

# INTEGRAL TRANSFORMS, CONVOLUTION PRODUCTS, AND FIRST VARIATIONS

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We establish the various relationships that exist among the integral transform  $\mathcal{F}_{\alpha,\beta}F$ , the convolution product  $(F * G)_\alpha$ , and the first variation  $\delta F$  for a class of functionals defined on  $K[0, T]$ , the space of complex-valued continuous functions on  $[0, T]$  which vanish at zero.

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**1. Introduction and definitions.** In a unifying paper [10], Lee defined an integral transform  $\mathcal{F}_{\alpha,\beta}$  of analytic functionals on an abstract Wiener space. For certain values of the parameters  $\alpha$  and  $\beta$  and for certain classes of functionals, the Fourier-Wiener transform [2], the Fourier-Feynman transform [3], and the Gauss transform are special cases of his integral transform  $\mathcal{F}_{\alpha,\beta}$ . In [5], Chang et al. established an interesting relationship between the integral transform and the convolution product for functionals on an abstract Wiener space. In this paper, we study the relationships that exist among the integral transform, the convolution product, and the first variation [1, 4, 9, 11].

Let  $C_0[0, T]$  denote one-parameter Wiener space, that is, the space of all real-valued continuous functions  $x(t)$  on  $[0, T]$  with  $x(0) = 0$ . Let  $\mathcal{M}$  denote the class of all Wiener measurable subsets of  $C_0[0, T]$  and let  $m$  denote Wiener measure. Then  $(C_0[0, T], \mathcal{M}, m)$  is a complete measure space and we denote the Wiener integral of a Wiener integrable functional  $F$  by

$$\int_{C_0[0, T]} F(x) m(dx). \quad (1.1)$$

Let  $K = K[0, T]$  be the space of complex-valued continuous functions defined on  $[0, T]$  which vanish at  $t = 0$ . Let  $\alpha$  and  $\beta$  be nonzero complex numbers. Next we state the definitions of the integral transform  $\mathcal{F}_{\alpha,\beta}F$ , the convolution product  $(F * G)_\alpha$ , and the first variation  $\delta F$  for functionals defined on  $K$ .

**DEFINITION 1.1.** Let  $F$  be a functional defined on  $K$ . Then the integral transform  $\mathcal{F}_{\alpha,\beta}F$  of  $F$  is defined by

$$\mathcal{F}_{\alpha,\beta}(F)(y) \equiv \mathcal{F}_{\alpha,\beta}F(y) \equiv \int_{C_0[0, T]} F(\alpha x + \beta y) m(dx), \quad y \in K, \quad (1.2)$$

if it exists [5, 8, 10].

**DEFINITION 1.2.** Let  $F$  and  $G$  be functionals defined on  $K$ . Then the convolution product  $(F * G)_\alpha$  of  $F$  and  $G$  is defined by

$$(F * G)_\alpha(\gamma) \equiv \int_{C_0[0,T]} F\left(\frac{\gamma + \alpha x}{\sqrt{2}}\right) G\left(\frac{\gamma - \alpha x}{\sqrt{2}}\right) m(dx), \quad \gamma \in K, \quad (1.3)$$

if it exists [5, 7, 13, 14].

**DEFINITION 1.3.** Let  $F$  be a functional defined on  $K$  and let  $w \in K$ . Then the first variation  $\delta F$  of  $F$  is defined by

$$\delta F(\gamma|w) \equiv \frac{\partial}{\partial t} F(\gamma + tw)|_{t=0}, \quad \gamma \in K, \quad (1.4)$$

if it exists [1, 4, 11].

Let  $\{\theta_1, \theta_2, \dots\}$  be a complete orthonormal set of real-valued functions in  $L_2[0, T]$ . Furthermore, assume that each  $\theta_j$  is of bounded variation on  $[0, T]$ . Also let  $\text{Var}(\theta_j, [0, T])$  denote the total variation of  $\theta_j$  on  $[0, T]$ . Then for each  $\gamma \in K$  and  $j \in \{1, 2, \dots\}$ , the Riemann-Stieltjes integral  $\langle \theta_j, \gamma \rangle \equiv \int_0^T \theta_j(t) d\gamma(t)$  exists. Furthermore,

$$|\langle \theta_j, \gamma \rangle| = \left| \theta_j(T)\gamma(T) - \int_0^T \gamma(t) d\theta_j(t) \right| \leq C_j \|\gamma\|_\infty \quad (1.5)$$

with

$$C_j = |\theta_j(T)| + \text{Var}(\theta_j, [0, T]). \quad (1.6)$$

Next we describe the class of functionals that we work with in this paper. Let  $E_0$  be the space of all functionals  $F: K \rightarrow \mathbb{C}$  of the form

$$F(\gamma) = f(\langle \theta_1, \gamma \rangle, \dots, \langle \theta_n, \gamma \rangle) \quad (1.7)$$

for some positive integer  $n$ , where  $f(\lambda_1, \dots, \lambda_n)$  is an entire function of the  $n$  complex variables  $\lambda_1, \dots, \lambda_n$  of exponential type; that is to say,

$$|f(\lambda_1, \dots, \lambda_n)| \leq A_F \exp \left\{ B_F \sum_{j=1}^n |\lambda_j| \right\} \quad (1.8)$$

for some positive constants  $A_F$  and  $B_F$ .

To simplify the expressions, we use the following notations. For  $\vec{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$  and  $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ , we write

$$\begin{aligned} \|\vec{u}\|^2 &= \sum_{j=1}^n u_j^2, & |\vec{u}| &= \sum_{j=1}^n |u_j|, & |\vec{\lambda}| &= \sum_{j=1}^n |\lambda_j|, & d\vec{u} &= du_1 \cdots du_n, \\ f(\alpha \vec{u} + \beta \vec{\lambda}) &= f(\alpha u_1 + \beta \lambda_1, \dots, \alpha u_n + \beta \lambda_n), \\ f(\langle \vec{\theta}, \gamma \rangle) &= f(\langle \theta_1, \gamma \rangle, \dots, \langle \theta_n, \gamma \rangle). \end{aligned} \quad (1.9)$$

Hence (1.7) and (1.8) can be expressed alternatively as

$$F(\gamma) = f(\langle \vec{\theta}, \gamma \rangle), \quad |f(\vec{\lambda})| \leq A_F \exp \{B_F |\vec{\lambda}|\}, \quad (1.10)$$

respectively. In addition, we use the notation

$$F_j(\gamma) = f_j(\langle \vec{\theta}, \gamma \rangle), \quad (1.11)$$

where  $f_j(\vec{\lambda}) = (\partial/\partial \lambda_j)f(\lambda_1, \dots, \lambda_n)$  for  $j = 1, \dots, n$ .

In Section 2, we show that if  $F$  and  $G$  are elements of  $E_0$ , then  $\mathcal{F}_{\alpha, \beta}F(\cdot)$ ,  $(F * G)_\alpha(\cdot)$ ,  $\delta F(\cdot|w)$ , and  $\delta F(\gamma|\cdot)$  are also elements of  $E_0$ . In Section 3, we examine all relationships involving exactly two of the three concepts of “integral transform,” “convolution product,” and “first variation,” while in Section 4, we examine all relationships involving all three of these concepts where each concept is used exactly once. For related work, see [2, 5, 7, 9, 10, 11, 13, 14] and for a detailed survey of previous work, see [12].

**REMARK 1.4.** For any  $F \in E_0$  and any  $G \in E_0$ , we can always express  $F$  by (1.7) and  $G$  by

$$G(x) = g(\langle \theta_1, x \rangle, \dots, \langle \theta_n, x \rangle) \quad (1.12)$$

using the same positive integer  $n$ , where  $f$  and  $g$  are entire functions of exponential type. For example, if  $F \in E_0$  is of the form

$$F(x) = r(\langle \theta_1, x \rangle, \langle \theta_2, x \rangle), \quad (1.13)$$

and  $G \in E_0$  is of the form

$$G(x) = s(\langle \theta_1, x \rangle, \langle \theta_3, x \rangle, \langle \theta_4, x \rangle), \quad (1.14)$$

where  $r(\lambda_1, \lambda_2)$  and  $s(\lambda_1, \lambda_3, \lambda_4)$  are entire functions of exponential type, then we can express  $F$  and  $G$  by (1.7) and (1.12) with  $n = 4$  by choosing  $f(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \equiv r(\lambda_1, \lambda_2)$  and  $g(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \equiv s(\lambda_1, \lambda_3, \lambda_4)$ . In addition, the positive constants  $A_F$ ,  $B_F$ ,  $A_G$ , and  $B_G$  remain fixed. Thus throughout this paper, we will always assume that  $F$  and  $G$  belong to  $E_0$  and are given by (1.7) and (1.12), respectively.

