

## NONLINEAR DIFFERENTIAL EQUATIONS AND ALGEBRAIC SYSTEMS

LLOYD K. WILLIAMS

Department of Mathematics  
Texas Southern University  
Houston, Texas 77004  
U.S.A.

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ABSTRACT. In this paper we obtain the general solution of scalar, first-order differential equations. The method is variation of parameters with asymptotic series and the theory of partial differential equations.

The result gives us a form like a differential quotient requiring only that a limit be taken. Like the familiar expression for the solution of linear, first order, ordinary equations, it is the same in all cases.

KEY WORDS AND PHRASES. *Riccati Equations, Abel Equations, Cauchy-Kowalewski Theorem, Cauchy-Kowalewski System, Universal Cauchy-Kowalewski System.*

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### 1. INTRODUCTION.

We present a unified treatment for the general scalar, first-order, ordinary differential equation

$$y' = G(x,y), \quad G \in C^I.$$

Particular examples are linear equations, Riccati equations and Abel equations.

## 2. PRELIMINARIES.

We begin with the differential system

$$\left\{ \begin{array}{l} v_1' = f(v_1, v_2) = -v_1 v_2 \\ v_2' = h(v_1, v_2) = v_1 - v_2 \\ v \neq 0 \end{array} \right\} \quad (2.1)$$

with general solution  $v_1 = v_1(x, c_1, c_2)$ ,  $v_2 = v_2(x, c_1, c_2)$ . Here  $c_1, c_2$  are arbitrary constants.

Now let  $x = x(t)$ . Then we get

$$\left\{ \begin{array}{l} \dot{v}_1 = U_1 \dot{x} \quad \cdot = \frac{d}{dt} \\ v_2 = U_2 \dot{x} \\ U_1 = f(v_1, v_2), U_2 = h(v_1, v_2) \\ v_1 \neq 0 \end{array} \right\} \quad (2.2)$$

We are now ready to present the algebraic system referred to in the title.

## 3. THE CAUCHY-KOWALEWSKI SYSTEM.

Let  $w_1 = w_1(t, \epsilon)$ ,  $w_2 = w_2(t, \epsilon)$  be two functions of  $t$  and  $\epsilon$  (at present unknown).

The functions  $v_1, v_2$  have been given by (2.1). Finally two more unknown functions  $K(w_1, w_2, t, \epsilon)$  and  $L(w_1, w_2, t, \epsilon)$  will be defined by partial differential equations later. They will contain another variable,  $\lambda$ . It will be possible to substitute an arbitrary  $G(w_1, t)$  for  $\lambda$  to solve specific equations.

DEFINITION. The system of algebraic equations

$$\left\{ \begin{array}{l} \text{(a)} \quad w_1 - K(w_1, w_2, t, \epsilon) V_1 = 0 \\ \text{(b)} \quad w_2^2 - L(w_1, w_2, t, \epsilon) - V_2 = 0 \\ \text{(c)} \quad x = w_1 + t w_2 \\ \quad \quad w_1 \neq 0 \end{array} \right. \quad (3.1)$$

is called the Cauchy-Kowalewski system, for a specific  $G(w_1, t)$ . Using  $\lambda$  we will get a universal system.

Under suitable conditions on the functions  $K$  and  $L$ , we can solve it for  $w_1 = w_1(t, \epsilon)$  and  $w_2 = w_2(t, \epsilon)$ . We proceed by defining these functions as solutions of appropriate partial differential equations. We will derive these functions  $L(w_1, w_2, t, \epsilon, \lambda)$  and  $K(w_1, w_2, t, \epsilon, \lambda)$  and regard them as fixed like universal constants.

#### 4. THE FIRST FUNCTION $K$ IN THE CAUCHY-KOWALEWSKI SYSTEM.

We differentiate 3(a-b) with respect to  $t$  to get expressions for  $\dot{w}_1, \dot{w}_2$ . Denoting the expression for  $\dot{w}_1$  by  $R$  we get

$$\dot{w}_1 = R \quad (4.1)$$

To simplify notation, let  $K = \alpha$  in (4.1) and get

$$\dot{w}_1 = R = \frac{A_1 L_2 + A_2 L_3 + A_3}{-A_2 L_1 + A_4 L_2 + A_5} \quad (4.1a)$$

Some of the  $A_i$ ,  $i = 1, \dots, 5$  are given explicitly later. These are not partial derivatives. By contrast,

$$L_1 = \frac{\partial L}{\partial w_1} \quad \text{etc.}$$

Now let  $z = L - w_2^2$  and note that from (2.1), 3(a-b) we have  $f = \frac{w_1}{\alpha} z$ ,  $h = \frac{w_1}{\alpha} + z$  in the new notation.

