COMPLETE RESIDUE SYSTEMS IN THE
RING OF MATRICES OF RATIONAL INTEGERS

JAU-SHYONG SHIUE

Department of Mathematical Sciences
National Chengchi University
Taipei, Taiwan
Republic of China

and

CHIE-PING HWANG

Department of Mathematics
National Central University
Chungli, Taiwan
Republic of China

ABSTRACT.   This paper deals with the characterizations of the complete residue

system mod. G, where G is any n×n matrix, in the ring of n×n matrices.

1.   INTRODUCTION.

    Let Z denote the ring of rational integers and Z(i) be the ring of

Gaussian integers. Jordan and Potratz [1] have exhibited several representations for the complete residue system (in short, C.R.S.) mod.r. in the ring of Gaussian integers. Also it is well known that the ring of Gaussian integers is isomorphic to the ring of 2×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, a, b in Z. This raises the question of characterizing the C.R.S. mod. G, where G is any $n \times n$ matrix, in the ring of $n \times n$ matrices of which we denote by $\text{Mat}_n(Z)$.

2. THE COMPLETE RESIDUE SYSTEM IN $\text{Mat}_n(Z)$.

First of all, we define $A|B$ mean there is a matrix C such that $B = CA$, and $A \equiv B$ mod. U means that $U|A - B$. Now we can give a definition of the C.R.S. mod. U in the ring of $\text{Mat}_n(Z)$.

DEFINITION. Let U be in $\text{Mat}_n(Z)$ with $\det U \neq 0$. Then a subset J of $\text{Mat}_n(Z)$ is called a C.R.S. mod. U if and only if for any A in $\text{Mat}_n(Z)$ there exists uniquely a matrix B in J such that $A \equiv B$ mod. U.

LEMMA 1. Let $G = \text{diag}(g_1, g_2, \ldots, g_n)$ with $g_i \neq 0$, $i = 1, 2, \ldots, n$. Let E_{ij} be the matrix units, then

$$I_{ik} = \{a \varepsilon Z : G \mid \sum_{m=i}^{n} \sum_{j=1}^{n} a_{mj} E_{mj} \text{ where } a_{mj} \text{ in } Z, a_{i1} = a_{i2} = \ldots = a_{ik-1} = 0, \ a_{ik} = a\}$$

are the principal ideals generated by a positive integer g_k, where $i, k = 1, 2, \ldots, n$.

PROOF. It is clear the I_{ik} are ideals in Z. But Z is a P.I.D., therefore I_{ik} are principal ideals generated by a positive integer d_{ik}. Since $g_k E_{ik} = E_{ik} G$, then g_k is in I_{ik}, i.e., $d_{ik}|g_k$. On the other hand, for d_{ik} in I_{ik} we have $\sum_{m=i}^{n} \sum_{j=1}^{n} a_{mj} E_{mj} = (t_{ik}) G$ for some (t_{ik}), where a_{mj} is in Z, $a_{i1} = a_{i2} = \ldots = a_{ik-1} = 0$, $a_{ik} = d_{ik}$. It follows that $d_{ik} = t_{ik} g_k$, i.e., $d_{ik} = |g_k|$. This completes the proof.

LEMMA 2. Let $G = \text{diag}(g_1, g_2, \ldots, g_n)$ with $g_k \neq 0$, $k = 1, 2, \ldots, n$. Then $J = \{(r_{ik}) : 0 \leq r_{ik} < |g_k|, \ i,k = 1, 2, \ldots, n\}$ forms a complete residue system mod. G.

PROOF. (1) For any $A = (a_{ik})$ in $\text{Mat}_n(Z)$, there exist p_{ik}, r_{ik} in Z such that $a_{ik} = p_{ik} |g_k| + r_{ik}$, where $0 \leq r_{ik} < |g_k|$. Therefore

$A - (p_{ik} \cdot |g_k|) = (r_{ik})$. But $|g_k| \cdot E_{ik} = |g_k| \cdot g_k^{-1} E_{ik} G$, and therefore

$G \mid A - (r_{ik})$. This shows that $A \equiv (r_{ik})$ mod. G.