**REMARK 1.5.** We considered various other classes of functionals before deciding to work exclusively with the class  $E_0$  throughout this paper. One very natural class we considered was  $L_2(C) \equiv L_2(C_0[0, T])$ , the space of all complex-valued functionals  $F$  satisfying

$$\int_{C_0[0, T]} |F(x)|^2 m(dx) < \infty. \quad (1.15)$$

However in [8], Kim and Skoug showed that  $L_2(C)$  is not invariant under the action of the integral transform operator. In fact, they showed that for every  $\beta \in \mathbb{C}$  with  $|\beta| > 1$ , there exists a functional  $F \in L_2(C)$  (the functional  $F$  depends on  $\beta$ ) with  $\mathcal{F}_{\alpha, \beta}(F) \notin L_2(C)$  even though  $\alpha^2 + \beta^2 = 1$ .

Another class of functionals we considered was

$$A = \{F \in L_2(C) : \mathcal{F}_{\alpha,\beta}(F) \in L_2(C) \ \forall \text{ nonzero } \alpha, \beta \in \mathbb{C}\}. \quad (1.16)$$

But for  $F \in A$ , the first variation  $\delta F$  of  $F$  may not exist; in fact, one needs some kind of a smoothness condition on  $F$  to even define  $\delta F$ .

As we will see in [Section 2](#),  $E_0$  is a very natural class of functionals in which to study the relationships that exist among the integral transform, the convolution product, and the first variation because for  $F$  and  $G$  in  $E_0$ ,  $\mathcal{F}_{\alpha,\beta}(F)$  and  $(F * G)_\alpha$  exist and belong to  $E_0$  for all nonzero complex numbers  $\alpha$  and  $\beta$ , while  $\delta F(\gamma|w)$  exists and belongs to  $E_0$  for all  $\gamma$  and  $w$  in  $K$ . In addition,  $E_0$  is a very rich class of functionals. Note that if  $E_0$  is given by (1.7), then the entire function  $f(\lambda_1, \dots, \lambda_n)$  is bounded if and only if it is a constant function. Thus many of the functionals in  $E_0$  are unbounded, while for example, all of the functionals considered in [11] are bounded.

The so-called “tame functionals,” that is, functionals of the form

$$G(x) = g(x(t_1), \dots, x(t_m)), \quad 0 < t_1 < \dots < t_m \leq T \quad (1.17)$$

as well as functionals of the form (1.7), played a major role in the development of Wiener space integration theory. But functionals of the form (1.17) are in  $E_0$  provided  $g(\lambda_1, \dots, \lambda_m)$  is an entire function of exponential growth. Included of course are all polynomials of  $m$  complex variables  $\lambda_1, \dots, \lambda_m$  for all positive integers  $m$ , as well as such polynomials in  $x(t_1), \dots, x(t_m)$  multiplied by functionals like  $\exp\{\sum_{j=1}^m a_j x_j(t)\}$ , and so forth.

**2. The integral transform, the convolution product, and the first variation of functionals in  $E_0$ .** In our first theorem, we show that if  $F$  is an element of  $E_0$ , then the integral transform of  $F$  exists and is an element of  $E_0$ .

**THEOREM 2.1.** *Let  $F \in E_0$  be given by (1.7). Then the integral transform  $\mathcal{F}_{\alpha,\beta}F$  exists, belongs to  $E_0$ , and is given by the formula*

$$\mathcal{F}_{\alpha,\beta}F(\gamma) = h(\langle \vec{\theta}, \gamma \rangle) \quad (2.1)$$

for  $\gamma \in K$ , where

$$h(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\alpha \vec{u} + \beta \vec{\lambda}) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u}. \quad (2.2)$$

**PROOF.** For each  $\gamma \in K$ , using a well-known Wiener integration theorem, we obtain

$$\begin{aligned} \mathcal{F}_{\alpha,\beta}F(\gamma) &= \int_{C_0[0,T]} f(\alpha \langle \vec{\theta}, x \rangle + \beta \langle \vec{\theta}, \gamma \rangle) m(dx) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\alpha \vec{u} + \beta \langle \vec{\theta}, \gamma \rangle) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u} \\ &= h(\langle \vec{\theta}, \gamma \rangle), \end{aligned} \quad (2.3)$$

where  $h$  is given by (2.2). By [6, Theorem 3.15],  $h(\vec{\lambda})$  is an entire function. Moreover, by inequality (1.8), we have

$$\begin{aligned} |h(\vec{\lambda})| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} A_F \exp \left\{ B_F |\alpha \vec{u} + \beta \vec{\lambda}| - \frac{1}{2} \|\vec{u}\|^2 \right\} d\vec{u} \\ &\leq A_{\mathcal{F}_{\alpha, \beta} F} \exp \{ B_{\mathcal{F}_{\alpha, \beta} F} |\vec{\lambda}| \}, \end{aligned} \quad (2.4)$$

where

$$A_{\mathcal{F}_{\alpha, \beta} F} = A_F \left( \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left\{ -\frac{u^2}{2} + B_F |\alpha u| \right\} du \right)^n < \infty \quad (2.5)$$

and  $B_{\mathcal{F}_{\alpha, \beta} F} = B_F |\beta|$ . Hence  $\mathcal{F}_{\alpha, \beta} F \in E_0$ .  $\square$

In our next theorem, we show that the convolution product of functionals from  $E_0$  is an element of  $E_0$ .

**THEOREM 2.2.** *Let  $F, G \in E_0$  be given by (1.7) and (1.12) with corresponding entire functions  $f$  and  $g$ . Then the convolution  $(F * G)_\alpha$  exists, belongs to  $E_0$ , and is given by the formula*

$$(F * G)_\alpha(\gamma) = k(\langle \vec{\theta}, \gamma \rangle) \quad (2.6)$$

for  $\gamma \in K$ , where

$$k(\vec{\lambda}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f \left( \frac{\vec{\lambda} + \alpha \vec{u}}{\sqrt{2}} \right) g \left( \frac{\vec{\lambda} - \alpha \vec{u}}{\sqrt{2}} \right) \exp \left\{ -\frac{1}{2} \|\vec{u}\|^2 \right\} d\vec{u}. \quad (2.7)$$

**PROOF.** For each  $\gamma \in K$ , using a well-known Wiener integration theorem, we obtain

$$\begin{aligned} (F * G)_\alpha(\gamma) &= \int_{C_0[0, T]} F \left( \frac{\gamma + \alpha x}{\sqrt{2}} \right) G \left( \frac{\gamma - \alpha x}{\sqrt{2}} \right) m(dx) \\ &= \int_{C_0[0, T]} f \left( \frac{\langle \vec{\theta}, \gamma \rangle + \alpha \langle \vec{\theta}, x \rangle}{\sqrt{2}} \right) g \left( \frac{\langle \vec{\theta}, \gamma \rangle - \alpha \langle \vec{\theta}, x \rangle}{\sqrt{2}} \right) m(dx) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f \left( \frac{\langle \vec{\theta}, \gamma \rangle + \alpha \vec{u}}{\sqrt{2}} \right) g \left( \frac{\langle \vec{\theta}, \gamma \rangle - \alpha \vec{u}}{\sqrt{2}} \right) \exp \left\{ -\frac{1}{2} \|\vec{u}\|^2 \right\} d\vec{u} \\ &= k(\langle \vec{\theta}, \gamma \rangle), \end{aligned} \quad (2.8)$$

where  $k$  is given by (2.7). By [6, Theorem 3.15],  $k(\vec{\lambda})$  is an entire function and

$$\begin{aligned} |k(\vec{\lambda})| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} A_F A_G \exp \left\{ \frac{B_F + B_G}{\sqrt{2}} (|\vec{\lambda}| + |\alpha| \|\vec{u}\|) - \frac{1}{2} \|\vec{u}\|^2 \right\} d\vec{u} \\ &= A_{(F * G)_\alpha} \exp \{ B_{(F * G)_\alpha} |\vec{\lambda}| \}, \end{aligned} \quad (2.9)$$

where  $B_{(F * G)_\alpha} = (B_F + B_G) / \sqrt{2}$  and

$$A_{(F * G)_\alpha} = A_F A_G \left( \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left\{ -\frac{u^2}{2} + B_{(F * G)_\alpha} |\alpha u| \right\} du \right)^n < \infty. \quad (2.10)$$

Hence  $(F * G)_\alpha \in E_0$ .  $\square$

In [Theorem 2.3](#), we fix  $w \in K$  and consider  $\delta F(y|w)$  as a function of  $y$ , while in [Theorem 2.4](#), we fix  $y \in K$  and consider  $\delta F(y|w)$  as a function of  $w$ .

