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A CLASS OF GENERALIZED UNIFORM ASYMPTOTIC EXPANSIONS

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ABSTRACT. In [1] E. L. Reiss obtained a multivariable expansion of the problem $L[y] \equiv y'' + 2\varepsilon y' + y = 0$, $t > 0$; $y(0;\varepsilon) = 0$, $y'(0,\varepsilon) = 1$ by means of a technique which does not employ the traditional method of using a secularity condition in obtaining higher order approximations. In this paper Reiss' technique is formalized and new results are established.

0. INTRODUCTION.

Consider the problem

$$L[y] = y'' + 2\varepsilon y' + y = 0, \quad (' = \frac{d}{dt}), \ t > 0; \tag{0.1}$$

$$y(0;\varepsilon) = \gamma, \ y'(0,\varepsilon) = \delta. \tag{0.2}$$

where γ and δ are constants independent of ε. This problem will be referred to as (P1). We seek a formal expansion of the form

$$y = \sum_{J=0}^{M} \varepsilon^J y^J(t_o,\ldots,t_N) + R^M(t;\varepsilon) \tag{0.3}$$

where the time scales are defined by

$$t_j = \varepsilon^j t, \quad j = 0, 1, \ldots, N \tag{0.4}$$

Our task is to determine the coefficients y^J so that (0.3) constitutes a "generalized uniform asymptotic expansion" in an interval of t. Definitions to be given later will make clear the nature of these expansions.

When (0.3) and (0.4) are substituted into (0.1) we get an expression of the form

$$L[y] = \sum_{\ell=o}^{M+2N} \varepsilon^\ell \left[\sum_{\substack{J+k=\ell \\ 0\le J\le M}} Z_k^J \right] + L[R^M] = 0 \tag{0.5}$$

where

$$\begin{aligned}
Z_k^J &= 0 \text{ if } k < 0 \\
Z_o^J &= y^J_{t_o t_o} + y^J \\
Z_k^J &= \sum_{\substack{i+j=k \\ 0\le i,j\le N}} y^J_{t_j t_i} + 2y^J_{t_{k-1}}, \quad 1 \le k \le N+1 \\
Z_k^J &= \sum_{\substack{i+j=k \\ 0\le i,j\le N}} y^J_{t_j t_i}, \quad N+2 \le k \le 2N \\
Z_k^J &= 0, \quad k > 2N
\end{aligned} \tag{0.6}$$

To determine the y^Js we set the coefficients of ε^ℓ to zero for $\ell = 0, \ldots, M$ while M successively assumes the values 0, 1, 2... . The solutions of the resulting equations involve arbitrary functions of N variables which must be found by imposing additional conditions. These conditions are supplied by further setting the coefficients of $\varepsilon^{M+1}, \ldots, \varepsilon^{M+N}$ to zero. Finally, to make the coefficients determinate (0.3) is substituted into (0.2) and the initial

values of the y^Js are chosen so that (0.2) is satisfied as "closely" as possible. Specifically, when (0.3) is substituted into (0.2) we get

$$\gamma = \sum_{J=o}^{M} \varepsilon^J y^J(\vec{0}) + R^M(0)$$

and

$$\delta = \sum_{\ell=0}^{M+N} \varepsilon^{\ell} \sum_{\substack{J+k=\ell \\ 0\le J\le M \\ 0\le k\le N}} y^J_{t_k}(\vec{0}) + \frac{dR^M}{dt}(0)$$

where $(\vec{0})$ denotes the vector $(0, 0, \ldots, 0)$ corresponding to $t_o = t_1 = \ldots = t_N = 0$. Equating the coefficients of like powers of ε in the above as far as possible we obtain

$$y^o(\vec{0}) = \gamma,\ y^o_{t_o}(\vec{0}) = \delta \tag{0.7}$$

$$y^J(\vec{0}) = 0;\quad \sum_{\substack{J+k=\ell \\ 0\le J\le M \\ 0\le k\le N}} y^J_{t_k}(\vec{0}) = 0;\ J, \ell = 1, \ldots, M \tag{0.8}$$