The following equation is of fundamental importance. We arbitrarily set

$$A_2 = 2w_1w_2\alpha_1 - w_1t\alpha_1 + w_1\alpha_3 + tfa^2 = \varepsilon \quad (4.2)$$

where  $K = \alpha = \alpha(L, w_1, w_2, t, \varepsilon)$  and  $\alpha_1 = \frac{\partial \alpha}{\partial L}$ , etc., for real  $\varepsilon > 0$ .

By the Cauchy-Kowalewski theorem [See e.g. (2.1)] let  $\alpha_0 = \alpha_0(L, w_1, w_2, t, \varepsilon)$  be an analytic solution of (4.2). Further, we will write

$$\bar{A}_i = A_i(\alpha_0), \quad i = 1, 2, 3, 4, 5.$$

Let  $\alpha_0 = \sum_{n=0}^{\infty} c_n \varepsilon^n$  where  $c_n = c_n(L, w_1, w_2, t)$  are analytic. Before imposing conditions on  $c_0$  we give the following definitions.

$$\text{DEFINITION.} \quad \lim_{\varepsilon \rightarrow 0} L \left[ \left( \frac{w_1}{\alpha} + z \right) (w_1\alpha_{04} - w_1w_2\alpha_{02} + w_2\alpha_0) \right] = S_1(L, w_1, w_2, t).$$

Two more of the  $\bar{A}_i$  will now be given explicitly.

$$\bar{A}_1 = \left( \frac{w_1}{\alpha_0} + z \right) (w_1\alpha_{04} - w_1w_2\alpha_{02} + w_2\alpha_0)$$

$$\bar{A}_4 = w_1\alpha_{02} + \alpha_0^2 f - \alpha_0 - w_1h\alpha_{01}$$

$$\text{DEFINITION.} \quad \lim_{\varepsilon \rightarrow 0} L \bar{A}_1 = \Delta.$$

$$\text{DEFINITION.} \quad \lim_{\varepsilon \rightarrow 0} L (\bar{A}_1 - G(w_1, t)\bar{A}_4) = S_2(L, w_1, w_2, t).$$

The conditions on  $c_0$  can be stated now as follows:

$$(1) \quad c_0 \neq 0, \quad (2) \quad S_1(L, w_1, w_2, t) \neq 0, \quad (3) \quad \Delta \neq 0.$$

Substituting  $\alpha_0 = \sum_{n=0}^{\infty} c_n \varepsilon^n$  in (4.2) we get

$$2w_1w_2c_{01} - w_1t\left(\frac{w_1}{c_0} + z\right)c_{01} + w_1c_{03} + tw_1zc_0 = 0 \quad (4.3)$$

of which some solutions are given

$$H[\beta(c_o, z, w_1), w_2 + \frac{1}{t} P(c_o, w_1, \beta(c_o, z, w_1))] = \text{constant} \quad (4.3a)$$

where

(1)  $H$  is arbitrary

(2)  $\beta$  satisfies the partial differential equation

$$c_o z \beta_1 + \left(\frac{w_1}{c_o} + z\right) \beta_2 = 0$$

$$(w_1 \beta_3 + c_o \beta_1 \neq 0)$$

(3)  $P$  is defined as follows: first solve  $\beta(c_o, w_1, z) = a$  for  $z = Q(c_o, w_1, a)$ . Then set

$$P = \int \frac{d c_o}{c_o Q(c_o, w_1, a)}.$$

THEOREM 1. The function  $H$  can be chosen analytic in (4.3a) so that conditions (2.1), (2.2), (3.1) hold for  $c_o$ .