**THEOREM 2.3.** *Let  $F \in E_0$  be given by (1.7) and let  $w \in K$ . Then*

$$\delta F(y|w) = p(\langle \vec{\theta}, y \rangle) \quad (2.11)$$

for  $y \in K$ , where

$$p(\vec{\lambda}) = \sum_{j=1}^n \langle \theta_j, w \rangle f_j(\vec{\lambda}). \quad (2.12)$$

Furthermore, as a function of  $y \in K$ ,  $\delta F(y|w)$  is an element of  $E_0$ .

**PROOF.** For  $y \in K$ ,

$$\begin{aligned} \delta F(y|w) &= \frac{\partial}{\partial t} f(\langle \vec{\theta}, y \rangle + t \langle \vec{\theta}, w \rangle) \big|_{t=0} \\ &= \sum_{j=1}^n \langle \theta_j, w \rangle f_j(\langle \vec{\theta}, y \rangle) = p(\langle \vec{\theta}, y \rangle), \end{aligned} \quad (2.13)$$

where  $p$  is given by (2.12). Since  $f(\vec{\lambda})$  is an entire function,  $f_j(\vec{\lambda})$  and so  $p(\vec{\lambda})$  are entire functions. By the Cauchy integral formula, we have

$$f_j(\lambda_1, \dots, \lambda_j, \dots, \lambda_n) = \frac{1}{2\pi i} \int_{|\zeta - \lambda_j| = 1} \frac{f(\lambda_1, \dots, \zeta, \dots, \lambda_n)}{(\zeta - \lambda_j)^2} d\zeta. \quad (2.14)$$

By inequality (1.8), for any  $\zeta$  with  $|\zeta - \lambda_j| = 1$ , we have

$$\begin{aligned} \left| \frac{f(\lambda_1, \dots, \zeta, \dots, \lambda_n)}{(\zeta - \lambda_j)^2} \right| &\leq A_F \exp \{B_F(|\lambda_1| + \dots + |\zeta| + \dots + |\lambda_n|)\} \\ &\leq A_F \exp \{B_F|\vec{\lambda}| + B_F\}. \end{aligned} \quad (2.15)$$

Hence

$$|f_j(\vec{\lambda})| \leq A_F e^{B_F} \exp \{B_F|\vec{\lambda}|\}, \quad (2.16)$$

and so

$$|p(\vec{\lambda})| \leq \sum_{j=1}^n |\langle \theta_j, w \rangle| |f_j(\vec{\lambda})| \leq A_{\delta F(\cdot|w)} \exp \{B_{\delta F(\cdot|w)}|\vec{\lambda}|\}, \quad (2.17)$$

where

$$A_{\delta F(\cdot|w)} = A_F e^{B_F} \|w\|_\infty \sum_{j=1}^n C_j < \infty \quad (2.18)$$

with  $C_j$  given by (1.6) and  $B_{\delta F(\cdot|w)} = B_F$ . □

**THEOREM 2.4.** *Let  $y \in K$  and let  $F \in E_0$  be given by (1.7). Then*

$$\delta F(y|w) = q(\langle \vec{\theta}, w \rangle) \quad (2.19)$$

for  $w \in K$ , where

$$q(\vec{\lambda}) = \sum_{j=1}^n \lambda_j f_j(\langle \vec{\theta}, y \rangle). \quad (2.20)$$

Furthermore, as a function of  $w$ ,  $\delta F(y|w)$  is an element of  $E_0$ .

**PROOF.** Equations (2.19) and (2.20) are immediate from the first part of the proof of Theorem 2.3. Clearly  $q(\vec{\lambda})$  is an entire function. Next, using (2.16) we obtain

$$\begin{aligned} |q(\vec{\lambda})| &\leq \sum_{j=1}^n |\lambda_j f_j(\langle \vec{\theta}, y \rangle)| \\ &\leq A_F e^{B_F} \exp \{B_F [|\langle \theta_1, y \rangle| + \cdots + |\langle \theta_n, y \rangle|]\} \sum_{j=1}^n |\lambda_j| \\ &< A_F e^{B_F} \exp \{B_F \|y\|_\infty [C_1 + \cdots + C_n]\} e^{|\vec{\lambda}|} \\ &= A_{\delta F(y|\cdot)} \exp \{B_{\delta F(y|\cdot)} |\vec{\lambda}|\}, \end{aligned} \quad (2.21)$$

where  $B_{\delta F(y|\cdot)} = 1$  and

$$A_{\delta F(y|\cdot)} = A_F e^{B_F} \exp \{B_F \|y\|_\infty [C_1 + \cdots + C_n]\}. \quad (2.22)$$

Hence, as a function of  $w$ ,  $\delta F(y|w) \in E_0$ .  $\square$

We finish this section with some observations which we use later in this paper. First of all, (1.2) implies that

$$\mathcal{F}_{\alpha, \beta} F\left(\frac{y}{\sqrt{2}}\right) = \mathcal{F}_{\alpha, \beta/\sqrt{2}} F(y) \quad (2.23)$$

for all  $y \in K$ . Next, a direct calculation using (1.4), (1.2), (2.11), and (2.23) shows that

$$\delta \mathcal{F}_{\alpha, \beta} F\left(\frac{y}{\sqrt{2}} \middle| \frac{w}{\sqrt{2}}\right) = \delta \mathcal{F}_{\alpha, \beta/\sqrt{2}} F(y|w) = \frac{\beta}{\sqrt{2}} \sum_{j=1}^n \langle \theta_j, w \rangle \mathcal{F}_{\alpha, \beta/\sqrt{2}} F_j(y) \quad (2.24)$$

for all  $y$  and  $w$  in  $K$ . Finally, by similar calculations, we obtain that

$$\mathcal{F}_{\alpha, \beta} (\delta F(\cdot|w))\left(\frac{y}{\sqrt{2}}\right) = \frac{\sqrt{2}}{\beta} \delta \mathcal{F}_{\alpha, \beta/\sqrt{2}} F(y|w) \quad (2.25)$$

for all  $\gamma$  and  $w$  in  $K$ , and for all  $\gamma \in K$ ,

$$(\mathcal{F}_{\alpha,\beta}F)_j(\gamma) = \beta \mathcal{F}_{\alpha,\beta}(F_j)(\gamma) = \beta \mathcal{F}_{\alpha,\beta}F_j(\gamma). \quad (2.26)$$

**3. Relationships involving two concepts.** In this section, we establish all of the various relationships involving exactly two of the three concepts of integral transform, convolution product, and first variation for functionals belonging to  $E_0$ . The seven distinct relationships, as well as alternative expressions for some of them, are given by (3.1), (3.2), (3.4), (3.7), (3.9), (3.11), and (3.13).

In view of Theorem 2.1 through Theorem 2.4, all of the functionals that occur in this section are elements of  $E_0$ . For example, let  $F$  and  $G$  be any functionals in  $E_0$ . Then by Theorem 2.2, the functional  $(F * G)_\alpha$  belongs to  $E_0$ , and hence by Theorem 2.1, the functional  $\mathcal{F}_{\alpha,\beta}(F * G)_\alpha$  also belongs to  $E_0$ . By similar arguments, all of the functionals that arise in (3.1) through (3.14) and (3.16) through (3.20) exist and belong to  $E_0$ .

Our first formula (3.1) is useful because it permits one to calculate  $\mathcal{F}_{\alpha,\beta}(F * G)_\alpha$  without ever actually calculating  $(F * G)_\alpha$ .

**FORMULA 3.1.** The integral transform of the convolution product of functionals from  $E_0$  is given by the formula

$$\mathcal{F}_{\alpha,\beta}(F * G)_\alpha(\gamma) = \mathcal{F}_{\alpha,\beta}F\left(\frac{\gamma}{\sqrt{2}}\right) \mathcal{F}_{\alpha,\beta}G\left(\frac{\gamma}{\sqrt{2}}\right) = \mathcal{F}_{\alpha,\beta/\sqrt{2}}F(\gamma) \mathcal{F}_{\alpha,\beta/\sqrt{2}}G(\gamma) \quad (3.1)$$

for all  $\gamma$  in  $K$ .

