

INCLUSION RESULTS FOR CONVOLUTION SUBMETHODS

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If B is a summability matrix, then the submethod B_λ is the matrix obtained by deleting a set of rows from the matrix B . Comparisons between Euler-Knopp submethods and the Borel summability method are made. Also, an equivalence result for convolution submethods is established. This result will necessarily apply to the submethods of the Euler-Knopp, Taylor, Meyer-König, and Borel matrix summability methods.

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1. Introduction and notation. Let E be an infinite subset of $\mathbb{N} \cup \{0\}$ and consider E as the range of a strictly increasing sequence of nonnegative integers, say $E := \{\lambda(n)\}_{n=0}^\infty$. If $B := (b_{n,k})$ is a summability matrix, then the submethod B_λ is the matrix whose nk th entry is $B_\lambda[n, k] := b_{\lambda(n), k}$. Thus, for a given sequence x , the B_λ -transform of x is the sequence $B_\lambda x$ with

$$(B_\lambda x)_n = (Bx)_{\lambda(n)} := \sum_{k=0}^{\infty} b_{\lambda(n), k} x_k. \quad (1.1)$$

Since B_λ is a row submatrix of B , it is regular (i.e., limit preserving) whenever B is regular.

Row submatrices have appeared throughout the literature [5, 6, 8, 12], but they were first studied as a class unto themselves by Goffman and Petersen [7], and later by Steele [14]. The class of Cesàro submethods has been studied by Armitage and Maddox [1] and Osikiewicz [11].

Let A and B be two summability matrices. If every sequence which is A -summable is also B -summable to the same limit, then B includes A , denoted by $A \subseteq B$. Also, B is called a triangle if $b_{n,k} = 0$ for all $k > n$ and $b_{n,n} \neq 0$ for all n . The following lemma extends [1, Theorem 1].

LEMMA 1.1. *Let B be a summability matrix and let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$.*

- (1) *If $F \setminus E$ is finite, then $B_\lambda \subseteq B_\rho$.*
- (2) *If B is a triangle and $B_\lambda \subseteq B_\rho$, then $F \setminus E$ is finite.*
- (3) *If B is a triangle, then B_λ is equivalent to B_ρ if and only if the symmetric difference $E \Delta F$ is finite.*

In particular, $B \subseteq B_\lambda$ for any λ .

PROOF. Assume $F \setminus E$ is finite and let x be a sequence that is B_λ -summable to L . Then there exists an N such that $\{\rho(n) : n \geq N\} \subseteq E$. That is, $\{\rho(n) : n \geq N\}$ is a

subsequence of $\{\lambda(n)\}$. Since $\lim_n (B_\lambda x)_n = \lim_n (Bx)_{\lambda(n)} = L$, we have $\lim_n (B_\rho x)_n = \lim_n (Bx)_{\rho(n)} = L$.

Now assume B is a triangle, and hence invertible, and $F \setminus E$ is infinite. Let $F \setminus E := \{\rho(n(j))\}_{j=0}^\infty$ with $\rho(n(j)) < \rho(n(j+1))$. Consider the sequence y defined by

$$y_k := \begin{cases} (-1)^j, & \text{if } k = \rho(n(j)) \text{ for some } j, \\ 0, & \text{otherwise,} \end{cases} \quad (1.2)$$

and let x be the sequence $B^{-1}y$. Then, for every n ,

$$(B_\lambda x)_n = (Bx)_{\lambda(n)} = (B(B^{-1}y))_{\lambda(n)} = y_{\lambda(n)} = 0. \quad (1.3)$$

Hence, $\lim_n (B_\lambda x)_n = 0$. However, for every j ,

$$(B_\rho x)_{n(j)} = (Bx)_{\rho(n(j))} = (B(B^{-1}y))_{\rho(n(j))} = y_{\rho(n(j))} = (-1)^j. \quad (1.4)$$

Thus x is not B_ρ -summable. Therefore B_ρ does not include B_λ , which completes the contrapositive of assertion (2). Lastly, assertion (3) follows from (1) and (2) since $E \triangle F := (E \setminus F) \cup (F \setminus E)$. \square