At the end of the formal process we are left with the following problem for the error term:

$$L[R^M] = r^M(t;\varepsilon) = -\sum_{\ell=M+N+1}^{M+2N} \left(\varepsilon^{\ell} \sum_{\substack{J+k=\ell \\ 0\le J\le M}} Z^J_k \right) \tag{0.9}$$

$$R^M(0) = 0,\ \frac{dR^M}{dt}(0) = Y^M = -\sum_{\ell=M+1}^{M+N} \varepsilon^{\ell} \left(\sum_{\substack{J+k=\ell \\ 0\le J\le M \\ 0\le k\le N}} y^J_{t_k}(\vec{0}) \right) \tag{0.10}$$

In this analysis we delineate explicitly the form of the coefficients y^J and establish the uniform validity of the formal expansions over the infinite interval $[0,\infty)$. Estimates of the magnitude of the error are made from (0.9) and (0.10).

1. GENERAL DEPENDENCE OF COEFFICIENTS ON TIME SCALES

By setting the coefficients of ε^{ℓ} to zero in (0.5) for $\ell = 0, \ldots, M, \ldots, M+N$ as M ranges through the values 0,1,2,... it can be shown that for each J we obtain

$$Z_k^J = 0, \; k = 0,\ldots, N-1 \tag{1.1a}$$

$$Z_N^J = -\{Z_{J+N}^o + \ldots + Z_{N+1}^{J-1}\} \tag{1.1b}$$

The (N+1) equations in (1.1) give the dependence of y^J on the time scales t_o, t_1, ..., t_N, respectively. It is worth noting that dependence on t_o, ..., t_{N-1} is the same for each J. We will now solve these equations to get the coefficients y^J. If we assume that N is infinite, then we can explore the dependence of y^J on t_o, t_1, t_2, ... by solving $Z_o^J = Z_1^J = Z_2^J = \ldots = 0$. From (0.6) we obtain the first four equations:

$$y^J_{t_o t_o} + y^J = 0 \tag{1.2}$$

$$y^J_{t_o t_1} + y^J_{t_o} = 0 \tag{1.3}$$

$$2y^J_{t_o t_2} + y^J_{t_1 t_1} + 2y^J_{t_1} = 0 \tag{1.4}$$

$$2y^J_{t_o t_3} + 2y^J_{t_1 t_2} + 2y^J_{t_2} = 0 \tag{1.5}$$

The equations can be solved one after another. From (1.2) we get

$$y^J = A^1(t_1,t_2,\ldots)e^{it_o} + B^1(t_1,t_2,\ldots)e^{-it_o} \tag{1.6}$$

Now substitute (1.6) into (1.3) and require that (1.3) be satisfied identically. We obtain

$$A^1 = A^2(t_2,t_3,\ldots)e^{-t_1} \; ; \; B^1 = B^2(t_2,t_3,\ldots)e^{-t_1} \; .$$

Hence

$$y^J = e^{-t_1}\left[A^2(t_2,\ldots)e^{it_o} + B^2(t_2,\ldots)e^{-it_o}\right] \tag{1.7}$$

Similarly, we substitute (1.7) into (1.4) and so on. Proceeding in this manner we find that y^J is sinusoidal in t_2 but is independent of t_3 (from (1.5)). If we carry the process still further we find that y^J is sinusoidal in t_4 but is independent of t_5.

By induction we establish

$$y^J = e^{-t_1}\left[A(t_{2(j+1),\ldots})\, e^{i(t_o+\ldots+\alpha_{2j}t_{2j})} + B(t_{2(j+1),\ldots})\, e^{-i(t_o+\ldots+\alpha_{2j}t_{2j})}\right],\ j = 1, 2, \ldots \tag{1.8}$$

$$\left.\begin{aligned} &\alpha_o = 1\ ,\ \alpha_2 = -1/2\ , \\ &\text{and} \\ &\alpha_{2j} = -\frac{1}{2}\left(\alpha_2\alpha_{2(j-1)} + \ldots + \alpha_{2(j-1)}\alpha_2\right),\ \text{if } j \geq 2 \end{aligned}\right\} \tag{1.9}$$

We have therefore shown, by directly solving the governing differential equations, that when N is infinite the coefficients y^J are sinusoidal in the even time scales t_o, t_2, ... and independent of the odd scales t_3, t_5, ... Dependence on t_1 is exponential as shown in (1.8).

