mathcal{I}$-open set is semiopen [2, Proposition 3(iii)]. The following Theorem 4.7(a) is a generalization of the above result and also shows that the condition *preclosed* is not necessary in [2, Proposition 5(i)], and Theorem 4.7(b) shows that [2, Proposition 3(iii)] is also true if we replace the condition τ^*-closed by semiclosed.

Theorem 4.7. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$.*
(a) *If A is τ^*-closed and quasi-$\mathcal{I}$-open, then A is regular closed.*
(b) *If A is semiclosed and quasi-$\mathcal{I}$-open, then A is semiopen and $A^* = A^*(\mathcal{N})$.*

Proof. (a) That A is τ^*-closed and quasi-$\mathcal{I}$-open implies $A = A^*$. Also, $A \in Q\mathcal{I}O(X) \Rightarrow A \subset \mathrm{cl}(\mathrm{int}(A^*)) \Rightarrow \mathrm{int}(A^*) \subset A^* \subset \mathrm{cl}(\mathrm{int}(A^*)) \Rightarrow \mathrm{cl}(\mathrm{int}(A^*)) \subset A^* \subset \mathrm{cl}(\mathrm{int}(A^*))$. Therefore, $A = A^* = \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(A))$ and so A and A^* are regular closed. (b) A is semiclosed $\Rightarrow \mathrm{int}(A) = \mathrm{int}(\mathrm{cl}(A))$ by [8, Proposition 1]. That A is quasi-$\mathcal{I}$-open implies $A \subset \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{int}(A))$ and so A is semiopen. By Theorem 4.1(b), $\mathrm{cl}(A) = A^*$. Since $\mathrm{int}(\mathrm{cl}(A)) \subset A \subset \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$, $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subset \mathrm{cl}(A) \subset \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ and so $A^* = \mathrm{cl}(A) = \mathrm{cl}(\mathrm{int}(A^*)) = A^*(\mathcal{N})$. □

The following theorem gives a characterization of quasi-$\mathcal{I}$-open sets.

Theorem 4.8. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. A is quasi-$\mathcal{I}$-open if and only if $A \subset A^*$ and $\mathrm{cl}_\alpha(A) = \mathrm{cl}(\mathrm{int}(A^*))$.*

Proof. Suppose $A \in Q\mathcal{I}O(X)$. Then $A \subset A^*$ and $\mathrm{cl}(A) = A^*$. Also $A \subset \mathrm{cl}(\mathrm{int}(A^*)) \Rightarrow A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \Rightarrow A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \Rightarrow \mathrm{cl}_\alpha(A) = \mathrm{cl}(\mathrm{int}(A^*))$, since $\mathrm{cl}_\alpha(A) = A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ [3]. Conversely, suppose the conditions hold. Then $\mathrm{cl}_\alpha(A) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$ and so $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) = \mathrm{cl}(\mathrm{int}(A^*))$. Therefore, A is quasi-$\mathcal{I}$-open. □

The quasi-$\mathcal{I}$-interior of a subset A in an ideal space $(X,\tau,\mathcal{I})$ is the largest quasi-$\mathcal{I}$-open set contained in A and is denoted by $\mathrm{qIint}(A)$. The following theorem deals with the properties of the quasi-$\mathcal{I}$-interior of subsets of ideal spaces. In [11], it was established that $\mathrm{Iint}(A) = \phi$ if and only if $A \in \tilde{\mathcal{I}}$. Theorem 4.9(c) is a partial generalization of this result.

Theorem 4.9. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. Then*
(a) $\mathrm{qIint}(A) = A \cap \mathrm{cl}(\mathrm{int}(A^*))$ *for every subset A of X,*
(b) *if A is α-closed, then* $\mathrm{qIint}(A) = \mathrm{cl}(\mathrm{int}(A^*))$ *and the converse holds if $A \subset A^*$,*
(c) $\mathrm{qIint}(A) = \phi$ *if and only if $A \in \tilde{\mathcal{I}}$.*

Proof. (a) $A \cap \mathrm{cl}(\mathrm{int}(A^*)) \subset \mathrm{cl}(\mathrm{int}(A^*)) = \mathrm{cl}(\mathrm{int}(\mathrm{int}(A^*))) = \mathrm{cl}(\mathrm{int}(A^* \cap (\mathrm{int} A^*))) \subset \mathrm{cl}(\mathrm{int}((A \cap \mathrm{int}(A^*))^*)) \subset \mathrm{cl}(\mathrm{int}((A \cap \mathrm{cl}(\mathrm{int}(A^*)))^*))$. Therefore, $A \cap \mathrm{cl}(\mathrm{int}(A^*))$ is a quasi-$\mathcal{I}$-open set contained in A and so $A \cap \mathrm{cl}(\mathrm{int}(A^*)) \subset \mathrm{qIint}(A)$. Since $\mathrm{qIint}(A)$ is quasi-$\mathcal{I}$-open, $\text{qIint}(A) \subset \text{cl}(\text{int}(\text{qIint}(A))^*) \subset \text{cl}(\text{int}(A^*))$ and so $A \cap \text{qIint}(A) \subset A \cap \text{cl}(\text{int}(A^*))$ which implies that $\text{qIint}(A) \subset A \cap \text{cl}(\text{int}(A^*))$. Hence $\text{qIint}(A) = A \cap \text{cl}(\text{int}(A^*))$.