PROOF. Let  $\gamma = w_2 + \frac{1}{t} P$  and then (4.3a) becomes  $H(\beta, \gamma) = \text{constant}$ . The partial derivatives of  $c_o$  are computed from (4.3a) and from them we see that  $H_\gamma \neq 0$  implies that  $\frac{\partial c_o}{\partial t} \neq 0$ , so condition (2.1) holds. Further,  $\Delta = \lim_{\epsilon \rightarrow 0} \frac{L \bar{A}_1}{\epsilon} = 0$  implies  $\left(\frac{P}{t} + w_2\right) H_\gamma = 0$ . So  $H_\gamma \neq 0$  implies  $\Delta \neq 0$ . Thus (2.1), (2.2) hold if merely  $H_\gamma \neq 0$ . Now  $S_1 = 0$  implies that  $tw_2(w_1 \beta_3 + c_o \beta_1) H_\beta + H_\gamma = 0$ . Since  $w_1 \beta_3 + c_o \beta_1 \neq 0$ , we can choose  $H$  so that  $S_1 \neq 0$ . This completes the proof.

Summarizing the results of this section,  $K = \alpha = \alpha_o$  can be defined as the solution of (4.2) where  $H$  is analytic,  $c_o \neq 0$ ,  $S_1 \neq 0$ , and  $\Delta \neq 0$ . To solve (3.1) however, we must define  $L$ .

## 5. SOLUTION OF THE CAUCHY-KOWALEWSKI SYSTEM.

To solve the system (3.1), we must now define the function  $L(w_1, w_2, t, \epsilon)$ .

Setting  $\dot{w} = G$ ,  $\alpha = \alpha_0$  and  $\bar{A}_2 = \epsilon$ , (4.2) in (4.1a) suggests defining  $L$  by

$$\epsilon GL_1 + (\bar{A}_1 - G\bar{A}_4)L_2 + \epsilon L_3 = G\bar{A}_5 - \bar{A}_3.$$

$L_1 = \frac{\partial L}{\partial w_1}$ , etc. This does not seem to be feasible. Instead, letting  $\epsilon$  tend to zero leads to

$$L_2 = \frac{\partial L}{\partial w_2} = \frac{G\bar{A}_5 - \bar{A}_3}{\bar{A}_1 - G\bar{A}_4} \quad (5.1)$$

This will be used to define  $L$ .

Let  $\lambda$  be a new variable and consider

$$L_2 = \frac{\lambda\bar{A}_5 - \bar{A}_3}{\bar{A}_1 - \lambda\bar{A}_4} \quad (5.2)$$

Note that the right side of (5.2) is analytic where  $w_1 \neq 0$  and  $\bar{A}_1 - \lambda\bar{A}_4 \neq 0$ . So let  $L = \bar{L}(w_1, w_2, t, \epsilon, \lambda) = P_1(w_2) + P_2(w_1, w_2, t, \epsilon, \lambda)$  be an analytic solution on (5.2) and assume that none of the expressions  $\Delta$ ,  $S_1$ ,  $c_0$  vanish when  $L \equiv P_1(w_2)$ .

Now since the value of  $\frac{\partial}{\partial w_2}(\bar{L}(w_1, w_2, t, \epsilon, \lambda))$  for  $\lambda = G(w_1, t)$  is the same as  $\frac{\partial}{\partial w_2}(\bar{L}(w_1, w_2, t, \epsilon, G(w_1, t)))$  we see that  $\bar{\bar{L}}(w_1, w_2, t, \epsilon) \equiv \bar{L}(w_1, w_2, t, \epsilon, G(w_1, t))$  is a solution of (5.1) for any  $G$ . Moreover  $\bar{\bar{L}} \in C^I$  since  $G$  is continuous and  $\bar{L}$  is analytic. Let  $K_G = \alpha_0(\bar{\bar{L}}, w_1, w_2, t)$  and  $L = \bar{\bar{L}}$ .

We now prove the solvability near suitable points of the Cauchy-Kowalewski system. The variable  $\lambda$  gives our functions the universal character referred to previously.

LEMMA I. Let  $(a, b, c)$  be such that  $S_1(P_1(b)a, b, c) \neq 0$ . Then, for small  $t$ , the Jacobian of (3.1) is nonzero at  $(a, b, c, \epsilon)$ .