A link between the form (1.8) and the exact solution of (P1) can be established. The exact solution is

$$y = \gamma e^{-\varepsilon t}\cos\left(\sqrt{1-\varepsilon^2}\,t\right) + \frac{(\delta+\gamma\varepsilon)}{1-\varepsilon^2}\, e^{-\varepsilon t}\sin\left(\sqrt{1-\varepsilon^2}\,t\right) \tag{1.10}$$

The desired link consists in showing from (1.9) and mathematical induction that α_{2j} is the coefficient of ε^{2j} in the binominal expansion of $(1-\varepsilon^2)^{1/2}$.

That is, we can write α_{2j} alternatively as follows

$$\alpha_{2j} = (-1)^j \begin{pmatrix} 1/2 \\ j \end{pmatrix} \tag{1.11}$$

Now let us consider the practical case in which N is finite. As can be easily seen from (1.1), dependence on the time scales $t_o,\ldots,t_{N-1}$ will be exactly as in the case where N is infinite. In order to be able to write expressions in a compact form, we will modify the definition for α_{2j} when $2j > N$. Let

$$n = \begin{cases} N/2 \text{ if } N \text{ is even} \\ \\ (N-1)/2 \text{ if } N \text{ is odd} \end{cases} \tag{1.12}$$

Then

$$\left.\begin{aligned} &\alpha_{2j} = 0 \text{ if } j < 0 \\ &\alpha_o = 1,\ \alpha_2 = -1/2 \\ &\alpha_{2j} = -\frac{1}{2}\left(\alpha_2\alpha_{2(j-1)} + \ldots + \alpha_{2(j-1)}\alpha_2\right) = (-1)^j \begin{pmatrix} \frac{1}{2} \\ j \end{pmatrix}, \quad 2 \le j \le n \\ &\alpha_{2n+2r} = -\frac{1}{2}\left(\alpha_{2r}\alpha_{2n} + \ldots + \alpha_{2n}\alpha_{2r}\right), \quad 1 \le r \le n \\ &\alpha_{2n+2n} = -\frac{1}{2}\left(\alpha_{2n}\right)^2 \\ &\alpha_{2n+2r} = 0, \quad r > n \end{aligned}\right\} \tag{1.13}$$

An immediate consequence of the results already established is that

$$y^J = A^J(t_1)e^{it_o} + B^J(t_1)e^{-it_o}, \quad N = 1 \tag{1.14}$$

$$y^J = e^{-t_1}\left[A^J(t_N)e^{i(t_o+\ldots+\alpha_{2(n-1)}t_{2(n-1)})} + B^J(t_N)e^{-i(t_o+\ldots+\alpha_{2(n-1)}t_{2(n-1)})}\right] \quad N(\geq 2) \text{ even} \tag{1.15}$$

$$y^J = e^{-t_1}\left[A^J(t_N)e^{i(t_o+\ldots+\alpha_{2n}t_{2n})} + B^J(t_N)e^{-i(t_o+\ldots+\alpha_{2n}t_{2n})}\right] \quad N(>1) \text{ odd} \tag{1.16}$$

In (1.15) and (1.16) n and the frequencies α_{2j} are given by (1.12) and (1.13), respectively.

After a lengthy but straightforward algebra it can be established that

$$y^J(t_o,\ldots,t_N) = e^{-t_1}\left[P_q(t_N)e^{i(t_o+\ldots+\alpha_{2n}t_{2n})} + \overline{P}_q(t_N)e^{-i(t_o+\ldots+\alpha_{2n}t_{2n})}\right] \tag{1.17}$$

where P_q are complex valued polynomials in t_N, the last time scale in the set

$$t_j = \varepsilon^j t, \quad j = 0,\ldots,N \tag{1.18}$$

and n and the α_{2i} are given by (1.12) and (1.13). The degrees of the polynomials depend on both N and J as follows

$$q = \begin{cases} J/2 & , \text{ if J is even and N is even} \\ (J-1)/2 & , \text{ if J is odd and N is even} \\ J & , \text{ if N is odd} \end{cases} \tag{1.19}$$

The following form of y^J is sometimes convenient to use:

$$y^J(t_o,\ldots,t_N) = e^{-t_1}\left[2\text{Re}\{P_q(t_N)\}\cos(t_o+\ldots+\alpha_{2n}t_{2n}) - 2\text{Im}\{P_q(t_N)\}\sin(t_o+\ldots+\alpha_{2n}t_{2n})\right] \tag{1.20}$$