**PROOF.** Formula 3.1 is a special case of [5, Theorem 3.1].  $\square$

**FORMULA 3.2.** The convolution product of the integral transform of functionals from  $E_0$  is given by the formula

$$\begin{aligned} & (\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G)_\alpha(\gamma) \\ &= (2\pi)^{-3n/2} \int_{\mathbb{R}^{3n}} f\left(\alpha\vec{r} + \frac{\beta}{\sqrt{2}}\langle\vec{\theta}, \gamma\rangle + \frac{\beta\alpha}{\sqrt{2}}\vec{u}\right) \\ & \quad \cdot g\left(\alpha\vec{s} + \frac{\beta}{\sqrt{2}}\langle\vec{\theta}, \gamma\rangle - \frac{\beta\alpha}{\sqrt{2}}\vec{u}\right) \exp\left\{-\frac{\|\vec{u}\|^2 + \|\vec{r}\|^2 + \|\vec{s}\|^2}{2}\right\} d\vec{u} d\vec{r} d\vec{s} \end{aligned} \quad (3.2)$$

for all  $\gamma$  in  $K$ .

**PROOF.** Using (1.3) and (1.2), we see that

$$\begin{aligned} & (\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G)_\alpha(\gamma) \\ &= \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}F\left(\frac{\gamma + \alpha x}{\sqrt{2}}\right) \mathcal{F}_{\alpha,\beta}G\left(\frac{\gamma - \alpha x}{\sqrt{2}}\right) m(dx) \\ &= \int_{C_0[0,T]} \left[ \int_{C_0[0,T]} F\left(\alpha z_1 + \frac{\beta(\gamma + \alpha x)}{\sqrt{2}}\right) m(dz_1) \right] \end{aligned}$$

$$\begin{aligned}
 & \cdot \left[ \int_{C_0[0,T]} G\left(\alpha z_2 + \frac{\beta(\gamma - \alpha x)}{\sqrt{2}}\right) m(dz_2) \right] m(dx) \\
 &= \int_{C_0^3[0,T]} f\left(\alpha \langle \vec{\theta}, z_1 \rangle + \frac{\beta}{\sqrt{2}} \langle \vec{\theta}, \gamma \rangle + \frac{\alpha\beta}{\sqrt{2}} \langle \vec{\theta}, x \rangle\right) \\
 & \quad \cdot g\left(\alpha \langle \vec{\theta}, z_2 \rangle + \frac{\beta}{\sqrt{2}} \langle \vec{\theta}, \gamma \rangle - \frac{\alpha\beta}{\sqrt{2}} \langle \vec{\theta}, x \rangle\right) m(dx) m(dz_1) m(dz_2).
 \end{aligned} \tag{3.3}$$

Formula (3.2) now follows upon evaluating the above Wiener integrals.  $\square$

**FORMULA 3.3.** The integral transform with respect to the first argument of the variation is given by the formula

$$\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))(\gamma) = \frac{1}{\beta} \delta \mathcal{F}_{\alpha,\beta} F(\gamma|w) = \sum_{j=1}^n \langle \theta_j, w \rangle \mathcal{F}_{\alpha,\beta} F_j(\gamma) \tag{3.4}$$

for all  $\gamma$  and  $w$  in  $K$ .

**PROOF.** By applying Theorem 2.1 to expression (2.11), we obtain

$$\begin{aligned}
 \mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w))(\gamma) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} p(\alpha \vec{u} + \beta \langle \vec{\theta}, \gamma \rangle) \exp\left\{-\frac{1}{2} \|\vec{u}\|^2\right\} d\vec{u} \\
 &= (2\pi)^{-n/2} \sum_{j=1}^n \langle \theta_j, w \rangle \int_{\mathbb{R}^n} f_j(\alpha \vec{u} + \beta \langle \vec{\theta}, \gamma \rangle) \exp\left\{-\frac{1}{2} \|\vec{u}\|^2\right\} d\vec{u}.
 \end{aligned} \tag{3.5}$$

On the other hand, by applying Theorem 2.3 to expression (2.1) and then using (2.2), we obtain

$$\begin{aligned}
 \frac{1}{\beta} \delta \mathcal{F}_{\alpha,\beta} F(\gamma|w) &= \frac{1}{\beta} \sum_{j=1}^n \langle \theta_j, w \rangle h_j(\langle \vec{\theta}, \gamma \rangle) \\
 &= \frac{1}{\beta} \sum_{j=1}^n \langle \theta_j, w \rangle (2\pi)^{-n/2} \beta \int_{\mathbb{R}^n} f_j(\alpha \vec{u} + \beta \langle \vec{\theta}, \gamma \rangle) \exp\left\{-\frac{1}{2} \|\vec{u}\|^2\right\} d\vec{u} \\
 &= (2\pi)^{-n/2} \sum_{j=1}^n \langle \theta_j, w \rangle \int_{\mathbb{R}^n} f_j(\alpha \vec{u} + \beta \langle \vec{\theta}, \gamma \rangle) \exp\left\{-\frac{1}{2} \|\vec{u}\|^2\right\} d\vec{u},
 \end{aligned} \tag{3.6}$$

and so (3.4) is established.  $\square$

**FORMULA 3.4.** The Integral transform with respect to the second argument of the variation is given by the formula

$$\mathcal{F}_{\alpha,\beta}(\delta F(\gamma|\cdot))(w) = \beta \delta F(\gamma|w) \tag{3.7}$$

for all  $\gamma$  and  $w$  in  $K$ .

**PROOF.** By applying [Theorem 2.1](#) to expression (2.19), we obtain

$$\begin{aligned}
 \mathcal{F}_{\alpha,\beta}(\delta F(\gamma|\cdot))(w) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} q(\alpha\vec{u} + \beta\langle\vec{\theta}, w\rangle) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u} \\
 &= (2\pi)^{-n/2} \sum_{j=1}^n \int_{\mathbb{R}^n} (\alpha u_j + \beta\langle\theta_j, w\rangle) f_j(\langle\vec{\theta}, \gamma\rangle) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u} \\
 &= \beta \sum_{j=1}^n \langle\theta_j, w\rangle f_j(\langle\vec{\theta}, \gamma\rangle) = \beta \delta F(\gamma|w).
 \end{aligned} \tag{3.8}$$

□

**FORMULA 3.5.** The first variation of the convolution product of functionals from  $E_0$  is given by the formula

$$\delta(F * G)_\alpha(\gamma|w) = \sum_{j=1}^n \frac{\langle\theta_j, w\rangle}{\sqrt{2}} [(F_j * G)_\alpha(\gamma) + (F * G_j)_\alpha(\gamma)] \tag{3.9}$$

for all  $\gamma$  and  $w$  in  $K$ .

**PROOF.** By applying [Theorem 2.3](#) to (2.6) and then using (2.7), we obtain

$$\begin{aligned}
 \delta(F * G)_\alpha(\gamma|w) &= \sum_{j=1}^n \langle\theta_j, w\rangle k_j(\langle\vec{\theta}, \gamma\rangle) \\
 &= (2\pi)^{-n/2} \sum_{j=1}^n \frac{\langle\theta_j, w\rangle}{\sqrt{2}} \int_{\mathbb{R}^n} \left[ f_j\left(\frac{\langle\vec{\theta}, \gamma\rangle + \alpha\vec{u}}{\sqrt{2}}\right) g\left(\frac{\langle\vec{\theta}, \gamma\rangle - \alpha\vec{u}}{\sqrt{2}}\right) \right. \\
 &\quad \left. + f\left(\frac{\langle\vec{\theta}, \gamma\rangle + \alpha\vec{u}}{\sqrt{2}}\right) g_j\left(\frac{\langle\vec{\theta}, \gamma\rangle - \alpha\vec{u}}{\sqrt{2}}\right) \right] \exp\left\{-\frac{1}{2}\|\vec{u}\|^2\right\} d\vec{u} \\
 &= \sum_{j=1}^n \frac{\langle\theta_j, w\rangle}{\sqrt{2}} [(F_j * G)_\alpha(\gamma) + (F * G_j)_\alpha(\gamma)].
 \end{aligned} \tag{3.10}$$

□

**FORMULA 3.6.** The convolution product, with respect to the first argument of the variation, of the variation of functionals from  $E_0$  is given by the formula

$$(\delta F(\cdot|w) * \delta G(\cdot|w))_\alpha(\gamma) = \sum_{j=1}^n \sum_{l=1}^n \langle\theta_j, w\rangle \langle\theta_l, w\rangle (F_j * G_l)_\alpha(\gamma) \tag{3.11}$$

for all  $\gamma$  and  $w$  in  $K$ .