To show the reason for the necessity of B being a triangle in assertion (2) of [Lemma 1.1](#), consider the matrix B whose nk th entry is

$$B[n, k] := \begin{cases} 0, & \text{if } n \text{ even and } k \neq \frac{n}{2}, \\ 1, & \text{if } n \text{ even and } k = \frac{n}{2}, \\ 0, & \text{if } n \text{ odd and } n \neq k, \\ 1, & \text{if } n \text{ odd and } n = k. \end{cases} \quad (1.5)$$

Then if $\lambda(n) := 2n$ and $\rho(n) := 2n+1$, $F \setminus E$ is infinite and $B_\lambda \subseteq B_\rho$.

2. Inclusion results for Euler-Knopp submethods. For $r \in \mathbb{C} \setminus \{0, 1\}$, the Euler-Knopp method of order r is given by the matrix E_r whose nk th entry is

$$E_r[n, k] := \begin{cases} \binom{n}{k} r^k (1-r)^{n-k}, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases} \quad (2.1)$$

For the case $r = 1$, E_1 is the identity matrix, and E_0 is the matrix whose nk th entry is

$$E_0[n, k] := \begin{cases} 1, & \text{if } k = 0, n = 0, 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases} \quad (2.2)$$

It is well known that E_r is regular if and only if $0 < r \leq 1$ (see [\[4\]](#)).

Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $r \in \mathbb{C} \setminus \{0, 1\}$. The submethod $E_{r,\lambda}$ is the matrix whose nk th entry is

$$E_{r,\lambda}[n, k] := \begin{cases} \binom{\lambda(n)}{k} r^k (1-r)^{\lambda(n)-k}, & \text{if } k \leq \lambda(n), \\ 0, & \text{if } k > \lambda(n). \end{cases} \quad (2.3)$$

Then $E_{r,\lambda}$ is regular if and only if E_r is regular.

By a direct application of [Lemma 1.1](#), we have the following inclusion result for the $E_{r,\lambda}$ methods.

LEMMA 2.1. *Let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$ and $r \neq 0$.*

- (1) *The method $E_{r,\lambda} \subseteq E_{r,\rho}$ if and only if $F \setminus E$ is finite.*
- (2) *The method $E_{r,\lambda}$ is equivalent to $E_{r,\rho}$ if and only if the symmetric difference $E \Delta F$ is finite.*

We now examine the relationship between $E_{r,\lambda}$ and the Borel summability method. Recall that a sequence x is Borel summable to L if

$$\lim_{t \rightarrow \infty} e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} = L. \quad (2.4)$$

THEOREM 2.2. *Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $r > 0$. Then the Borel summability method includes $E_{r,\lambda}$ if and only if $S := (\mathbb{N} \cup \{0\}) \setminus E$ is finite.*

PROOF. If S is finite, then by [Lemma 2.1](#), E_r and $E_{r,\lambda}$ are equivalent. But the Borel summability method includes E_r for $r > 0$ (see [4]). Hence, it also includes $E_{r,\lambda}$. If S is infinite, then it may be written as a strictly increasing sequence of nonnegative integers, say $S := \{\rho(m)\}_{m=0}^{\infty}$. If $M_n := \max_{0 \leq k \leq n} |E_r[n, k]|$, consider the sequence y defined by

$$y_n := \begin{cases} (\rho(m)!)^2 (\rho(m)+1) M_{\rho(m)}, & \text{if } n = \rho(m), \\ 0, & \text{otherwise,} \end{cases} \quad (2.5)$$

and let x be the sequence $E_r^{-1}y$; that is, $y = E_r x$ and

$$\lim_{n \rightarrow \infty} (E_{r,\lambda} x)_n = \lim_{n \rightarrow \infty} (E_r x)_{\lambda(n)} = \lim_{n \rightarrow \infty} y_{\lambda(n)} = 0. \quad (2.6)$$

Hence, x is $E_{r,\lambda}$ -summable to 0. Now observe that for a given n ,

$$|y_n| = |(E_r x)_n| \leq \sum_{k=0}^n |E_r[n, k]| |x_k| \leq M_n \sum_{k=0}^n |x_k|. \quad (2.7)$$