(2) If $(r_{ik}) \equiv (s_{ik})$ mod. G, where $0 \leq r_{ik}, s_{ik} < |g_k|$, then $G \mid (r_{ik} - s_{ik})$, i.e., $r_{11} - s_{11}$ is in I_{11} (by Lemma 1). This implies that $g_1 \mid (r_{11} - s_{11})$, and so $r_{11} = s_{11}$, for $0 \leq | r_{11} - s_{11}| < |g_1|$. It follows that $r_{12} - s_{12}$ is in I_{12}. Therefore $g_2 \mid (r_{12} - s_{12})$ and $r_{12} = s_{12}$, for $0 \leq |r_{12} - s_{12}| < |g_2|$. Continuing in this way, we must have $r_{ik} = s_{ik}$, for all $i, k = 1, 2, \ldots, n$.

THEOREM 1. If G is a n×n matrix with $\det G \neq 0$, and if U and V are unimodular n×n matrices such that $UGV = \text{diag}(g_1, g_2, \ldots, g_n)$, then

$J = \{(r_{ik})V^{-1} : 0 \leq r_{ik} < |g_k|, \ i, k = 1, 2, \ldots, n\}$ forms a complete residue system mod. G.

PROOF. (1) By Lemma 2, for any n×n matrix A, there exists a matrix (r_{ik}) with $0 \leq r_{ik} < |g_k|$ such that $AV \equiv (r_{ik})$ mod. UGV., i.e., $A \equiv (r_{ik})V^{-1}$ mod. G.

(2) Let $(r_{ik})V^{-1} \equiv (s_{ik})V^{-1}$ mod. G, where $0 \leq r_{ik}, s_{ik} < |g_k|$. It follows that $(r_{ik}) \equiv (s_{ik})$ mod. UGV. Therefore $(r_{ik}) = (s_{ik})$.

COROLLARY 1. If J forms a C.R.S. mod. G, and U and V are unimodular n×n matrices, then $\{URV : R \text{ in } J\}$ forms a C.R.S. mod. GV.

COROLLARY 2. If G is a n×n matrix with $\det G \neq 0$, then the cardinality of the C.R.S. mod. G is $|\det G|^n$.

3. THE COMPLETE RESIDUE SYSTEM IN $Mat_2(Z)$.

By restricting the order of the matrix we may relax the condition on the diagonable matrix.

LEMMA 3.   Let $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \varepsilon\ Mat_2(Z)$ with $\det U \neq 0$, then

(1) $I_o = \{a\ \varepsilon\ Z : U\ |\ \begin{pmatrix} a & \alpha \\ \beta & r \end{pmatrix}$ for some $\alpha,\ \beta,\ r\ \varepsilon\ Z\}$ and

$I'_o = \{a\ \varepsilon\ Z : U\ |\ \begin{pmatrix} 0 & 0 \\ a & \delta \end{pmatrix}$ for some $\delta\ \varepsilon\ Z\}$ are nonzero principal ideals

of Z generated by a positive integer $d = $ g.c.d.(u_1, u_2). Moreover $I_o = I'_o$.

(2) $I_1 = \{a\ \varepsilon\ Z : U\ |\ \begin{pmatrix} 0 & a \\ \beta & r \end{pmatrix}$ for some $\beta,\ r\ \varepsilon\ Z\}$ and

$I'_1 = \{a\ \varepsilon\ Z : U\ |\ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\}$ are nonzero principal ideals of Z generated by

a positive integer $\dfrac{|\det U|}{d}$. Moreover, $I_1 = I'_1$.

PROOF.   (1) $a\ \varepsilon\ I_o$ implies $U\ |\ \begin{pmatrix} a & \alpha \\ \beta & r \end{pmatrix}$ for some $\alpha,\ \beta,\ r\ \varepsilon\ Z$, and then

$U\ |\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & \alpha \\ \beta & r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & \alpha \end{pmatrix}$, i.e., $a\ \varepsilon\ I'_o$. This shows that $I_o \subseteq I'_o$.

