

MAPPING PROPERTIES FOR CONVOLUTIONS INVOLVING HYPERGEOMETRIC FUNCTIONS

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For $\mu \geq 0$, we consider a linear operator $L_\mu : A \rightarrow A$ defined by the convolution $f_\mu * f$, where $f_\mu = (1 - \mu)z_2 F_1(a, b, c; z) + \mu z(z_2 F_1(a, b, c; z))'$. Let $\varphi^*(A, B)$ denote the class of normalized functions f which are analytic in the open unit disk and satisfy the condition $zf'/f \prec (1 + Az)/1 + Bz$, $-1 \leq A < B \leq 1$, and let $R_\eta(\beta)$ denote the class of normalized analytic functions f for which there exists a number $\eta \in (-\pi/2, \pi/2)$ such that $\operatorname{Re}(e^{i\eta}(f'(z) - \beta)) > 0$, $(\beta < 1)$. The main object of this paper is to establish the connection between $R_\eta(\beta)$ and $\varphi^*(A, B)$ involving the operator $L_\mu(f)$. Furthermore, we treat the convolution $I = \int_0^z (f_\mu(t)/t) dt * f(z)$ for $f \in R_\eta(\beta)$.

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1. Introduction. Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and S denotes the subclass of functions in A which are univalent in U . Moreover, let $S^*(\alpha)$ and $K(\alpha)$ be the subclasses of S consisting, respectively, of functions which are starlike of order α and convex of order α , where $0 \leq \alpha < 1$ in U . Clearly, we have $S^*(\alpha) \subseteq S^*(0) = S^*$, where S^* denotes the class of functions in A which are starlike in U and $K(\alpha) \subseteq K(0) = K$, where K denotes the class of functions in A which are convex in U , and we mention the well-known inclusion chain $K \subset S^*(1/2) \subset S^* \subset S$. For the analytic functions g and h on U with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function w on U such that $w(0) = 0$, $|w(z)| < 1$, and $g(z) = h(w(z))$ for $z \in U$. We denote this subordinated relation by

$$g \prec h \quad \text{or} \quad g(z) \prec h(z) \quad (z \in U). \quad (1.2)$$

For $-1 \leq A < B \leq 1$, a function p , which is analytic in U with $p(0) = 1$, is said to belong to the class $P(A, B)$ if

$$p(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in U). \quad (1.3)$$

The above condition means that p takes the values in the disk with a center $(1-AB)/(1-B^2)$ and a radius $|A-B|/(1-B^2)$. The boundary circle cuts the real axis at the points $(1+A)/(1+B)$ and $(1-A)/(1-B)$. A function $f \in A$ is said to be in $\varphi^*(A, B)$ if $zf'/f \in P(A, B)$, and in $K(A, B)$ if $zf' \in \varphi^*(A, B)$. The class $\varphi^*(A, B)$ was introduced by N. Shukla and P. Shukla [4]. Also, Janowski [2] introduced the class $P(A, B)$. For the fixed natural number n , the subclass $P_n(A, B)$ of $P(A, B)$ containing functions p of the form $p(z) = 1 + p_n z^n + \dots$, $z \in U$, was defined by Stankiewicz and Waniurski [7]. In addition, Stankiewicz and Trojnar-Spelina [6] investigated a function $p(z) = 1 - p_n z^n - \dots$ belongs to the class $R(n, A, B)$, where $A \in R$ and $B \in [0, 1]$ if $p(z) \prec (1 + Az)/(1 - Bz)$. Let $R_\eta(\beta)$ denote the class of functions $f \in A$ for which there exists a number $\eta \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re}[e^{i\eta}(f'(z) - \beta)] > 0 \quad (z \in U, \beta < 1). \quad (1.4)$$

Clearly, we have $R_\eta(\beta) \subset S$ ($0 \leq \beta < 1$). Furthermore, if a function f of the form (1.1) belongs to the class $R_\eta(\beta)$, then

$$|a_n| \leq \frac{2(1-\beta)\cos\eta}{n} \quad (n \in N \setminus \{1\}). \quad (1.5)$$

The class $R_\eta(\beta)$ was studied by Kanas and Srivastava [3].