Thus, for $n = \rho(m)$, we have

$$\begin{aligned} (\rho(m)!)^{1/\rho(m)} &= \left(\frac{1}{\rho(m)!} \cdot \frac{1}{\rho(m)+1} \cdot \frac{|\mathcal{Y}_{\rho(m)}|}{M_{\rho(m)}} \right)^{1/\rho(m)} \\ &\leq \left(\frac{1}{\rho(m)!} \cdot \frac{1}{\rho(m)+1} \sum_{k=0}^{\rho(m)} |x_k| \right)^{1/\rho(m)}. \end{aligned} \quad (2.8)$$

Since $\limsup_m (\rho(m)!)^{1/\rho(m)} = \infty$,

$$\limsup_{m \rightarrow \infty} \left(\frac{1}{\rho(m)!} \cdot \frac{1}{\rho(m)+1} \sum_{k=0}^{\rho(m)} |x_k| \right)^{1/\rho(m)} = \infty, \quad (2.9)$$

and it follows that $\limsup_n (|x_n|/n!)^{1/n} = \infty$. Thus, $\sum_{k=0}^{\infty} (x_k/k!)t^k$ diverges for all nonzero t and hence x is not Borel summable. \square

THEOREM 2.3. *There exists a sequence which is Borel summable but not $E_{r,\lambda}$ -summable for any λ and $r > 0$.*

PROOF. Let $r > 0$ and consider the sequence x defined by

$$x_n := n \left(-\frac{1}{r} \right) \left(1 - \frac{2}{r} \right)^{n-1}. \quad (2.10)$$

Then it can be shown that $(E_{r,\lambda}x)_n = (-1)^{\lambda(n)}\lambda(n)$. Hence x is not $E_{r,\lambda}$ -summable for any λ . However,

$$\begin{aligned} e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} &= e^{-t} \sum_{k=1}^{\infty} \left[k \left(-\frac{1}{r} \right) \left(1 - \frac{2}{r} \right)^{k-1} \right] \frac{t^k}{k!} \\ &= \left(-\frac{1}{r} \right) e^{-t} \sum_{k=1}^{\infty} \left(1 - \frac{2}{r} \right)^{k-1} \frac{t^k}{(k-1)!} \\ &= \left(-\frac{1}{r} \right) t e^{-t} \sum_{k=0}^{\infty} \left(1 - \frac{2}{r} \right)^k \frac{t^k}{k!} \\ &= \left(-\frac{1}{r} \right) t e^{-t} e^{(1-2/r)t} \\ &= \left(-\frac{1}{r} \right) t e^{-(2/r)t}. \end{aligned} \quad (2.11)$$

Since $r > 0$,

$$\lim_{t \rightarrow \infty} e^{-t} \sum_{k=0}^{\infty} x_k \frac{t^k}{k!} = \lim_{t \rightarrow \infty} \left(-\frac{1}{r} \right) t e^{-(2/r)t} = 0, \quad (2.12)$$

and hence x is Borel summable to 0. \square

3. Convolution methods. Let p and q be sequences of real numbers with $p_k \geq 0$, $q_k \geq 0$, $\sum_{k=0}^{\infty} p_k = 1$, and $\sum_{k=0}^{\infty} q_k = 1$. The convolution summability method is given by the matrix $C^* := (c_{n,k})$ whose nk th entry is

$$c_{n,k} := \begin{cases} q_k, & \text{if } n = 0, \\ \sum_{j=0}^k c_{n-1,j} p_{k-j}, & \text{if } n \geq 1. \end{cases} \quad (3.1)$$

It is clear that C^* is a nonnegative matrix such that for every n , $\sum_{k=0}^{\infty} c_{n,k} = 1$. Some classical summability matrices are examples of the matrix C^* . If $0 \leq r \leq 1$, $p := \{1-r, r, 0, 0, \dots\}$, and $q := \{1, 0, 0, \dots\}$, then C^* is the Euler-Knopp method of order r . If $0 \leq r < 1$, $p := \{0, (1-r), (1-r)r, (1-r)r^2, \dots\}$, and $q := \{(1-r), (1-r)r, (1-r)r^2, \dots\}$, then C^* is the Taylor method of order r , denoted by T_r . If $0 < r < 1$ and $p := q := \{(1-r), (1-r)r, (1-r)r^2, \dots\}$, then C^* is the Meyer-König method of order r , denoted by S_r . If $p := q := \{1/k!e\}$, then C^* is the Borel matrix method B^* . Similar forms of the convolution method are known by different names, such as the random-walk method and Sonnenschein method. (Further information on all of these methods may be found in [3, 4, 13].)