**PROOF.** Applying the additive distribution properties of the convolution product to the expressions

$$\delta F(\gamma|w) = \sum_{j=1}^n \langle\theta_j, w\rangle F_j(\gamma), \quad \delta G(\gamma|w) = \sum_{l=1}^n \langle\theta_l, w\rangle G_l(\gamma) \tag{3.12}$$

yields (3.11) as desired. □

**FORMULA 3.7.** The convolution product, with respect to the second argument of the variation, of the variation of functionals from  $E_0$  is given by the formula

$$(\delta F(\gamma|\cdot) * \delta G(\gamma|\cdot))_\alpha(w) = \frac{1}{2} \delta F(\gamma|w) \delta G(\gamma|w) - \frac{\alpha^2}{2} \sum_{j=1}^n F_j(\gamma) G_j(\gamma) \quad (3.13)$$

for all  $\gamma$  and  $w$  in  $K$ .

**PROOF.** Upon applying [Theorem 2.2](#) to the expressions

$$\delta F(\gamma|w) = \sum_{j=1}^n \langle \theta_j, w \rangle f_j(\langle \vec{\theta}, \gamma \rangle), \quad \delta G(\gamma|w) = \sum_{l=1}^n \langle \theta_l, w \rangle g_l(\langle \vec{\theta}, \gamma \rangle), \quad (3.14)$$

and using the fact that

$$\int_{\mathbb{R}^n} u_j u_l \exp \left\{ -\frac{1}{2} \|\vec{u}\|^2 \right\} d\vec{u} = \begin{cases} (2\pi)^{n/2} & \text{if } j = l, \\ 0 & \text{if } j \neq l, \end{cases} \quad (3.15)$$

we obtain

$$\begin{aligned} & (\delta F(\gamma|\cdot) * \delta G(\gamma|\cdot))_\alpha(w) \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left[ \sum_{j=1}^n \frac{\langle \theta_j, w \rangle + \alpha u_j}{\sqrt{2}} f_j(\langle \vec{\theta}, \gamma \rangle) \right] \\ & \quad \cdot \left[ \sum_{l=1}^n \frac{\langle \theta_l, w \rangle - \alpha u_l}{\sqrt{2}} g_l(\langle \vec{\theta}, \gamma \rangle) \right] \exp \left\{ -\frac{1}{2} \|\vec{u}\|^2 \right\} d\vec{u} \\ &= \frac{1}{2} (2\pi)^{-n/2} \sum_{j=1}^n \sum_{l=1}^n f_j(\langle \vec{\theta}, \gamma \rangle) g_l(\langle \vec{\theta}, \gamma \rangle) \\ & \quad \cdot \int_{\mathbb{R}^n} (\langle \theta_j, w \rangle + \alpha u_j) (\langle \theta_l, w \rangle - \alpha u_l) \exp \left\{ -\frac{1}{2} \|\vec{u}\|^2 \right\} d\vec{u} \\ &= \frac{1}{2} \sum_{j=1}^n \sum_{l=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle f_j(\langle \vec{\theta}, \gamma \rangle) g_l(\langle \vec{\theta}, \gamma \rangle) - \frac{\alpha^2}{2} \sum_{j=1}^n f_j(\langle \vec{\theta}, \gamma \rangle) g_j(\langle \vec{\theta}, \gamma \rangle) \\ &= \frac{1}{2} \left[ \sum_{j=1}^n \langle \theta_j, w \rangle f_j(\langle \vec{\theta}, \gamma \rangle) \right] \left[ \sum_{l=1}^n \langle \theta_l, w \rangle g_l(\langle \vec{\theta}, \gamma \rangle) \right] - \frac{\alpha^2}{2} \sum_{j=1}^n F_j(\gamma) G_j(\gamma) \\ &= \frac{1}{2} \delta F(\gamma|w) \delta G(\gamma|w) - \frac{\alpha^2}{2} \sum_{j=1}^n F_j(\gamma) G_j(\gamma). \end{aligned} \quad (3.16)$$

□

Finally, letting  $G = F$  in (3.1), (3.9), (3.11), and (3.13) yields the formulas

$$\mathcal{F}_{\alpha,\beta}(F * F)_\alpha(\gamma) = [\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(\gamma)]^2, \quad (3.17)$$

$$\delta(F * F)_\alpha(\gamma|w) = \sqrt{2} \sum_{j=1}^n \langle \theta_j, w \rangle (F * F_j)_\alpha(\gamma), \quad (3.18)$$

$$(\delta F(\cdot|w) * \delta F(\cdot|w))_\alpha(\gamma) = \sum_{j=1}^n \sum_{l=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle (F_j * F_l)_\alpha(\gamma), \quad (3.19)$$

$$(\delta F(\gamma|\cdot) * \delta F(\gamma|\cdot))_\alpha(w) = \frac{1}{2} [\delta F(\gamma|w)]^2 - \frac{\alpha^2}{2} \sum_{j=1}^n [F_j(\gamma)]^2 \quad (3.20)$$

for all  $\gamma$  and  $w$  in  $K$ .

It is interesting to note that the left-hand side of each of the formulas (3.1), (3.2), (3.4), (3.7), (3.9), (3.11), (3.13), (3.17), (3.18), (3.19), and (3.20) involve exactly two of the operations of transform, convolution and first variation, while each right-hand side involves at most one of these three operations.

**4. Relationships involving three concepts.** In this section, we examine all of the various relationships involving the integral transform, the convolution product, and the first variation, where each concept is used exactly once. There are more than six possibilities since one can take the transform or the convolution with respect to either the first or the second argument of the variation. However, in view of formula (3.4) and (3.7), there are some repetitions. To exhaust all possibilities, we need to take the variation of the expressions in (3.1) and (3.2), the convolution of the expressions in (3.4) and (3.7), and the transform of the expressions in formulas (3.9), (3.11), and (3.13). It turns out that there are ten distinct formulas, and these are given by (4.1) through (4.10) below. We omit the details of the calculations used to obtain (4.1) through (4.10) because the techniques needed are similar to those used above in Sections 2 and 3.

Again, because of the theorems in Section 2, all of the functionals that arise in this section are automatically elements of  $E_0$ . As usual,  $F$  and  $G$  in  $E_0$  are given by (1.7) and (1.12), respectively.

**FORMULA 4.1.** Taking the first variation of the expressions in (3.1) or taking the transform of the expressions in (3.9) with respect to the first argument of the variation and then using (2.23) and (2.24) yields the formula

$$\begin{aligned} \delta \mathcal{F}_{\alpha,\beta}(F * G)_\alpha(\gamma|w) &= \beta \mathcal{F}_{\alpha,\beta} \delta(F * G)_\alpha(\cdot|w)(\gamma) \\ &= \mathcal{F}_{\alpha,\beta} F \left( \frac{\gamma}{\sqrt{2}} \right) \delta \mathcal{F}_{\alpha,\beta} G \left( \frac{\gamma}{\sqrt{2}} \middle| \frac{w}{\sqrt{2}} \right) \\ &\quad + \delta \mathcal{F}_{\alpha,\beta} F \left( \frac{\gamma}{\sqrt{2}} \middle| \frac{w}{\sqrt{2}} \right) \mathcal{F}_{\alpha,\beta} G \left( \frac{\gamma}{\sqrt{2}} \right) \\ &= \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(\gamma) \delta \mathcal{F}_{\alpha,\beta/\sqrt{2}} G(\gamma|w) \\ &\quad + \delta \mathcal{F}_{\alpha,\beta/\sqrt{2}} F(\gamma|w) \mathcal{F}_{\alpha,\beta/\sqrt{2}} G(\gamma) \end{aligned} \quad (4.1)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.2.** Taking the first variation of the expressions in (3.2) or replacing  $F$  with  $\mathcal{F}_{\alpha,\beta}F$  and  $G$  with  $\mathcal{F}_{\alpha,\beta}G$  in (3.9) yields the formula

$$\begin{aligned} & \delta(\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G)_\alpha(\gamma|w) \\ &= \frac{\beta}{\sqrt{2}} \sum_{j=1}^n \langle \theta_j, w \rangle [(\mathcal{F}_{\alpha,\beta}F_j * \mathcal{F}_{\alpha,\beta}G)_\alpha(\gamma) + (\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G_j)_\alpha(\gamma)] \end{aligned} \quad (4.2)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.3.** Taking the integral transform of the expressions in (3.9) with respect to the second argument of the variation yields the formula

$$\begin{aligned} & \mathcal{F}_{\alpha,\beta}\delta(F * G)_\alpha(\gamma|\cdot)(w) = \beta\delta(F * G)_\alpha(\gamma|w) \\ &= \frac{\beta}{\sqrt{2}} \sum_{j=1}^n \langle \theta_j, w \rangle [(F_j * G)_\alpha(\gamma) + (F * G_j)_\alpha(\gamma)] \end{aligned} \quad (4.3)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.4.** Taking the integral transform of the expressions in (3.11) with respect to the first argument of the variation and then using (3.1) and (2.25) yields the formula