The hypergeometric function ${}_2F_1(a, b, c; z)$ is given as a power series, converging in U , in the following way

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad (1.6)$$

where a , b , and c are complex numbers with $c \neq 0, -1, -2, \dots$, and $(\lambda)_n$ denotes the Pochhammer symbol (or the generalized factorial since $(1)_n = n!$) defined, in terms of the Gamma function Γ , by

$$\begin{aligned} (\lambda)_n &:= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1 & \text{if } n = 0, \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & \text{if } n \in N = \{1, 2, \dots\}. \end{cases} \end{aligned} \quad (1.7)$$

Note that ${}_2F_1(a, b, c; z)$, for $a = c$ and $b = 1$ (or, alternatively, for $a = 1$ and $b = c$), reduces to the relatively more familiar geometric function. We also

note that ${}_2F_1(a, b, c; 1)$ converges for $\operatorname{Re}(c - a - b) > 0$ and is related to the Gamma functions by

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \quad (1.8)$$

The Hadamard product (or convolution) of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \quad (1.9)$$

N. Shukla and P. Shukla [4] studied the mapping properties of a function f_{μ} to be as given in

$$f_{\mu}(z) = (1 - \mu)z {}_2F_1(a, b, c; z) + \mu z (z {}_2F_1(a, b, c; z))' \quad (\mu \geq 0), \quad (1.10)$$

and investigated the geometric properties of an integral operator of the form

$$I(z) = \int_0^z \frac{f_{\mu}(t)}{t} dt. \quad (1.11)$$

We now consider a linear operator $L_{\mu} : A \rightarrow A$ defined by

$$L_{\mu}(f) = f_{\mu}(z) * f(z). \quad (1.12)$$

For $\mu = 0$ in (1.12), $L_{\mu}(f) = [I_{a,b,c}(f)](z)$, which was introduced by Hohlov [1]. Also, Kanas and Srivastava [3], and Srivastava and Owa [5] showed that the operator $I_{a,b,c}(f)$ is the natural extensions of the Alexander, Libera, Bernardi, and Carlson-Shaffer operators. In this paper, we find a relation between $R_{\eta}(\beta)$ and $\varphi^*(A, B)$ involving the operator $L_{\mu}(f)$. Furthermore, we study to obtain some conditions for the starlikeness and convexity of the convolution of I and f , which are given by (1.11) and (1.1), respectively, for $f \in R_{\eta}(\beta)$.

2. Main results. We make use of the following lemma.

LEMMA 2.1 [4]. *Sufficient conditions for f of the form (1.1) to be in $\varphi^*(A, B)$ and $K(A, B)$ are*

$$\begin{aligned} \sum_{n=2}^{\infty} [(1+B)n - (A+1)] |a_n| &\leq B - A, \\ \sum_{n=2}^{\infty} n[(1+B)n - (A+1)] |a_n| &\leq B - A, \end{aligned} \quad (2.1)$$

respectively.

THEOREM 2.2. *Let $a > 1$, $b > 1$, and $c > a + b + 1$. If $f \in R_\eta(\beta)$ and the inequality*

$$\begin{aligned} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[(1+B) \left(1 + \frac{\mu ab}{c-a-b-1} \right) - (A+1) \left(\mu - \frac{(\mu-1)(c-a-b)}{(a-1)(b-1)} \right) \right] \\ & \leq (B-A) \left(\frac{1}{2(1-\beta)\cos\eta} + 1 \right) + \frac{(A+1)(\mu-1)(c-1)}{(a-1)(b-1)} \end{aligned} \quad (2.2)$$

is satisfied, then $L_\mu(f) \in \varphi^*(A, B)$.