2. VALIDITY OF THE EXPANSIONS.

We seek a "generalized uniform asymptotic expansion" (see [1] and [3]) in an interval D, which is assumed to be generally finite and possibly infinite. More specifically, we use the following definition

DEFINITION 1

Let $\{S_j(t;\varepsilon)\}$ be a sequence of functions defined for $t \in D$ and $0 \leq \varepsilon \leq \varepsilon_{M^o}$ for some $\varepsilon_o > 0$, and let $\{f_j(\varepsilon)\}$, $\varepsilon \to 0$, be an asymptotic sequence. Then $\sum_{j=o} S_j(t;\varepsilon)$ is a generalized Uniform asymptotic expansion of $y(t;\varepsilon)$ in D to (M+1) terms if the remainders satisfy

$$R^k(t;\varepsilon) = y(t;\varepsilon) - \sum_{j=o}^{k} S_j(t;\varepsilon) = 0(f_{k+1}) \tag{2.1}$$

for each $k \in \{0,1,\ldots,M\}$.

In our analysis the element of the sequence, $S_j(t;\varepsilon)$ will be in the form $\varepsilon^J y^J(t_o,\ldots,t_N)$ and $f_k = \varepsilon^k$. We wish to show that

$$\sum_{J=o}^{M} \varepsilon^J y^J(t_o,\ldots,t_N)$$

is a uniform asymptotic expansion of the solution of (P1) in the sense of (2.1) where the interval D is infinite. To do this we need estimates of the error terms $R^M(t;\varepsilon)$.

Since L is a differential operator with constant coefficients, an integral representation of the solution of (0.9) can be obtained by means of the appropriate Green's function. With the integral representation estimates can be made as follows:

$$\left|R^M(t;\varepsilon)\right| \leq \alpha\left|Y^M\right| + (\beta/\varepsilon)\left|\left|r^M(t;\varepsilon)\right|\right|, \quad t \geq 0 \tag{2.2}$$

where

$$\left|\left| \cdot \right|\right| = \max_{t \geq 0} \left| \cdot \right| \tag{2.3}$$

and α and β are positive constants.

One of the goals of our analysis is to determine the link between $|R^M|$ and the initial conditions of (P1). This is achieved by expressing Y^M in terms of those initial conditions. The following lemma gives the required expressions.

Lemma.

The quantity $\frac{dR^M}{dt}(0) = Y^M$ can be written in the following terms

$$\frac{dR^M}{dt}(0) = Y_e^M = \gamma \sum_{0 \leq s \leq [(N-1)/2]} \Pi_{2s+1} \varepsilon^{M+(2s+1)}$$

$$+ \delta \sum_{1 \leq r \leq [N/2]} \Pi_{2r} \varepsilon^{M+2r}, \quad \text{if M is even} \tag{2.4}$$

$$\frac{dR^M}{dt}(0) = Y_o^M = \delta \sum_{0 \leq s \leq [(N-1)/2]} \Lambda_{2s+1} \varepsilon^{M+(2s+1)}$$

$$+ \gamma \sum_{1 \leq r \leq [N/2]} \Lambda_{2r} \varepsilon^{M+2r}, \quad \text{if M is odd} \tag{2.5}$$

The constants Π_j and Λ_j depend on the quantities α_{2i}. [x] is defined as the greatest integer less than or equal to x.

The proof of the lemma is lengthy but straightforward and is omitted for brevity.

Before considering the main theorem, let us introduce an additional notation which will enable us to state the theorem in a compact form. Define the function $z_\ell(t;\varepsilon)$ as follows:

$$z_\ell(t;\varepsilon) = - \sum_{\substack{J+k=\ell \\ 0\leq J\leq M}} Z_k^J \tag{2.6}$$

Using (2.6) we can rewrite equation (0.9) in the form

$$L[R^M] = r^M(t;\varepsilon) = \sum_{\ell=M+N+1}^{M+2N} \varepsilon^\ell z_\ell(t;\varepsilon) \tag{0.9b}$$

Now let us state the theorem.