$$\begin{aligned} & \mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w) * \delta G(\cdot|w))_\alpha(\gamma) = \mathcal{F}_{\alpha,\beta}\delta F(\cdot|w)\left(\frac{\gamma}{\sqrt{2}}\right)\mathcal{F}_{\alpha,\beta}\delta G(\cdot|w)\left(\frac{\gamma}{\sqrt{2}}\right) \\ &= \frac{2}{\beta^2}\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}F(\gamma|w)\delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}G(\gamma|w) \end{aligned} \quad (4.4)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.5.** Taking the integral transform of the expressions in (3.11) with respect to the second argument of the variation yields the formula

$$\begin{aligned} & \int_{C_0[0,T]} (\delta F(\cdot|\beta w + \alpha x) * \delta G(\cdot|\beta w + \alpha x))_\alpha(\gamma)m(dx) \\ &= \beta^2(\delta F(\cdot|w) * \delta G(\cdot|w))_\alpha(\gamma) + \alpha^2 \sum_{j=1}^n (F_j * G_j)_\alpha(\gamma) \end{aligned} \quad (4.5)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.6.** Taking the integral transform of the expressions in (3.13) with respect to the first argument of the variation yields the formula

$$\begin{aligned} & \int_{C_0[0,T]} (\delta F(\beta\gamma + \alpha x|\cdot) * \delta G(\beta\gamma + \alpha x|\cdot))_\alpha(w)m(dx) \\ &= \frac{1}{2} \sum_{j=1}^n \sum_{l=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle \mathcal{F}_{\alpha,\beta}(F_j G_l)(\gamma) - \frac{\alpha^2}{2} \sum_{j=1}^n \mathcal{F}_{\alpha,\beta}(F_j G_j)(\gamma) \end{aligned} \quad (4.6)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.7.** Taking the integral transform of the expressions in (3.13) with respect to the second argument of the variation yields the formula

$$\mathcal{F}_{\alpha,\beta}(\delta F(\gamma|\cdot) * \delta G(\gamma|\cdot))_{\alpha}(w) = \frac{\beta^2}{2} \delta F(\gamma|w) \delta G(\gamma|w) \quad (4.7)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.8.** Taking the convolution product of the expressions in (3.4) with respect to the first argument of the variation yields the formula

$$\begin{aligned} (\mathcal{F}_{\alpha,\beta} \delta F(\cdot|w) * \mathcal{F}_{\alpha,\beta} \delta G(\cdot|w))_{\alpha}(\gamma) &= \frac{1}{\beta^2} (\delta \mathcal{F}_{\alpha,\beta} F(\cdot|w) * \delta \mathcal{F}_{\alpha,\beta} G(\cdot|w))_{\alpha}(\gamma) \\ &= \sum_{j=1}^n \sum_{l=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle (\mathcal{F}_{\alpha,\beta} F_j * \mathcal{F}_{\alpha,\beta} G_l)_{\alpha}(\gamma) \end{aligned} \quad (4.8)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.9.** Taking the convolution product of the expressions in (3.4) with respect to the second argument of the variation, or replacing  $F$  with  $\mathcal{F}_{\alpha,\beta} F$  and  $G$  with  $\mathcal{F}_{\alpha,\beta} G$  in (3.13) and using (2.26) yields the formula

$$\begin{aligned} &(\delta \mathcal{F}_{\alpha,\beta} F(\gamma|\cdot) * \delta \mathcal{F}_{\alpha,\beta} G(\gamma|\cdot))_{\alpha}(w) \\ &= \frac{1}{2} \delta \mathcal{F}_{\alpha,\beta} F(\gamma|w) \delta \mathcal{F}_{\alpha,\beta} G(\gamma|w) - \frac{\alpha^2}{2} \sum_{j=1}^n (\mathcal{F}_{\alpha,\beta} F)_j(\gamma) (\mathcal{F}_{\alpha,\beta} G)_j(\gamma) \\ &= \frac{1}{2} \delta \mathcal{F}_{\alpha,\beta} F(\gamma|w) \delta \mathcal{F}_{\alpha,\beta} G(\gamma|w) - \frac{\alpha^2 \beta^2}{2} \sum_{j=1}^n \mathcal{F}_{\alpha,\beta} F_j(\gamma) \mathcal{F}_{\alpha,\beta} G_j(\gamma) \end{aligned} \quad (4.9)$$

for all  $\gamma$  and  $w$  in  $K$ .

**FORMULA 4.10.** Taking the convolution product of the expressions in formula (3.7) with respect to the second argument of the variation yields the formula

$$\begin{aligned} (\mathcal{F}_{\alpha,\beta} \delta F(\gamma|\cdot) * \mathcal{F}_{\alpha,\beta} \delta G(\gamma|\cdot))_{\alpha}(w) &= \beta^2 (\delta F(\gamma|\cdot) * \delta G(\gamma|\cdot))_{\alpha}(w) \\ &= \frac{\beta^2}{2} \left[ \delta F(\gamma|w) \delta G(\gamma|w) - \alpha^2 \sum_{j=1}^n F_j(\gamma) G_j(\gamma) \right] \end{aligned} \quad (4.10)$$

for all  $\gamma$  and  $w$  in  $K$ .

For completeness, note that taking the convolution product of the expressions in (3.7) with respect to the first argument of the variation, does not yield a new formula; we simply get formula (3.11) again.

Again it is interesting to note that the left-hand side of each of the formulas (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10) involve all three of the operations of transform, convolution, and first variation, while each right-hand side involves at most two. Also note that formulas (3.1), (3.13), (4.4), (4.7), (4.9), and (4.10) are useful

because they permit one to calculate  $\mathcal{F}_{\alpha,\beta}(F * G)_\alpha(\gamma)$ ,  $(\delta F(\gamma|\cdot) * \delta G(\gamma|\cdot))_\alpha(w)$ , ..., and  $(\mathcal{F}_{\alpha,\beta}\delta F(\gamma|\cdot) * \mathcal{F}_{\alpha,\beta}\delta G(\gamma|\cdot))_\alpha(w)$  without actually having to calculate the convolution products on the left-hand sides of formulas (3.1), (3.13), ..., and (4.10). It is usually harder to calculate convolution products than transforms and first variations.

**5. Further results.** It is well known, see for example [5, 10], that for all  $F \in E_0$ , all  $\gamma \in K$ , and all  $a, b$ , and  $c$  in  $\mathbb{C}$ ,

$$\begin{aligned} & \int_{C_0[0,T]} \left( \int_{C_0[0,T]} F(aw + bx + c\gamma) m(dw) \right) m(dx) \\ &= \int_{C_0[0,T]} F(\sqrt{a^2 + b^2}z + c\gamma) m(dz) \\ &= \int_{C_0[0,T]} \left( \int_{C_0[0,T]} F(aw + bx + c\gamma) m(dx) \right) m(dw), \end{aligned} \quad (5.1)$$

and that

$$\mathcal{F}_{\alpha,\beta}(\mathcal{F}_{\alpha',\beta'}F)(\gamma) = F(\gamma) = \mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F)(\gamma) \quad (5.2)$$

provided  $\beta\beta' = 1$  and  $\alpha^2 + (\beta\alpha')^2 = 0$ . In particular, for all  $\gamma \in K$ ,

$$\mathcal{F}_{\alpha,\beta}(\mathcal{F}_{i\alpha/\beta,1/\beta}F)(\gamma) = F(\gamma) = \mathcal{F}_{i\alpha/\beta,1/\beta}(\mathcal{F}_{\alpha,\beta}F)(\gamma) \quad (5.3)$$

for all nonzero complex numbers  $\alpha$  and  $\beta$ .

If in (1.3) we replace  $\alpha$  with  $i\alpha/\beta$ , then (5.3) enables us to express the convolution product of the transforms of  $F$  and  $G$  as a transform of the product of  $F$  with  $G$ .