If C^* is the convolution method formed from the sequences p and q , then let

$$\mu := \sum_{j=0}^{\infty} j p_j, \quad \nu := \sum_{j=0}^{\infty} j q_j. \quad (3.2)$$

We note here that for the remainder of this work, p and q are nonnegative sequences whose sums are 1, and μ and ν represent the sums in (3.2). Also, $c_{n,k} := 0$ whenever $k < 0$.

We next present some preliminary results concerning the convolution method.

LEMMA 3.1. *The convolution method C^* is regular if and only if $p_0 < 1$.*

PROOF. See [9]. □

LEMMA 3.2. *If $\mu < \infty$ and $\nu < \infty$, then for every n ,*

$$\sum_{k=0}^{\infty} k c_{n,k} = n\mu + \nu. \quad (3.3)$$

PROOF. Note that for $n = 0$, the result holds. So assume the result holds for some integer $n > 0$. Then

$$\begin{aligned} \sum_{k=0}^{\infty} k c_{n+1,k} &= \sum_{k=0}^{\infty} k \left(\sum_{j=0}^k c_{n,j} p_{k-j} \right) = \sum_{j=0}^{\infty} c_{n,j} \sum_{k=j}^{\infty} k p_{k-j} \\ &= \sum_{j=0}^{\infty} c_{n,j} \left(\sum_{i=0}^{\infty} i p_i + j \sum_{i=0}^{\infty} p_i \right) = \sum_{j=0}^{\infty} \mu c_{n,j} + \sum_{j=0}^{\infty} j c_{n,j} = (n+1)\mu + \nu. \end{aligned} \quad (3.4)$$

By induction, the result follows. □

LEMMA 3.3. *Let C^* be the convolution method formed from the sequences p and q and $D^* := (d_{n,k})$ the convolution method formed from the sequences p and $\tilde{q} := \{1, 0, 0, \dots\}$. Then for nonnegative integers n, k , and j ,*

$$c_{n+j,k} = \sum_{i=0}^k c_{n,k-i} d_{j,i}. \quad (3.5)$$

The proof of this lemma is a straightforward induction argument left to the reader.

LEMMA 3.4. *Let C^* be the convolution method formed from the sequences p and q . If $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty} (j - \mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$, then*

$$\sum_{k=0}^{\infty} |c_{n,k+1} - c_{n,k}| = O\left(\frac{1}{\sqrt{n}}\right). \quad (3.6)$$

PROOF. Let $D^* := (d_{n,k})$ be the convolution method formed from the sequences p and $\tilde{q} := \{1, 0, 0, \dots\}$. We first prove that the result holds for D^* .

Let $\phi(t) := (\sqrt{2\pi}e^{t^2/2})^{-1}$ and $x_{n,k} := (k - n\mu)/\sigma\sqrt{n}$, where $\sigma^2 := \sum_{j=0}^{\infty} (j - \mu)^2 p_j$. Then

$$\begin{aligned} \sqrt{n} \sum_{k=0}^{\infty} |d_{n,k+1} - d_{n,k}| &\leq \sqrt{n} \sum_{k=0}^{\infty} \left| d_{n,k+1} - \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k+1}) \right| \\ &\quad + \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k+1}) - \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k}) \right| \\ &\quad + \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k}) - d_{n,k} \right|. \end{aligned} \quad (3.7)$$