PROOF. If the Jacobian of (3.1) = 0, then

$$-\bar{A}_2 L_1 + \bar{A}_4 L_2 + \bar{A}_5 = 0 \quad (5.3)$$

The subsidiary equations of (5.3) are:

$$\frac{dw_1}{-\bar{A}_2} = \frac{dw_2}{\bar{A}_4} = \frac{dL}{-\bar{A}_5}, \quad \text{so that} \quad \frac{dL}{dw_2} = \frac{-\bar{A}_5}{\bar{A}_4}$$

$$\text{But from (5.1),} \quad \frac{dL}{dw_2} = \frac{\bar{GA}_5 - \bar{A}_3}{\bar{A}_1 - \bar{GA}_4}.$$

$$\text{Thus } \bar{A}_1 \bar{A}_5 - \bar{A}_3 \bar{A}_4 = 0.$$

$$\text{But } \bar{A}_1 \bar{A}_5 - \bar{A}_3 \bar{A}_4 = (w_1 \alpha_{04} - w_1 w_2 \alpha_{02} + w_2 \alpha_0) \left( \frac{w_1}{\alpha_0} + z \right) \epsilon. \quad \text{So}$$

$$\lim_{\epsilon \rightarrow 0} (w_1 \alpha_{04} - w_1 w_2 \alpha_{02} + w_2 \alpha_0) \left( \frac{w_1}{\alpha_0} + z \right) = 0. \quad \text{However}$$

$$\lim_{\epsilon \rightarrow 0} (w_1 \alpha_{04} - w_1 w_2 \alpha_{02} + w_2 \alpha_0) \left( \frac{w_1}{\alpha_0} + z \right) = S_1(P_1(w_2), w_1, w_2, t) \neq 0 \quad \text{and the}$$

proof is complete.

We next consider continuity in order to apply the implicit function theorem to (3.1). We first observe that  $\lim_{\epsilon \rightarrow 0} \bar{A}_1 \neq 0$ . If  $\lim_{\epsilon \rightarrow 0} \bar{A}_4 = 0$ , then

$$\lim_{\epsilon \rightarrow 0} (\bar{A}_1 - \bar{GA}_4) \neq 0.$$

Now consider the case where  $\lim_{\epsilon \rightarrow 0} \bar{A}_4 \neq 0$ , but  $\lim_{\epsilon \rightarrow 0} (\bar{A}_1 - \bar{GA}_4) = 0$ .

LEMMA II. There is at most one function  $G$  such that  $\lim_{\epsilon \rightarrow 0} (\bar{A}_1 - \bar{GA}_4) = 0$ .

PROOF.  $\bar{L}(w_1, w_2, t, \epsilon) = \bar{L}(w_1, w_2, t, G(w_1, t)) = P_1(w_2) + \epsilon P_2(w_1, w_2, t, \epsilon, G(w_1, t))$ .

So it and its partials with respect to  $w_1, w_2, t$  do not contain  $G$  as  $\epsilon \rightarrow 0$ . Since

$$\alpha_0 = \sum_{n=0}^{\infty} c_n(L, w_1, w_2, t) = c_0(L, w_1, w_2, t) + c_1(L, w_1, w_2, t) + c_2(L, w_1, w_2, t) \epsilon^2 + \dots,$$

the same holds for it.

Thus  $\lim_{\epsilon \rightarrow 0} \bar{A}_1$  and  $\lim_{\epsilon \rightarrow 0} \bar{A}_4$  are independent of  $G$ .

So  $G = \frac{\lim_{\epsilon \rightarrow 0} \bar{A}_1}{\lim_{\epsilon \rightarrow 0} \bar{A}_4}$ . This completes the proof.

In the sequel, we ignore this possible exception and assume that

$\lim_{\epsilon \rightarrow 0} (\bar{A}_1 - G\bar{A}_4) \neq 0$  for any  $G$ .

LEMMA III. If  $(a, b, c)$  is such that  $S_2(P_1(b), a, b, c) \neq 0$ , there is an  $\epsilon > 0$  such that the left sides of (3.1) are  $C^I$  at  $(a, b, c, \epsilon)$ .