PROOF. By Lemma 2.1, it suffices to show that

$$T_1 := \sum_{n=2}^{\infty} [(1+B)n - (A+1)] \left| \frac{(1+(n-1)\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \right| \leq B - A. \quad (2.3)$$

Since $f \in R_\eta(\beta)$ and $|a_n| \leq 2(1-\beta)\cos\eta/n$. Hence,

$$\begin{aligned} T_1 & \leq \sum_{n=2}^{\infty} [(1+B)n - (A+1)] \frac{(1+(n-1)\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \frac{2(1-\beta)\cos\eta}{n} \\ & = 2(1-\beta)\cos\eta \left\{ (1+B) \left(\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} - 1 \right) \right. \\ & \quad - \frac{(A+1)(c-1)}{(a-1)(b-1)} \left(\sum_{n=0}^{\infty} \frac{(a-1)_n(b-1)_n}{(c-1)_n(1)_n} - 1 - \frac{(a-1)(b-1)}{c-1} \right) \\ & \quad + \frac{(1+B)\mu ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\ & \quad \left. - (A+1)\mu \left[\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} - 1 \right. \right. \\ & \quad \left. \left. - \frac{c-1}{(a-1)(b-1)} \left(\sum_{n=0}^{\infty} \frac{(a-1)_n(b-1)_n}{(c)_n(1)_n} - 1 - \frac{(a-1)(b-1)}{c-1} \right) \right] \right\} \\ & = 2(1-\beta)\cos\eta \left\{ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[(1+B) \left(1 + \frac{\mu ab}{c-a-b-1} \right) \right. \right. \\ & \quad \left. \left. + (A+1) \left(\mu - \frac{(\mu-1)(c-a-b)}{(a-1)(b-1)} \right) \right] \right. \\ & \quad \left. - \left[1 + B - (A+1) \left(1 - \frac{(\mu-1)(c-1)}{(a-1)(b-1)} \right) \right] \right\}. \end{aligned} \quad (2.4)$$

Now, this last expression is bounded above by $B - A$ if (2.2) holds. \square

If we take $\mu = 0$, $A = 2\alpha - 1$, and $B = 1$ in [Theorem 2.2](#), we have the following corollary.

COROLLARY 2.3. *Let $a > 1$, $b > 1$, and $c > a + b + 1$. If $f \in R_\eta(\beta)$ and the inequality*

$$\begin{aligned} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[1 - \frac{\alpha(c-a-b)}{(a-1)(b-1)} \right] \\ & \leq (1-\alpha) \left(\frac{1}{2(1-\beta)\cos\eta} + 1 \right) - \frac{\alpha(c-1)}{(a-1)(b-1)} \end{aligned} \quad (2.5)$$

is satisfied, then $z_2F_1(a, b, c; z) * f \in S^*(\alpha)$.

If we take $\alpha = 0$, $\beta = 0$, and $\eta = 0$ in [Corollary 2.3](#), we get the following corollary.

COROLLARY 2.4. *Let $a > 1$, $b > 1$, and $c > a + b + 1$. If $f \in S$, and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{3}{2} \quad (2.6)$$

is satisfied, then $z_2F_1(a, b, c; z) * f \in S^*$.

THEOREM 2.5. *Let $a > 0$, $b > 0$, and $c > a + b + 2$. If $f \in R_\eta(\beta)$, and the inequality*

$$\begin{aligned} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[B - A + ((1+B)(1+2\mu) \right. \\ & \quad \left. - (A+1)\mu) \frac{ab}{c-a-b-1} + \frac{(1+B)\mu(a)_2(b)_2}{(c-a-b-2)_2} \right] \\ & \leq (B-A) \left(\frac{1}{2(1-\beta)\cos\eta} + 1 \right) \end{aligned} \quad (2.7)$$

is satisfied, then $L_\mu(f) \in K(A, B)$.

PROOF. The proof follows from [Lemma 2.1](#). Using the method of the proof of [Theorem 2.2](#), we omit the details involved. \square

For $\mu = 0$, $A = 2\alpha - 1$, and $B = 1$, [Theorem 2.5](#) yields the following corollary.