The first and the third terms on the right-hand side of the inequality are bounded by a result of Bikjalis and Jasjunas [2]. For the middle term, the mean value theorem yields

$$\begin{aligned} \sqrt{n} \sum_{k=0}^{\infty} \left| \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k+1}) - \frac{1}{\sigma\sqrt{n}} \phi(x_{n,k}) \right| &= \frac{1}{\sigma} \sum_{k=0}^{\infty} |\phi'(\xi_{n,k})| (x_{n,k+1} - x_{n,k}) \\ &< \frac{K}{\sigma} \int_{\mathbb{R}} |\phi'(t)| dt < \infty, \end{aligned} \quad (3.8)$$

where $\xi_{n,k} \in (x_{n,k}, x_{n,k+1})$ and $K > 0$ is some constant. Thus, the result holds for the convolution method D^* . Then, by Lemma 3.3,

$$\begin{aligned} \sum_{k=0}^{\infty} |c_{n,k+1} - c_{n,k}| &= \sum_{k=0}^{\infty} \left| \sum_{i=0}^{k+1} q_{k+1-i} d_{n,i} - \sum_{i=0}^k q_{k-i} d_{n,i} \right| \\ &= \sum_{k=0}^{\infty} \left| q_{k+1} d_{n,0} + \sum_{i=1}^{k+1} q_{k+1-i} d_{n,i} - \sum_{i=0}^k q_{k-i} d_{n,i} \right| \end{aligned}$$

$$\begin{aligned}
&\leq p_0^n \sum_{k=0}^{\infty} q_{k+1} + \sum_{k=0}^{\infty} \sum_{i=0}^k q_{k-i} |d_{n,i+1} - d_{n,i}| \\
&\leq p_0^n + \sum_{i=0}^{\infty} |d_{n,i+1} - d_{n,i}| \sum_{k=i}^{\infty} q_{k-i} \\
&= p_0^n + \sum_{i=0}^{\infty} |d_{n,i+1} - d_{n,i}| = O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned} \tag{3.9}$$

□

4. Equivalence results for convolution submethods. Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$. The convolution submethod C_{λ}^* is the matrix whose nk th entry is

$$C_{\lambda}^*[n, k] := C^*[\lambda(n), k]. \tag{4.1}$$

LEMMA 4.1. *The convolution submethod C_{λ}^* is regular if and only if $p_0 < 1$.*

PROOF. If $p_0 < 1$, then C^* is regular and hence C_{λ}^* is also regular. Conversely, if C_{λ}^* is regular and $p_0 = 1$, then $C_{\lambda}^*[n, k] = q_k$ for all n and k . Since $\sum_{k=0}^{\infty} q_k = 1$, there exists a \hat{k} such that $q_{\hat{k}} \neq 0$. Then $\lim_n C_{\lambda}^*[n, \hat{k}] = q_{\hat{k}} \neq 0$, which contradicts the regularity of C_{λ}^* . □

The following theorem compares C_{λ}^* with C^* for bounded sequences.

THEOREM 4.2. *Let C^* be the convolution method formed from the sequences p and q with $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty} (j - \mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$. Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$. If*

$$\lim_{n \rightarrow \infty} \frac{\lambda(n+1) - \lambda(n)}{\sqrt{\lambda(n)}} = 0, \tag{4.2}$$

then C^ and C_{λ}^* are equivalent for bounded sequences.*

PROOF. By Lemma 1.1, $C^* \subseteq C_{\lambda}^*$ for any λ . So assume $\lim_n (\lambda(n+1) - \lambda(n)) / \sqrt{\lambda(n)} = 0$ and let x be a bounded sequence that is C_{λ}^* -summable to L . Consider the set $S := \{\rho(n)\} := (\mathbb{N} \cup \{0\}) \setminus E$. If S is finite, then Lemma 1.1 shows that C_{λ}^* and C^* are equivalent for all sequences. So assume S is infinite. Then there exists an N such that for $n \geq N$, $\rho(n) > \lambda(0)$. Since E and S are disjoint, for $n \geq N$, there exists an integer m such that $\lambda(m) < \rho(n) < \lambda(m+1)$. We write $\rho(n) := \lambda(m) + j$, where $0 < j < \lambda(m+1) - \lambda(m)$. Then, for $n \geq N$,

$$\begin{aligned}
|(C_{\rho}^* x)_n - (C_{\lambda}^* x)_m| &= \left| \sum_{k=0}^{\infty} c_{\rho(n), k} x_k - \sum_{k=0}^{\infty} c_{\lambda(m), k} x_k \right| \\
&= \left| \sum_{k=0}^{\infty} c_{\lambda(m)+j, k} x_k - \sum_{k=0}^{\infty} c_{\lambda(m), k} x_k \right|.
\end{aligned} \tag{4.3}$$