**THEOREM 5.1.** *Let  $\alpha$  and  $\beta$  be nonzero complex numbers and let  $F$  and  $G$  be functionals from  $E_0$  given by (1.7) and (1.12), respectively. Then for all  $\gamma \in K$ ,*

$$\begin{aligned} (\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G)_{i\alpha/\beta}(\gamma) &= \mathcal{F}_{\alpha,\beta}\left(F\left(\frac{\cdot}{\sqrt{2}}\right)G\left(\frac{\cdot}{\sqrt{2}}\right)\right)(\gamma) \\ &= \mathcal{F}_{\alpha,\beta/\sqrt{2}}(FG)(\gamma). \end{aligned} \quad (5.4)$$

**PROOF.** Let  $\alpha' = i\alpha/\beta$  and  $\beta' = 1/\beta$ . Using (3.1), it follows that the formula

$$\mathcal{F}_{\alpha',\beta'}(L_1 * L_2)_{\alpha'}(\gamma) = \mathcal{F}_{\alpha',\beta'}L_1\left(\frac{\gamma}{\sqrt{2}}\right)\mathcal{F}_{\alpha',\beta'}L_2\left(\frac{\gamma}{\sqrt{2}}\right) \quad (5.5)$$

holds for all  $L_1$  and  $L_2$  in  $E_0$  and all  $\gamma \in K$ . Letting  $L_1 = \mathcal{F}_{\alpha,\beta}F$  and  $L_2 = \mathcal{F}_{\alpha,\beta}G$  in (5.5) and then using (5.3) yields the formula

$$\begin{aligned} \mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G)_{\alpha'}(\gamma) &= \mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F)\left(\frac{\gamma}{\sqrt{2}}\right)\mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}G)\left(\frac{\gamma}{\sqrt{2}}\right) \\ &= F\left(\frac{\gamma}{\sqrt{2}}\right)G\left(\frac{\gamma}{\sqrt{2}}\right) \end{aligned} \quad (5.6)$$

for all  $\gamma \in K$ . Next taking the integral transform  $\mathcal{F}_{\alpha,\beta}$  of each side of (5.6) yields formula (5.4) as desired.  $\square$

**THEOREM 5.2.** Let  $\alpha, \beta, F$ , and  $G$  be as in [Theorem 5.1](#). Then for all  $y$  and  $w$  in  $K$ ,

$$\delta(\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G)_{i\alpha/\beta}(y|w) = \frac{\beta}{\sqrt{2}}\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w)G(\cdot) + F(\cdot)\delta G(\cdot|w))\left(\frac{y}{\sqrt{2}}\right). \quad (5.7)$$

**PROOF.** Using [\(5.4\)](#) and [\(2.25\)](#), we see that for all  $y$  and  $w$  in  $K$ ,

$$\begin{aligned} \delta(\mathcal{F}_{\alpha,\beta}F * \mathcal{F}_{\alpha,\beta}G)_{i\alpha/\beta}(y|w) &= \delta\mathcal{F}_{\alpha,\beta}\left(F\left(\frac{\cdot}{\sqrt{2}}\right)G\left(\frac{\cdot}{\sqrt{2}}\right)\right)(y|w) \\ &= \delta\mathcal{F}_{\alpha,\beta/\sqrt{2}}(F(\cdot)G(\cdot))(y|w) \\ &= \frac{\beta}{\sqrt{2}}\mathcal{F}_{\alpha,\beta}(\delta F(\cdot|w)G(\cdot) + F(\cdot)\delta G(\cdot|w))\left(\frac{y}{\sqrt{2}}\right). \end{aligned} \quad (5.8)$$

□

Next, using [\(5.4\)](#), we obtain the following analogue of [Formula 4.8](#).

**THEOREM 5.3.** Let  $F, G, \alpha$ , and  $\beta$  be as in [Theorem 5.1](#). Then for all  $y$  and  $w$  in  $K$ ,

$$\begin{aligned} &(\delta\mathcal{F}_{\alpha,\beta}F(\cdot|w) * \delta\mathcal{F}_{\alpha,\beta}G(\cdot|w))_{i\alpha/\beta}(y) \\ &= \beta^2 \sum_{l=1}^n \sum_{j=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle \mathcal{F}_{\alpha,\beta/\sqrt{2}}(F_j G_l)(y). \end{aligned} \quad (5.9)$$

**PROOF.** Using [\(3.4\)](#), [\(5.4\)](#), [Theorem 2.3](#), and [\(2.23\)](#), we obtain

$$\begin{aligned} &(\delta\mathcal{F}_{\alpha,\beta}F(\cdot|w) * \delta\mathcal{F}_{\alpha,\beta}G(\cdot|w))_{i\alpha/\beta}(y) \\ &= \beta^2 (\mathcal{F}_{\alpha,\beta}\delta F(\cdot|w) * \mathcal{F}_{\alpha,\beta}\delta G(\cdot|w))_{i\alpha/\beta}(y) \\ &= \beta^2 \mathcal{F}_{\alpha,\beta}\left(\delta F\left(\frac{\cdot}{\sqrt{2}}\middle|w\right)\delta G\left(\frac{\cdot}{\sqrt{2}}\middle|w\right)\right)(y) \\ &= \beta^2 \mathcal{F}_{\alpha,\beta}\left(\left[\sum_{j=1}^n \langle \theta_j, w \rangle F_j\left(\frac{\cdot}{\sqrt{2}}\right)\right]\left[\sum_{l=1}^n \langle \theta_l, w \rangle G_l\left(\frac{\cdot}{\sqrt{2}}\right)\right]\right)(y) \\ &= \beta^2 \sum_{l=1}^n \sum_{j=1}^n \langle \theta_j, w \rangle \langle \theta_l, w \rangle \mathcal{F}_{\alpha,\beta/\sqrt{2}}(F_j G_l)(y) \end{aligned} \quad (5.10)$$

for all  $y$  and  $w$  in  $K$ . □

It is interesting to note that we can obtain analogues of [Formulas 4.9](#) and [4.10](#) directly by use of [\(3.13\)](#) and [\(3.7\)](#) rather than using [Theorem 5.1](#) as we did in [Theorem 5.3](#) to obtain an analogue of [Formula 4.8](#).

**THEOREM 5.4.** Let  $F, G, \alpha$ , and  $\beta$  be as in [Theorem 5.1](#). Then for all  $y$  and  $w$  in  $K$ ,

$$\begin{aligned} &(\delta\mathcal{F}_{\alpha,\beta}F(y|\cdot) * \delta\mathcal{F}_{\alpha,\beta}G(y|\cdot))_{i\alpha/\beta}(w) \\ &= \frac{1}{2}\delta\mathcal{F}_{\alpha,\beta}F(y|w)\delta\mathcal{F}_{\alpha,\beta}G(y|w) + \frac{\alpha^2}{2} \sum_{j=1}^n \mathcal{F}_{\alpha,\beta}F_j(y)\mathcal{F}_{\alpha,\beta}G_j(y), \\ &(\mathcal{F}_{\alpha,\beta}\delta F(y|\cdot) * \mathcal{F}_{\alpha,\beta}\delta G(y|\cdot))_{i\alpha/\beta}(w) \\ &= \frac{\beta^2}{2}\delta F(y|w)\delta G(y|w) + \frac{\alpha^2}{2} \sum_{j=1}^n F_j(y)G_j(y). \end{aligned} \quad (5.11)$$

**EXAMPLE 5.5.** Next, we briefly discuss the functionals  $F(x) = \sum_{j=1}^n \langle \theta_j, x \rangle$ ,  $G(x) = \exp\{F(x)\}$ ,  $H(x) = F(x) \exp\{F(x)\}$ ,  $M(x) = [F(x)]^2 = [\sum_{j=1}^n \langle \theta_j, x \rangle]^2$ , and  $N(x) = \sum_{j=1}^n [\langle \theta_j, x \rangle]^2$ , all of which are elements of  $E_0$ . The following formulas follow quite readily for all  $\gamma$  and  $w$  in  $K$ :

$$\mathcal{F}_{\alpha,\beta}F(\gamma) = \beta F(\gamma), \quad (5.12)$$

$$\delta F(\gamma|w) = F(w), \quad (5.13)$$

$$\delta \mathcal{F}_{\alpha,\beta}F(\gamma|w) = \beta F(w), \quad (5.14)$$

$$\mathcal{F}_{\alpha,\beta}G(\gamma) = \exp\left\{\frac{n\alpha^2}{2} + \beta F(\gamma)\right\}, \quad (5.15)$$

$$\delta G(\gamma|w) = F(w) \exp\{F(\gamma)\}, \quad (5.16)$$

$$\delta \mathcal{F}_{\alpha,\beta}G(\gamma|w) = \beta F(w) \exp\left\{\frac{n\alpha^2}{2} + \beta F(\gamma)\right\}, \quad (5.17)$$

$$\mathcal{F}_{\alpha,\beta}H(\gamma) = [n\alpha^2 + \beta F(\gamma)] \exp\left\{\frac{n\alpha^2}{2} + \beta F(\gamma)\right\}, \quad (5.18)$$