COROLLARY 2.6. *Let $a > 0$, $b > 0$, and $c > a + b + 2$. If $f \in R_\eta(\beta)$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[1 - \alpha + \frac{ab}{c-a-b-1} \right] \leq (1-\alpha) \left(\frac{1}{2(1-\beta)\cos\eta} + 1 \right) \quad (2.8)$$

*is satisfied, then $z_2F_1(a, b, c; z) * f \in K(\alpha)$.*

For $\alpha = 0$, $\beta = 0$, and $\eta = 0$, Corollary 2.6 yields the following corollary.

COROLLARY 2.7. *Let $a > 0$, $b > 0$, and $c > a + b + 1$. If $f \in S$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[1 + \frac{ab}{c-a-b-1} \right] \leq \frac{3}{2} \quad (2.9)$$

*is satisfied, then $z_2F_1(a, b, c; z) * f \in K$.*

In our next theorems, we find the sufficient conditions for $I * f$ to be in $\varphi^*(A, B)$ and $K(A, B)$. From the definition of I given by (1.11), we obtain

$$I(z) = z + \sum_{n=2}^{\infty} \frac{((1-\mu)+n\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n \quad (\mu \geq 0, z \in U). \quad (2.10)$$

THEOREM 2.8. *Let $a > 1$, $b > 1$, and $c > a+b$. If $f \in R_\eta(\beta)$ and the inequality*

$$\begin{aligned} (1+B-(A+1)\mu)_2F_1(a, b, c; 1) - (A+1)(1-\mu)_4F_3(a, b, 1, 1, c, 2, 2; 1) \\ \leq (B-A) \left(\frac{1}{2(1-\beta)\cos\eta} + 1 \right) \end{aligned} \quad (2.11)$$

*is satisfied, then $I * f \in \varphi^*(A, B)$.*

PROOF. By Lemma 2.1, it satisfies to show that

$$T_2 := \sum_{n=2}^{\infty} ((1+B)n - (A+1)) \left| \frac{(1-\mu+n\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} a_n \right| \leq B - A. \quad (2.12)$$

Suppose that $f \in R_\eta(\beta)$. Then by (1.5) we observe that

$$\begin{aligned}
T_2 &\leq \sum_{n=2}^{\infty} ((1+B)n - (A+1)) \frac{(1-\mu+n\mu)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} \frac{2(1-\beta)\cos\eta}{n} \\
&= 2(1-\beta)\cos\eta \left\{ ((1+B)(1-\mu) - (A+1)\mu) \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} \right. \\
&\quad - (A+1)(1-\mu) \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n+1}} \\
&\quad \left. + (1+B)\mu \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \right\} \\
&= 2(1-\beta)\cos\eta \left\{ ((1+B)(1-\mu) - (A+1)\mu) \left(\frac{c-1}{(a-1)(b-1)} + {}_2F_1(a, b, c; 1) \right) \right. \\
&\quad - (A+1)(1-\mu) {}_4F_3(a, b, 1, 1, c, 2, 2; 1) \\
&\quad \left. + (1+B)\mu {}_2F_1(a, b, c; 1) \right. \\
&\quad \left. - \left[((1+B)(1-\mu) - (A+1)\mu) \frac{c-1}{(a-1)(b-1)} + B - A \right] \right\} \\
&\leq B - A
\end{aligned} \tag{2.13}$$

by (2.11). This completes the proof. \square

Taking $\mu = 0$, $A = 2\alpha - 1$, and $B = 1$ in [Theorem 2.8](#), we see the following corollary.