By [Lemma 3.3](#), this becomes

$$\begin{aligned}
|(C_\rho^* x)_n - (C_\lambda^* x)_m| &= \left| \sum_{k=0}^{\infty} \left(\sum_{i=0}^{\infty} c_{\lambda(m), k-i} d_{j,i} \right) x_k - \sum_{k=0}^{\infty} c_{\lambda(m), k} x_k \right| \\
&= \left| \sum_{k=0}^{\infty} x_k \left[\left(\sum_{i=0}^{\infty} c_{\lambda(m), k-i} d_{j,i} \right) - \left(\sum_{i=0}^{\infty} c_{\lambda(m), k} d_{j,i} \right) \right] \right| \\
&\leq \|x\|_\infty \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} d_{j,i} |c_{\lambda(m), k-i} - c_{\lambda(m), k}| \\
&= \|x\|_\infty \sum_{i=0}^{\infty} d_{j,i} \sum_{k=0}^{\infty} \left| \sum_{l=0}^{i-1} c_{\lambda(m), k-l} - c_{\lambda(m), k-l-1} \right| \\
&\leq \|x\|_\infty \sum_{i=0}^{\infty} d_{j,i} \sum_{k=0}^{\infty} \sum_{l=0}^{i-1} |c_{\lambda(m), k-l} - c_{\lambda(m), k-l-1}| \\
&= \frac{\|x\|_\infty}{\sqrt{\lambda(m)}} \sum_{i=0}^{\infty} d_{j,i} \sum_{l=0}^{i-1} \sqrt{\lambda(m)} \sum_{k=0}^{\infty} |c_{\lambda(m), k-l} - c_{\lambda(m), k-l-1}|. \tag{4.4}
\end{aligned}$$

By [Lemma 3.4](#), there exists an $M > 0$ such that

$$\sqrt{\lambda(m)} \sum_{k=0}^{\infty} |c_{\lambda(m), k-l} - c_{\lambda(m), k-l-1}| < M. \tag{4.5}$$

Then, by [Lemma 3.2](#),

$$|(C_\rho^* x)_n - (C_\lambda^* x)_m| \leq \frac{\|x\|_\infty}{\sqrt{\lambda(m)}} \sum_{i=0}^{\infty} d_{j,i} \sum_{l=0}^{i-1} M = \frac{\|x\|_\infty M}{\sqrt{\lambda(m)}} \sum_{i=0}^{\infty} i d_{j,i} \leq \frac{\|x\|_\infty M}{\sqrt{\lambda(m)}} \cdot j \mu. \tag{4.6}$$

Since $0 < j < \lambda(m+1) - \lambda(m)$,

$$|(C_\rho^* x)_n - (C_\lambda^* x)_m| < \|x\|_\infty M \mu \cdot \frac{\lambda(m+1) - \lambda(m)}{\sqrt{\lambda(m)}} = o(1). \tag{4.7}$$

Thus,

$$0 \leq |(C_\rho^* x)_n - L| \leq |(C_\rho^* x)_n - (C_\lambda^* x)_m| + |(C_\lambda^* x)_m - L| = o(1) + o(1) = o(1). \tag{4.8}$$

Therefore, the sequence $C^* x$ may be partitioned into two disjoint subsequences, namely $(C_\lambda^* x)_n = (C^* x)_{\lambda(n)}$ and $(C_\rho^* x)_n = (C^* x)_{\rho(n)}$, each having the common limit L . Thus, x must be C^* -summable to L , and hence C^* and C_λ^* are equivalent for bounded sequences. \square

The following theorem is a well-known result due to Meyer-König (see [\[10, Theorem 25\]](#)).