On the other hand, $b\ \varepsilon\ I'_o$ implies $U\ |\ \begin{pmatrix} 0 & 0 \\ b & \delta \end{pmatrix}$ for some $\delta\ \varepsilon\ Z$ and then

$U\ |\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b & \delta \end{pmatrix} = \begin{pmatrix} b & \delta \\ 0 & 0 \end{pmatrix}$, i.e., $b\ \varepsilon\ I_o$. Therefore $I_o = I'_o$. It is

clear that I_o is an ideal of Z. Now $\det U\ \varepsilon\ I_o$, for $U\ |\ \begin{pmatrix} \det U & 0 \\ 0 & \det U \end{pmatrix}$.

Thus I_o is a nonzero ideal of Z. But Z is a P.I.D., therefore I_o is an ideal

generated by a positive integer d. Since $U|U$ implies $U\ |\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}U = \begin{pmatrix} U_{21} & U_{22} \\ 0 & 0 \end{pmatrix}$,

we have $U_{11},\ U_{12}\ \varepsilon\ I_o$, and then $d|U_{11},\ d|U_{21}$. By $d\ \varepsilon\ I_o$, we have

$U^-\ |\ \begin{pmatrix} 0 & 0 \\ d & \delta \end{pmatrix}$, i.e., $\begin{pmatrix} 0 & 0 \\ d & \delta \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}U$ for some $\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \varepsilon\ Mat_2(Z)$.

Therefore $d = t_{21}U_{11} + t_{22}U_{21}$. If $x|U_{11}$ and $x|U_{21}$, then $x|d$. Thus

$d = $ g.c.d.$(U_{11},\ U_{21})$.

(2) $a \in I_1$ implies $U \mid \begin{pmatrix} 0 & a \\ \beta & r \end{pmatrix}$ for some $\beta, r \in Z$ and then

$U \mid \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ \beta & r \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$, i.e., $a \in I_1'$. Thus $I_1 \subseteq I_1'$. Conversely,

if $b \in I_1'$, then $U \mid \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ and so $U \mid \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, i.e.,

$b \in I_1$. It is also clear that I_1 is an ideal of Z. Now $\frac{\det U}{d} \in I_1$ for all

U such that $\begin{pmatrix} 0 & 0 \\ 0 & \frac{\det U}{d} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{-u_{21}}{d} & \frac{u_{12}}{d} \end{pmatrix} U$, and then I_1 is a nonzero ideal of

Z. But Z is a P.I.D., and then I_1 is an ideal generated by a positive integer

g. Now $\frac{\det U}{d} \in I_1$ implies $\frac{\det U}{d} \in I_1$, i.e., $g \mid \frac{|\det U|}{d}$. By $g \in I_1$, we have

$U \mid \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}$, i.e., $\det U \mid \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -gu_{21} & gu_{11} \end{pmatrix}$, and then

$\det U \mid gU_{21}$, $\det U \mid gU_{11}$.

By the proof of (1), we have $d = t_{21}U_{11} + t_{22}U_{21}$, and then

$$gd = t_{21}(gU_{11}) + t_{22}(gu_{21}) \text{ or } \frac{|\det U|}{d} \mid g. \text{ Therefore } g = \frac{|\det U|}{d}. \text{ This}$$

completes the proof of (2).

THEOREM 2. Let $U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in \text{Mat}_2(Z)$ with $\det U \neq 0$, let

$d = \text{g.c.d.}(u_{11}, u_{21})$. Then $J = \{R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \in \text{Mat}_2(Z) : 0 \leq r_{11},$

$r_{21} < d, 0 \leq r_{12}, r_{22} < \frac{|\det U|}{d}\}$ is a complete residue system (mod. U) in

$\text{Mat}_2(Z)$.