Theorem

If the time scales are given by (0.4), then for $N \geq 1$ (0.3) is a (M+1) - term - (N+1) - time generalized uniform asymptotic expansion of the solution of (P1) in the interval $[0,\infty)$. The error $R^M(t;\varepsilon)$ is bounded in magnitude as follows:

$$\begin{aligned} \left|R^M(t;\varepsilon)\right| &\leq A_1|Y_e^M| + (B_1/\varepsilon)\ ||r^M(t;\varepsilon)||, \text{ if M is even} \\ &\leq A_2|Y_o^M| + (B_2/\varepsilon)\ ||r^M(t;\varepsilon)||, \text{ if M is odd} \end{aligned} \tag{2.7}$$

where Y_e^M and Y_o^M are given by (2.4) and (2.5), respectively;

$$||r^M(t;\varepsilon)|| \leq \sum_{\ell=M+N+1}^{M+2N} \varepsilon^\ell\ ||z_\ell(t;\varepsilon)|| \tag{2.8}$$

and A_1, A_2, B_1, B_2 are positive constants.

Proof

From equations (0.9), (2.2) - (2.5) we see that the theorem will be proved if we can show that

$$||z_\ell(t;\varepsilon)|| = || \sum_{\substack{J+k=\ell \\ 0\leq J\leq M}} Z_k^J\ || \tag{2.9}$$

exists for all $t \geq 0$. From (0.6) we see that for each ℓ, z_ℓ is a sum of terms like y^J, $y^J_{t_k}$ and $y^J_{t_j t_i}$; these terms are therefore of the form

$$t_N^q e^{-t_1} \left\{ \begin{array}{l} \sin(t_o+\ldots+\alpha_{2n}t_{2n}) \\ \cos(t_o+\ldots+\alpha_{2n}t_{2n}) \end{array} \right\} \tag{2.10}$$

where $q \leq M$. With $t_1 = \varepsilon t$ and $t_N = \varepsilon^N t$ it can be shown that the quantities in (2.10) are bounded by $(\varepsilon^{N-q})q^q e^{-q}$ for $t \geq 0$. Therefore $||z_\ell(t;\varepsilon)||$ exists. (2.7) follows from the lemma and equation (2.2) while (2.8) follows from (0.9b) and the triangle inequality. Thus the theorem is proved. Q.E.D.

The following corollaries are easily established.

Corollary 2.

If $N = 1$, then $R^M = 0(\varepsilon^{M+1})$ for all M.

Corollary 3.

Let N be greater than 1 and $\gamma = 0$. Then,

(I) $R^M = 0(\varepsilon^{M+1})$ if M is odd

(II) $R^M = 0(\varepsilon^{M+2})$ if M is even

(III) $y^J = 0$ when J is odd and N is even.

Corollary 4.

Let N be greater than 1 and $\delta = 0$. Then

(IV) $R^M = 0(\varepsilon^{M+2})$ if M is odd

(V) $R^M = 0(\varepsilon^{M+1})$ if M is even

3. GENERAL RESULTS

The first set of results to be presented in this section concerns the role of the number of time scales in the expansions. We give and clarify the following results originally stated by Reiss [1]:

> There are a minimum number of time scales, namely two, that are required to obtain a uniform asymptotic expansion. (RI)

For a fixed number of terms in the expansion three or more time scales give uniform expansions with the smallest estimate of the remainder. In particular, three time scales will always give uniform expansions with the smallest estimate of the remainder. (RII)

From (RII) it is seen that there may be no justification in using N>2 (i.e. more than three time scales). However, expansions using N>2 may be desirable for certain purposes as is the case when more accurate frequencies are desired. Recall the error bound given in (2.7) and (2.8). One of the terms in this bound consists of a sum of terms of the form

$$(\varepsilon^{N+\ell-q})\ q^q e^{-q},\quad 0\leq q\leq M,\quad M+N+1 \leq \ell \leq M+2N$$

Thus for fixed M this term can be decreased indefinitely by simply increasing the number of time scales. Only the term depending on the initial conditions cannot be so diminished. In a given initial value problem (M fixed) is it legitimate to ask if we can select the least number of time scales which give the smallest estimate of the remainder? Levine and Lubot [4] and Obi [2] have explored this question in some detail. The answer is in the affirmative. In many cases three is the least number, but with certain initial conditions (which depend upon ε) one can indeed increase the accuracy indefinitely by merely increasing the number of time scales (even for M=0).