PROOF. Based on analytic properties of  $V_1, V_2, \bar{L}, K_G$  and the nonvanishing of  $S_2$ , we will not give details.

Choosing constant values for  $w_1, w_2$  in (3.1), we can get  $c_1(\epsilon), c_2(\epsilon)$  so that left sides vanishes and apply the implicit function theorem to (3.1). Then we solve for  $w_1(t, \epsilon)$  and  $w_2(t, \epsilon)$ . Here  $c_1, c_2$  come from equation (2.1) of section 2.

## 6. THE PRINCIPAL DIFFERENTIAL EQUATION.

We now consider the differential equation

$$\frac{dy}{dx} = y' = g(x, y) \quad (6.1)$$

DEFINITION.  $W_1(t) = \lim_{\epsilon \rightarrow 0} w_1(t, \epsilon)$ .

It will be shown that  $W_1(t)$  satisfies (6.1). Of course we change  $y, x$  to  $W_1, t$  respectively.

We begin this process with

THEOREM II. Let  $S_1 \neq 0$  at  $(\bar{w}_1, \bar{w}_2, \bar{t})$ . Then  $\frac{d}{dt} w_1(t, \epsilon) \rightarrow G(\bar{w}_1, \bar{t})$  as  $\epsilon \rightarrow 0$ .

PROOF.  $\bar{L} = P_1(w_2) + \epsilon P_2(w_1, w_2, t, G(w_1, t))$  so that  $\frac{\partial \bar{L}}{\partial w_1} \rightarrow 0$  as  $\epsilon \rightarrow 0$  and also  $\frac{\partial \bar{L}}{\partial t} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Thus  $\bar{L}_1, \bar{L}_3 \rightarrow 0$  as  $\epsilon \rightarrow 0$ .



Now (5.1)  $\bar{A}_1 L_2 + \bar{A}_3 = G(\bar{A}_4 L_2 + \bar{A}_5)$ .

$$\text{Also } \bar{A}_4 L_2 + \bar{A}_5 = \frac{\bar{A}_1 \bar{A}_5 - \bar{A}_3 \bar{A}_4}{\bar{A}_1 - G \bar{A}_4} = \frac{S_1 \epsilon}{\bar{A}_1 - G \bar{A}_4}.$$

$$\text{Thus } R = \frac{\bar{A}_1 L_2 + \bar{A}_2 L_3 + \bar{A}_3}{-\bar{A}_2 L_1 + \bar{A}_4 L_2 + \bar{A}_5} = \frac{\epsilon L_3 + \bar{A}_1 L_2 + \bar{A}_3}{-\epsilon L_1 + \bar{A}_4 L_2 + \bar{A}_5}.$$

$$\text{So } \dot{w}_1 = \frac{\epsilon L_3 + G(\bar{A}_4 L_2 + \bar{A}_5)}{-\epsilon L_1 + (\bar{A}_4 L_2 + \bar{A}_5)} = \frac{(\bar{A}_1 - G \bar{A}_4) L_3 + G S_1}{-(\bar{A}_1 - G \bar{A}_4) L_1 + S_1}.$$

Therefore  $\dot{w}_1 \rightarrow \frac{G S_1}{S_1}$  as  $\epsilon \rightarrow 0$  and  $S_1 \neq 0$ . This completes the proof.

By the last theorem,  $L_{\epsilon \rightarrow 0} \frac{d}{dt} w_1(t, \epsilon) = L_{\epsilon \rightarrow 0} G(w_1(t, \epsilon), t) = G(L_{\epsilon \rightarrow 0} w_1(t, \epsilon), t) =$

$G(w_1(t), t)$ .