THEOREM 4.3. *The methods E_r ($0 < r < 1$), S_r ($0 < r < 1$), T_r ($0 < r < 1$), and the Borel method are equivalent for bounded sequences.*

Since the Euler-Knopp methods of order $0 < r < 1$, Taylor methods of order $0 < r < 1$, Meyer-König methods of order $0 < r < 1$, and the Borel matrix method all have generating sequences satisfying the conditions in [Theorem 4.2](#), the following corollary is immediate.

COROLLARY 4.4. *Let $E := \{\lambda(n)\}$ be an infinite subset of $\mathbb{N} \cup \{0\}$ and $0 < r < 1$. If λ satisfies condition (4.6), then $E_{r,\lambda}$, E_r , $T_{r,\lambda}$, T_r , $S_{r,\lambda}$, S_r , B_λ^* , B^* , and the Borel method are all equivalent for bounded sequences.*

The next theorem presents an equivalence relationship between the C_λ^* submethods.

THEOREM 4.5. *Let C^* be the convolution method formed from the sequences p and q with $\mu < \infty$, $\nu < \infty$, $0 < \sum_{j=0}^{\infty} (j - \mu)^2 p_j$, and $\sum_{j=0}^{\infty} j^3 p_j < \infty$. Let $E := \{\lambda(n)\}$ and $F := \{\rho(n)\}$ be infinite subsets of $\mathbb{N} \cup \{0\}$. If*

$$\lim_{n \rightarrow \infty} \frac{\rho(n) - \lambda(n)}{\sqrt{\lambda(n)}} = 0, \quad (4.9)$$

then C_λ^ and C_ρ^* are equivalent for bounded sequences.*

PROOF. Let x be a bounded sequence and consider the sequences $M(n) := \max\{\lambda(n), \rho(n)\}$ and $m(n) := \min\{\lambda(n), \rho(n)\}$. We write $M(n) := m(n) + j$, where $j := M(n) - m(n)$. For $n \geq 1$, we have

$$\begin{aligned} |(C_\rho^* x)_n - (C_\lambda^* x)_n| &= \left| \sum_{k=0}^{\infty} c_{\rho(n),k} x_k - \sum_{k=0}^{\infty} c_{\lambda(n),k} x_k \right| \\ &= \left| \sum_{k=0}^{\infty} c_{M(n),k} x_k - \sum_{k=0}^{\infty} c_{m(n),k} x_k \right| \\ &= \left| \sum_{k=0}^{\infty} c_{m(n)+j,k} x_k - \sum_{k=0}^{\infty} c_{m(n),k} x_k \right|. \end{aligned} \quad (4.10)$$

Then, as in the proof of [Theorem 4.2](#), we have

$$\begin{aligned} |(C_\rho^* x)_n - (C_\lambda^* x)_n| &\leq O(1) \frac{j}{\sqrt{m(n)}} = O(1) \frac{M(n) - m(n)}{\sqrt{m(n)}} \\ &= O(1) \sqrt{\frac{\lambda(n)}{m(n)}} \frac{|\rho(n) - \lambda(n)|}{\sqrt{\lambda(n)}} \\ &= O(1) \cdot O(1) \cdot o(1) = o(1). \end{aligned} \quad (4.11)$$

Then if x is C_λ^* -summable to L ,

$$\begin{aligned} 0 \leq |(C_\rho^* x)_n - L| &\leq |(C_\rho^* x)_n - (C_\lambda^* x)_n| + |(C_\lambda^* x)_n - L| \\ &= o(1) + o(1) = o(1). \end{aligned} \quad (4.12)$$

Similarly, if x is C_ρ^* -summable to L , then

$$\begin{aligned} 0 &\leq |(C_\lambda^* x)_n - L| \leq |(C_\rho^* x)_n - (C_\lambda^* x)_n| + |(C_\rho^* x)_n - L| \\ &= o(1) + o(1) = o(1). \end{aligned} \quad (4.13)$$

Thus, C_λ^* and C_ρ^* are equivalent for bounded sequences. \square

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