COROLLARY 2.9. *Let $a > 1$, $b > 1$, and $c > a + b$. If $f \in R_\eta(\beta)$ and the inequality*

$${}_2F_1(a, b, c; 1) - \alpha {}_4F_3(a, b, 1, 1, c, 2, 2; 1) \leq (1-\alpha) \left(\frac{1}{2(1-\beta)\cos\eta} + 1 \right) \tag{2.14}$$

*is satisfied, then $\int_0^z {}_2F_1(a, b, c; t) dt * f \in S^*(\alpha)$.*

Taking $\alpha = 0$, $\beta = 0$, and $\eta = 0$ in [Corollary 2.9](#), we get the following corollary.

COROLLARY 2.10. *Let $a > 1$, $b > 1$, and $c > a + b$. If $f \in S$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{3}{2} \tag{2.15}$$

*is satisfied, then $\int_0^z {}_2F_1(a, b, c; t) dt * f \in S^*$.*

THEOREM 2.11. *Let $a > 1$, $b > 1$, and $c > a + b + 1$. If $f \in R_\eta(\beta)$ and the inequality*

$$\begin{aligned} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[(1+B) \left(1 + \frac{\mu ab}{c-a-b-1} \right) \right. \\ & \quad \left. + (A+1) \left(\mu \left(\frac{c-a-b}{(a-1)(b-1)} - 1 \right) - \frac{c-a-b}{(a-1)(b-1)} \right) \right] \quad (2.16) \\ & \leq (B-A) \left(\frac{1}{2(1-\beta)\cos\eta} + 1 \right) - \frac{(1-\mu)(A+1)(c-1)}{(a-1)(b-1)} \end{aligned}$$

*is satisfied, then $I * f \in K(A, B)$.*

PROOF. The proof follows from [Lemma 2.1](#) and by applying similar method as in the proof of [Theorem 2.8](#); we omit the details involved. \square

If we let $\mu = 0$, $A = 2\alpha - 1$, and $B = 1$ in [Theorem 2.11](#), we get the following corollary.

COROLLARY 2.12. *Let $a > 1$, $b > 1$, and $c > a + b + 1$. If $f \in R_\eta(\beta)$ and the inequality*

$$\begin{aligned} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[1 - \frac{\alpha(c-a-b)}{(a-1)(b-1)} \right] \\ & \leq (1-\alpha) \left(\frac{1}{2(1-\beta)\cos\eta} + 1 \right) - \frac{\alpha(c-1)}{(a-1)(b-1)} \quad (2.17) \end{aligned}$$

*is satisfied, then $\int_0^z {}_2F_1(a, b, c; t) dt * f \in K(\alpha)$.*

If we let $\alpha = 0$, $\beta = 0$, and $\eta = 0$ in [Corollary 2.12](#), we have the following corollary.

COROLLARY 2.13. *Let $a > 1$, $b > 1$, and $c > a + b + 1$. If $f \in S$ and the inequality*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq \frac{3}{2} \quad (2.18)$$

*is satisfied, then $\int_0^z {}_2F_1(a, b, c; t) dt * f \in K$.*

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REFERENCES

- [1] Ju. E. Hohlov, *Operators and operations on the class of univalent functions*, Izv. Vyssh. Uchebn. Zaved. Mat. **10**(1978) (1978), 83-89 (Russian).
- [2] W. Janowski, *Some extremal problems for certain families of analytic functions. I*, Ann. Polon. Math. **28** (1973), 297-326.
- [3] S. Kanas and H. M. Srivastava, *Linear operators associated with k -uniformly convex functions*, Integral Transform. Spec. Funct. **9** (2000), no. 2, 121-132.

- [4] N. Shukla and P. Shukla, *Mapping properties of analytic function defined by hypergeometric function. II*, Soochow J. Math. 25 (1999), no. 1, 29–36.
- [5] H. M. Srivastava and S. Owa (eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing, New Jersey, 1992.
- [6] J. Stankiewicz and L. Trojnar-Spelina, *Some parametric family of functions*, Zeszyty Nauk. Politech. Rzeszowskiej Mat. Fiz. 14 (1992), 45–54.
- [7] J. Stankiewicz and J. Waniurski, *Some classes of functions subordinate to linear transformation and their applications*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 28 (1974), 85–94.

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