But also it is true [2: P.461] that

$$L_{\epsilon \rightarrow 0} \frac{d}{dt} w_1(t, \epsilon) = \frac{d}{dt} (L_{\epsilon \rightarrow 0} w_1(t, \epsilon)) = w_1'(t).$$

$$\text{So } w_1'(t) = G(w_1(t), t) \quad (6.2)$$

## 7. PARTICULAR AND GENERAL SOLUTIONS OF $y' = G(x, y)$ .

7(a) PARTICULAR SOLUTIONS. Let  $J(w_1, t) \in C^I$ ,

$$L^*(w_1, w_2, t) = \bar{L}(w_1, w_2, t, \epsilon, J(w_1, t)) \text{ and } \alpha^*(w_1, w_2, t) = \alpha_0(L^*, w_1, w_2, t).$$

Let  $Q$  be the set of points in  $(w_1, w_2, t)$ -space where

$$(1) \ w_1 \neq 0 \quad (2) \ c_0 \neq 0 \quad (3) \ S_1 \neq 0 \quad (4) \ S_2 \neq 0.$$

Let  $\bar{Q}$  be the projection of  $Q$  on the  $(w_1, t)$  plane.

### The Universal Cauchy-Kowalewski System

DEFINITION.  $\bar{\alpha}(w_1, w_2, t, \epsilon, \lambda) = \alpha_0(\bar{L}, w_1, w_2, t)$ .

DEFINITION.  $F_1 \equiv w_1 - \bar{\alpha} V_1(w_1 + t w_2, c_1, c_2)$ .

DEFINITION.  $F_2 \equiv w_2^2 - \bar{L}(w_1, w_2, t, \epsilon, \lambda) - v_2(w_1 + tw_2, c_1, c_2)$ .

DEFINITION.  $F_3 \equiv \bar{A}_1 - J(w_1, t)\bar{A}_4$  with  $\lambda$  replaced by  $J(w_1, t)$ .

DEFINITION. The system 
$$\begin{cases} F_1 = 0 \\ F_2 = 0 \\ F_3 \neq 0 \end{cases}$$
 is also called the Universal

Cauchy-Kowalewski System.

We refer to it in the following

THEOREM III. Let  $P \in \bar{Q}$ . There is a region in which the solution through  $P$  of  $\dot{w}_1 = J(w_1, t)$  is determined as follows:

- (1) In  $F_1, F_2$  replace  $\lambda$  by  $J(w_1, t)$  and  $c_1, c_2$  by suitable functions of  $\epsilon$ .
- (2) Equate the results in (2.1) to zero.
- (3) Solve the resulting system for  $w_1(t, \epsilon)$  and  $w_2(t, \epsilon)$ .
- (4) Take the limit of  $w_1(t, \epsilon)$  as  $\epsilon \rightarrow 0$ .

PROOF. Let  $P = (a, t_0)$ ,  $P \in \bar{Q}$ . Since  $c_0(P_1(b), a, b, t_0) \neq 0$ , there is an  $\epsilon$  such that  $\alpha^*(a, b, t_0, \epsilon) \neq 0$ . Let  $(\bar{v}_1, \bar{v}_2)$  be a solution of (2.1) such that

$$\left\{ \begin{array}{l} \bar{v}_1(a + t_0 b) = \frac{a}{\alpha^*(a, b, t_0, \epsilon)} \\ \bar{v}_2(a + t_0 b) = b^2 - L^*(a, b, t_0, \epsilon) \end{array} \right\}$$

Solve the system:

$$\left\{ \begin{array}{l} (1) \quad v_1(a + t_0 b, c_1, c_2) - \frac{a}{\alpha^*(a, b, t_0, \epsilon)} = 0 \\ (2) \quad v_2(a + t_0 b, c_1, c_2) - b^2 + L^*(a, b, t_0, \epsilon) = 0 \end{array} \right\}$$

to get suitable  $c_1 = c_1(\epsilon)$ ,  $c_2 = c_2(\epsilon)$ .

Since  $S_1 \neq 0$  our system has nonzero Jacobian. We solve for  $w_1(t, \epsilon)$  and get the result.

7(b) GENERAL SOLUTIONS. Alternatively, eliminating  $w_2$  from the Universal Cauchy-Kowalewski System we get

$$X(w_1, t, \epsilon, \lambda, c_1, c_2) = 0 \quad (7.1)$$

where  $c_1, c_2$  are constants.

The general solution of a specific equation is obtained as follows:

- (1) Replace  $\lambda$  by  $G(w_1, t)$  in (7.1).
- (2) Take the limit as  $\epsilon \rightarrow 0$  of the result.

$X$  is derived from  $L$  and  $K$  and is like the familiar differential quotient in generality.

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