PROOF. (1) From $d \in I_o$, $\frac{|\det U|}{d} \in I_1$, we have

$U \mid \begin{pmatrix} d & \alpha \\ \beta & r \end{pmatrix}$, $U \mid \begin{pmatrix} 0 & 0 \\ d & \eta \end{pmatrix}$, $U \mid \begin{pmatrix} 0 & \frac{|\det U|}{d} \\ \epsilon & \delta \end{pmatrix}$, $U \mid \begin{pmatrix} 0 & 0 \\ 0 & \frac{|\det U|}{d} \end{pmatrix}$, i.e.,

there exists $T_i \in \text{Mat}_2(Z)$, $i = 1,2,3,4$ such that

$$\begin{pmatrix} d & \alpha \\ \beta & r \end{pmatrix} = T_1 U, \quad \begin{pmatrix} 0 & \dfrac{|\det U|}{d} \\ \varepsilon & \delta \end{pmatrix} = T_2 U, \quad \begin{pmatrix} 0 & 0 \\ d & \eta \end{pmatrix} = T_3 U, \quad \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{|\det U|}{d} \end{pmatrix} = T_4 U.$$

For any matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{Mat}_2(Z)$, there exists p_{11}, $r_{11} \in Z$ such

that $a_{11} = p_{11} d + r_{11}$ where $0 \leq r_{11} < d$. Thus $A - p_{11} T_1 U = \begin{pmatrix} r_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$,

for some $b_{12}, b_{21}, b_{22} \in Z$. Moreover, $b_{12} = P_{12} \dfrac{|\det U|}{d} + r_{12}$ for some

P_{12}, $r_{12} \in Z$, $0 \leq r_{12} < \dfrac{|\det U|}{d}$. Then $A - p_{11} T_1 U - p_{12} T_2 U = \begin{pmatrix} r_{11} & r_{12} \\ c_{21} & c_{22} \end{pmatrix}$

for some $c_{21}, c_{22} \in Z$. Again $c_{21} = p_{21} - d + r_{21}$ for some $p_{21}, r_{21} \in Z$,

$0 \leq r_{21} < d$. Then $A - p_{11} T_1 U - p_{12} T_2 U - p_{21} T_3 U = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ for some

$d_{22} \in Z$. Finally $d_{22} = P_{22} \dfrac{|\det U|}{d} + r_{22}$ for some $p_{22}, r_{22} \in Z$, $0 \leq r_{22} < \dfrac{|\det U|}{d}$,

implies $A - p_{11} T_1 U - p_{12} T_2 U - p_{21} T_3 U - p_{22} T_4 U = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ or

$U \,\Big|\, A - \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$, where $0 \leq r_{11}$, $r_{21} < d$, $0 \leq r_{22}$, $r_{12} < \dfrac{|\det U|}{d}$.

This proves that for any matrix $A \in \text{Mat}_2(Z)$ there exists $R \in J_2$ such that
$A \equiv R (\text{mod. } U)$.

(2)   Assume that $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \equiv \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ (mod. U) where

$0 \leq r_{11}$, r_{21}, s_{11}, $s_{21} < d$, $0 \leq r_{12}$, r_{22}, s_{12}, $s_{22} < \dfrac{|\det U|}{d}$.

This implies

$$U \left| \begin{pmatrix} r_{11}-s_{11} & r_{12}-s_{12} \\ r_{21}-s_{21} & r_{22}-s_{22} \end{pmatrix} \right. , \text{ i.e., } r_{11} - s_{11} \epsilon I_o, \text{ or } d \mid r_{11} - s_{11}.$$

Now $0 \le |r_{11} - s_{11}| < d$, $r_{11} = s_{11}$. It follows that $U \left| \begin{pmatrix} 0 & r_{12}-s_{12} \\ r_{21}-s_{21} & r_{22}-s_{22} \end{pmatrix} \right.$,

i.e., $r_{12} - s_{12} \epsilon I_1$, or $\dfrac{|\det U|}{d} \left| (r_{12} - s_{12}) \right.$. But $0 \le |r_{12}-s_{12}| < \dfrac{|\det U|}{d}$,

so that $r_{12} = s_{12}$.

It follows that

$$U \left| \begin{pmatrix} 0 & 0 \\ r_{21}-s_{21} & r_{22}-s_{22} \end{pmatrix} \right. , \text{ i.e., } r_{21} - s_{21} \epsilon I_o \text{ or } d \mid (r_{21} - s_{21}).$$

Also $0 \le |r_{21}-s_{21}| < d$, so that $r_{21} = s_{21}$. This implies that $U \left| \begin{pmatrix} 0 & 0 \\ 0 & r_{22}-s_{22} \end{pmatrix} \right.$,

i.e., $r_{22} - s_{22} \epsilon I_1$ or $\dfrac{|\det U|}{d} \left| (r_{22} - s_{22}) \right.$. Finally $0 \le |r_{22}-s_{22}| < \dfrac{|\det U|}{d}$,

so that $r_{22} = s_{22}$, i.e., $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$. This proves that any

two elements in J_2 are incongruent.