(b) A is α-closed $\Rightarrow \text{cl}(\text{int}(\text{cl}(A))) \subset A \Rightarrow \text{cl}(\text{int}(A^*)) \subset A \Rightarrow \text{qIint}(A) = \text{cl}(\text{int}(A^*))$. Conversely, if $A \subset A^*$, then $A^* = \text{cl}(A)$. $\text{qIint}(A) = \text{cl}(\text{int}(A^*)) \Rightarrow \text{cl}(\text{int}(A^*)) \subset A$ and so $\text{cl}(\text{int}(\text{cl}(A))) \subset A$ and so A is α-closed.

(c) $\text{qIint}(A) = \phi \Rightarrow A \cap \text{cl}(\text{int}(A^*)) = \phi \Rightarrow A \cap \text{int}(A^*) = \phi \Rightarrow \text{Iint}(A) = \phi \Rightarrow A \in \tilde{\mathcal{I}}$. Conversely, $A \in \tilde{\mathcal{I}} \Rightarrow \text{int}(A^*) = \phi \Rightarrow \text{cl}(\text{int}(A^*)) = \phi \Rightarrow A \cap \text{cl}(\text{int}(A^*)) = \phi \Rightarrow \text{qIint}(A) = \phi$. □

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References

[1] M. E. Abd El-Monsef, E. F. Lashien, and A. A. Nasef, *On I-open sets and I-continuous functions*, Kyungpook Math. J. **32** (1992), no. 1, 21–30.
[2] M. E. Abd El-Monsef, R. A. Mahmoud, and A. A. Nasef, *On quasi I-openness and quasi I-continuity*, Tamkang J. Math. **31** (2000), no. 2, 101–108.
[3] D. Andrijević, *Some properties of the topology of α-sets*, Mat. Vesnik **36** (1984), no. 1, 1–10.
[4] ______, *Semipreopen sets*, Mat. Vesnik **38** (1986), no. 1, 24–32.
[5] J. Dontchev, *On pre-$\mathcal{I}$-open sets and a decomposition of I-continuity*, Banyan Math. J. **2** (1996).
[6] J. Dontchev, M. Ganster, and D. A. Rose, *Ideal resolvability*, Topology Appl. **93** (1999), no. 1, 1–16.
[7] M. Ganster, *Preopen sets and resolvable spaces*, Kyungpook Math. J. **27** (1987), no. 2, 135–143.
[8] M. Ganster, F. Gressl, and I. L. Reilly, *On a decomposition of continuity*, General Topology and Applications (Staten Island, NY, 1989), Lecture Notes in Pure and Appl. Math., vol. 134, Dekker, New York, 1991, pp. 67–72.
[9] E. Hayashi, *Topologies defined by local properties*, Math. Ann. **156** (1964), 205–215.
[10] D. Janković and T. R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly **97** (1990), no. 4, 295–310.
[11] ______, *Compatible extensions of ideals*, Boll. Un. Mat. Ital. B (7) **6** (1992), no. 3, 453–465.
[12] K. Kuratowski, *Topology. Vol. I*, Academic Press, New York, 1966.
[13] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70** (1963), 36–41.
[14] A. S. Mashhour, M. E. Abd El-Monsef, and S. N. El-Deep, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.
[15] O. Njåstad, *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.
[16] I. L. Reilly and M. K. Vamanamurthy, *On α-continuity in topological spaces*, Acta Math. Hungar. **45** (1985), no. 1-2, 27–32.
[17] D. A. Rose and D. Janković, *On functions having the property of Baire*, Real Anal. Exchange **19** (1993/1994), no. 2, 589–597.
[18] D. Sivaraj, *A note on S-closed spaces*, Acta Math. Hungar. **44** (1984), no. 3-4, 207–213.
[19] R. Vaidyanathaswamy, *The localisation theory in set-topology*, Proc. Indian Acad. Sci. Sect. A **20** (1944), 51–61.
[20] ______, *Set Topology*, 2nd ed., Chelsea Publishing, New York, 1960.

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