$$\delta H(\gamma|w) = [1 + F(\gamma)]F(w) \exp\{F(\gamma)\}, \quad (5.19)$$

$$\delta \mathcal{F}_{\alpha,\beta}H(\gamma|w) = \beta F(w) [n\alpha^2 + \beta F(\gamma) + 1] \exp\left\{\frac{n\alpha^2}{2} + \beta F(\gamma)\right\}, \quad (5.20)$$

$$\mathcal{F}_{\alpha,\beta}M(\gamma) = n\alpha^2 + [\beta F(\gamma)]^2, \quad (5.21)$$

$$\delta M(\gamma|w) = 2F(w)F(\gamma), \quad (5.22)$$

$$\delta \mathcal{F}_{\alpha,\beta}M(\gamma|w) = 2\beta^2 F(w)F(\gamma), \quad (5.23)$$

$$\mathcal{F}_{\alpha,\beta}N(\gamma) = n\alpha^2 + \beta^2 N(\gamma), \quad (5.24)$$

$$\delta N(\gamma|w) = 2 \sum_{j=1}^n \langle \theta_j, w \rangle \langle \theta_j, \gamma \rangle = \sum_{j=1}^n N_j(\gamma) F_j(w), \quad (5.25)$$

$$\delta \mathcal{F}_{\alpha,\beta}N(\gamma|w) = \beta^2 \delta N(\gamma|w) = \beta^2 \sum_{j=1}^n N_j(\gamma) F_j(w). \quad (5.26)$$

Finally, note that by using the various formulas in Sections 3 and 4 together with the formulas (5.12) through (5.26), we can immediately write down many additional formulas involving the specific functionals  $F$ ,  $G$ ,  $H$ ,  $M$ , and  $N$  defined above in Example 5.5. For example, using (3.1), (5.15), and (5.21), we observe that

$$\mathcal{F}_{\alpha,\beta}(M * G)_\alpha(\gamma) = \left[n\alpha^2 + \frac{\beta^2}{2} F^2(\gamma)\right] \exp\left\{\frac{n\alpha^2}{2} + \frac{\beta}{\sqrt{2}} F(\gamma)\right\}, \quad (5.27)$$

and hence using (5.13), (5.16), and (5.22),

$$\begin{aligned} \delta \mathcal{F}_{\alpha,\beta}(M * G)_\alpha(\gamma|w) &= \left[n\alpha^2 + \frac{\beta^2}{2} F^2(\gamma)\right] \frac{\beta}{\sqrt{2}} F(w) \exp\left\{\frac{n\alpha^2}{2} + \frac{\beta}{\sqrt{2}} F(\gamma)\right\} \\ &\quad + \beta^2 F(\gamma) F(w) \exp\left\{\frac{n\alpha^2}{2} + \frac{\beta}{\sqrt{2}} F(\gamma)\right\}. \end{aligned} \quad (5.28)$$

**REMARK 5.6.** For  $\sigma \in [0, 1)$ , let  $E_\sigma$  be the space of all functionals  $F : K \rightarrow \mathbb{C}$  of the form (1.7) for some positive integer  $n$ , where  $f(\lambda_1, \dots, \lambda_n)$  is an entire function of the

$n$  complex variables  $\lambda_1, \dots, \lambda_n$  such that

$$|f(\lambda_1, \dots, \lambda_n)| \leq A_F \exp \left\{ B_F \sum_{j=1}^n |\lambda_j|^{1+\sigma} \right\} \quad (5.29)$$

for some positive constants  $A_F$  and  $B_F$ . Note that if  $\sigma = 0$ , then  $E_\sigma = E_0$  and for  $0 < \sigma_1 < \sigma_2 < 1$ ,  $E_0 \subsetneq E_{\sigma_1} \subsetneq E_{\sigma_2} \subsetneq L_2(C_0[0, T])$ .

A careful examination of the proofs of Theorems 2.1, 2.2, 2.3, and 2.4 shows that the conclusions of all four of these theorems hold for all  $F$  and  $G$  in  $E_\sigma$ ,  $0 \leq \sigma < 1$ . For example, to show that the conclusions of Theorem 2.1 hold for  $E_\sigma$ , let  $F \in E_\sigma$  be given by (1.7) with  $f$  satisfying (5.29). Then proceeding as in the proof of Theorem 2.1, we obtain that  $\mathcal{F}_{\alpha, \beta} F$  is given by (2.1) with  $h$  defined by (2.2) satisfying

$$|h(\lambda_1, \dots, \lambda_n)| \leq A_{\mathcal{F}_{\alpha, \beta} F} \exp \left\{ B_{\mathcal{F}_{\alpha, \beta} F} \sum_{j=1}^n |\lambda_j|^{1+\sigma} \right\} \quad (5.30)$$

with

$$A_{\mathcal{F}_{\alpha, \beta} F} = A_F \left( \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left\{ -\frac{u^2}{2} + B_F (2|\alpha\mu|)^{1+\sigma} \right\} du \right)^n < \infty, \quad (5.31)$$

and with  $B_{\mathcal{F}_{\alpha, \beta} F} = B_F (2|\beta|)^{1+\sigma}$ . Hence  $\mathcal{F}_{\alpha, \beta} F$  exists and belongs to  $E_\sigma$ .

**SOME POSSIBLE EXTENSIONS.** It seems likely that using the functionals in  $E_0$  (or  $E_\sigma$ ) as building blocks, one could show that the results established in this paper hold for larger classes of functionals.

For example, let  $\{F_m\}_{m=1}^\infty$  be a sequence from  $E_0$  such that  $\lim_{m \rightarrow \infty} F_m(\gamma)$  exists for all  $\gamma \in K$  and let  $F(\gamma) = \lim_{m \rightarrow \infty} F_m(\gamma)$ . Now the condition

$$|F_m(\gamma)| \leq A \exp \{B \|\gamma\|_\infty\} \quad (5.32)$$

for all  $\gamma \in K$  and all  $m = 1, 2, \dots$  ensures the existence of the integral transform  $\mathcal{F}_{\alpha, \beta} F$  since by the dominated convergence theorem,

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathcal{F}_{\alpha, \beta} F_m(\gamma) &= \lim_{m \rightarrow \infty} \int_{C_0[0, T]} F_m(\alpha x + \beta \gamma) m(dx) \\ &= \int_{C_0[0, T]} F(\alpha x + \beta \gamma) m(dx) \\ &= \mathcal{F}_{\alpha, \beta} F(\gamma) \end{aligned} \quad (5.33)$$

for each  $\gamma \in K$ . Example 5.7 shows that  $F$  need not belong to  $E_\sigma$  for any  $\sigma \in [0, 1)$ .

It seems as though finding appropriate conditions to put on the sequences  $\{F_m\}_{m=1}^\infty$  and  $\{G_m\}_{m=1}^\infty$  from  $E_0$  to ensure the existence of  $(F * G)_\alpha$  should not be too difficult. However to proceed further, a major key would be to find appropriate conditions to put on the functionals  $\{F_m\}_{m=1}^\infty$  in order to ensure the existence of  $\delta F$ .

**EXAMPLE 5.7.** Let  $\{\theta_j\}_{j=1}^\infty$  be a complete orthonormal sequence of functions in  $L_2[0, T]$ , each of bounded variation on  $[0, T]$ . For  $m = 1, 2, \dots$  and  $y \in K$ , let

$$F_m(y) = \exp \left\{ \sum_{j=1}^m \frac{\langle \theta_j, y \rangle}{2^j C_j} \right\} \quad (5.34)$$

with  $C_j$  given by (1.6). Clearly  $F_m \in E_0$  for each  $m = 1, 2, \dots$

Also for each  $m = 1, 2, \dots$  and each  $y \in K$ ,

$$|F_m(y)| \leq \exp \left\{ \|y\|_\infty \sum_{j=1}^m \frac{1}{2^j} \right\} \leq \exp \{ \|y\|_\infty \}. \quad (5.35)$$

But  $\lim_{m \rightarrow \infty} F_m(y) = \exp \{ \sum_{j=1}^\infty (\langle \theta_j, y \rangle / 2^j C_j) \} \equiv F(y)$  is not an element of  $E_0$  (or  $E_\sigma$  for  $0 \leq \sigma < 1$ ) because it depends upon  $\langle \theta_m, y \rangle$  for every  $m \in \{1, 2, \dots\}$  and so it cannot be written in the form (1.7) for any positive integer  $n$ ; recall that  $\{\theta_j\}_{j=1}^\infty$  is a complete orthonormal set of functions in  $L_2[0, T]$ .

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