COROLLARY 3. Let $U \epsilon \text{Mat}_2(Z)$ with $\det U \ne 0$. Then the cardinality of the complete residue system (mod. U) is $|\det U|^2$.

REMARK. If we consider the ring of 3×3 matrices, the corresponding results will read as follows, the proofs will be as in Lemma 3 and Theorem 2, with possible minor changes.

LEMMA 4. Let $u = \left(u_{ij} \right) \epsilon \text{Mat}_3(Z)$ with $\det U \ne 0$. Then

$$(1) \quad I_o = \{a \epsilon Z : U \begin{pmatrix} a & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \epsilon Z\},$$

$$I_o' = \{a \in Z : U \mid \begin{pmatrix} 0 & 0 & 0 \\ a & \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in Z\}.$$

$$I_o'' = \{a \in Z : U \mid \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{32}, \alpha_{33} \in Z\}$$

are nonzero principal ideals of Z generated by the positive integer

$g_o = \text{g.c.d.} (u_{11}, u_{21}, u_{31})$. Moreover, $I_o = I_o' = I_o''$.

$$(2) \quad I_2 = \{a \in Z : U \begin{pmatrix} 0 & 0 & a \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in Z\},$$

$$I_2' = \{a \in Z : U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in Z\},$$

$$I_2'' = \{a \in Z : U \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix} \}$$

are nonzero principal ideals of Z generated by the positive integer

$g_2 = \dfrac{|\det U|}{g'}$, where $g' = \text{g.c.d.}(\text{cofu}_{13}, \text{cofu}_{23}, \text{cofu}_{33})$, and

cofu_{ij} is the cofactor of the element u_{ij}. Moreover, $I_2 = I_2' = I_2''$.

$$(3) \quad I_1 = \{a \in Z : U \mid \begin{pmatrix} 0 & a_1 & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in Z\}$$

$$I_1' = \{a \in Z : U \mid \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{ij} \in Z\}$$

$$I_1'' = \{a \in Z : U \mid \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_{33} \end{pmatrix} \text{ for some } \alpha_{33} \in Z\}$$

are nonzero principal ideals of Z generated by the positive integer $g_1 = \dfrac{g'}{g_o}$.

Moreover, $I_1 = I_1' = I_1''$.

THEOREM 3. Let $U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \; \epsilon \; \text{Mat}_3(Z)$ with $\det U \neq 0$, let

$g_o = \text{g.c.d.}(u_{11}, u_{21}, u_{31})$, $g' = \text{g.c.d.}(\text{cofu}_{13}, \text{cofu}_{23}, \text{cofu}_{33})$. Then

$J_3 = \{R = [r_{ij}] \; \epsilon \; \text{Mat}_3(Z) : 0 \leq r_{ij} < g_{j-1} \; i,j = 1,2,3\}$ is a complete

residue system (mod. U) where $g_1 = \dfrac{g'}{g_o}$, $g_2 = \dfrac{|\det U|}{g'}$.

REFERENCE

1. Jordan, J. H. and C. J. Potratz. Complete Residue Systems in the
    Gaussian Integers, <u>Math. Mag.</u> <u>38</u> (1965) 1-12.

Special Issue on
Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/bvp/guidelines.html. Authors should follow the Boundary Value Problems manuscript format described at the journal site http://www.hindawi.com/journals/bvp/. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	May 1, 2009
First Round of Reviews	August 1, 2009
Publication Date	November 1, 2009

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de Santiago de Compostela, Santiago de Compostela 15782, Spain; juanjose.nieto.roig@usc.es

Guest Editor

Donal O'Regan, Department of Mathematics, National University of Ireland, Galway, Ireland; donal.oregan@nuigalway.ie