We now turn to the role of the odd scales $\varepsilon^3 t$, $\varepsilon^5 t$, ... in the multi-time expansions. Reiss had reasoned that since these scales do not appear in the expansions of the exact solution the multi-time method need be applied using only the scales

$$t_o = t,\ t_1 = \varepsilon t, \ldots,\ t_k = \varepsilon^{2(k-1)} t,\quad k = 2,3,\ldots N$$

The present analysis clearly shows that expansions involving the odd scales

exist. We reproduce some of them:

$$y = y^0(t_0,t_1,t_2,t_3) + \varepsilon y^1(t_0,\ldots,t_3) + O(\varepsilon^2)$$
$$= e^{-t_1}\sin(t_0 - \tfrac{1}{2}t_2) + \varepsilon\left[-\tfrac{1}{8}t_3 e^{-t_1}\cos(t_0 - \tfrac{1}{2}t_2)\right] + O(\varepsilon^2) \quad (3.1)$$

$$y = y^0(t_0,\ldots,t_3) + \varepsilon y^1(t_0,\ldots,t_3) + \varepsilon^2 y^2(t_0,\ldots,t_3) + O(\varepsilon^4)$$
$$= e^{-t_1}\sin(t_0 - \tfrac{1}{2}t_2) + \varepsilon\left[-\tfrac{1}{8}t_3 e^{-t_1}\cos(t_0 - \tfrac{1}{2}t_2)\right] \quad (3.2)$$
$$+ \varepsilon^2\left[\tfrac{1}{2}(1 - \tfrac{1}{64}t_3^2)e^{-t_1}\sin(t_0 - \tfrac{1}{2}t_2)\right] + O(\varepsilon^4)$$

In these expansions $\gamma=0$, $\delta=1$ and t_k is given by (0.4). Observe that $R^0 = O(\varepsilon^2)$ and $R^1 = O(\varepsilon^2)$ so that Corollary 3 still holds. This will appear confusing at first until we further observe that in reality y^1 is "small". Since $t_3 e^{-t_1} = \varepsilon^3 t e^{-\varepsilon t}$ and $te^{-\varepsilon t}$ is uniformly bounded we see that $\varepsilon y^1 = O(\varepsilon^4)$. Hence this term does not improve the order of magnitude estimate of the error. The approach taken in this analysis has thus revealed a larger class of generalized uniform asymptotic expansions. The results of [1] can be recovered in one of two ways. The first is to work with the complete set (0.4) but with N even. In this case dependence on the odd scales drops out in direct computations. Another way is to seek expansions of the form

$$y = y^0(t_0,\ldots,t_3) + \varepsilon^2 y^2(t_0,\ldots,t_3) + \bar{R}^2$$

That is, we assume beforehand that the coefficient with the odd superscript vanishes. In this case, however, we obtain the same expansion as would result if we used only t_0,t_1,t_2. (This should be regarded as a coincidence since such results do not carry over when we consider equations with variable coefficients [2]).

The role of the initial conditions in determining the approximations is very interesting. When $N(\geq 2)$ is even the coefficients y^J, J an odd integer, are all proportional to $\gamma = y(0)$. These coefficients therefore vanish when $\gamma = 0$ but not otherwise (constrast Corollaries 3 and 4). The analysis in [1] considered only the case $\gamma = 0$ and the incomplete picture thus presented certainly gave rise to the false impression that the odd coefficients will vanish for other cases.

Another important set of observations concerning the influence of the number of time scales is fundamental to the understanding of multi-time expansions. This analysis shows clearly two kinds of dependence on individual time scales. The first kind is definitely a property of the differential equation involved. Thus if we assume that N is infinite only t_o, t_1 and the even scales appear in our expansions. These scales are precisely those which will appear in the uniformly valid expansions of the exact solution. The second kind of dependence may be characterized as a truncation effect. This is a consequence of using only a finite number of time scales and manifests itself in the form of polynominals in the last scale, t_N. It is due to this effect that we are able to get a class of expansions involving the other odd scales. These expansions were not anticipated, and in fact, were discounted in [1].

4. FINAL REMARKS

The general results reported in this paper are common to a broad class of generalized uniform asymptotic expansions. The basic feature of the method reported in [1] is found to be the imposition of secularity conditions by picking approximations of a certain form. The most general form of such approximations have been dealt with in [5] and [6] where it has been shown that time

scales may be derived from the governing differential equation.

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