

SOME FIXED POINTS OF EXPANSION MAPPINGS

H. K. PATHAK

Department of Mathematics,
Kalyan Mahavidyalaya,
Bhilai Nagar (M.P.) 490006, India

S. M. KANG

Department of mathematics,
Gyeongsang National University,
Chinju 660-701, Korea

J. W. RYU

Department of mathematics,
Dong-A University,
Pusan 604-714, Korea

(Received February 1, 1995 and in revised form April 19, 1994)

ABSTRACT. Wang et al. [11] proved some fixed point theorems on expansion mappings, which correspond some contractive mappings. Recently, several authors generalized their results by some ways.

In this paper, we give some fixed point theorems for expansion mappings, which improve the results of some authors.

KEY WORDS AND PHRASES. Expansion mappings and fixed points.

1992 AMS SUBJECT CLASSIFICATION CODE. 54H25.

1. INTRODUCTION.

Rhoades [8] summarized contractive mappings of some types and discussed on fixed points. Wang et al. [11] proved some fixed point theorems on expansion mappings, which correspond to some contractive mappings in [8]. Recently, by using functions, Khan et al. [5] generalized the results of [11], and Park and Rhoades [7] proved some fixed point theorems for expansion mappings. Also, Rhoades [9] and Taniguchi [10] generalized the results of [11] for pairs of mappings. Furthermore, Kang [3] and Kang and Rhoades [4] extend the results obtained by Khan et al. [5], Rhoades [9] and Taniguchi [10].

In this paper, we give some fixed point theorems for expansion mappings, which improve the results of Kang [3], Khan et al. [5], Rhoades [9] and Taniguchi [10].

2. THE MAIN THEOREMS.

Throughout this paper, following Boyd and Wong [1], let \mathcal{F} be the family of mappings such that for each $\phi \in \mathcal{F}$, $\phi : [0, \infty) \rightarrow [0, \infty)$ is upper semi-continuous from the right and non-decreasing in each coordinate variable with $\phi(t) < t$ for all $t > 0$.

We also need the following Lemma due to Matkoski [6] in the proof of our main theorems.

LEMMA. If $\phi(t) < t$ for every $t > 0$, then $\lim_{n \rightarrow \infty} \phi^n(t) = 0$, where $\phi^n(t)$ denotes the composition of $\phi(t)$ with n -times.

Now, we prove some common fixed point theorems.

THEOREM 2.1. Let S and T be mappings from a metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists $\phi \in \mathcal{F}$ such that for each x, y in X , at least one of the following conditions holds:

$$\phi(d(S^2x, TSy)) \geq d(Sx, Sy). \quad (2.1)$$

$$\phi(d(S^2x, TSy)) \geq \frac{1}{2}[d(Sx, Sy) + d(TSy, Sy)]. \quad (2.2)$$

$$\phi(d(S^2x, TSy)) \geq \frac{1}{2}[d(Sx, S^2x) + d(Sx, Sy)]. \quad (2.3)$$

$$\phi(d(S^2x, TSy)) \geq \frac{1}{3}[d(Sx, S^2x) + d(TSy, Sy) + d(Sx, Sy)]. \quad (2.4)$$

Then either S or T has a fixed point, or S and T have a common fixed point.

PROOF. Let x_0 be an arbitrary point in X . Since $S(X) \subseteq S^2(X)$ and $S(X) \subseteq TS(X)$, we have for $x_0 \in X$, there exists a point x_1 in X such that $S^2x_1 = Sx_0 = y_0$, say, and for this point x_1 , there exists a point x_2 in X such that $TSx_2 = Sx_1 = y_1$, say. Inductively, we can define a sequence $\{y_n\}$ in $S(X)$ such that

$$S^2x_{2n+1} = Sx_{2n} = y_{2n} \quad \text{and} \quad TSx_{2n+2} = Sx_{2n+1} = y_{2n+1}.$$

It is easy to show that, for each of the inequalities (2.1)~(2.4), that we have $\phi(d(y_{2n}, y_{2n+1})) \geq d(y_{2n+1}, y_{2n+2})$. Then one can show that $\phi(d(y_{2n+1}, y_{2n+2})) \geq d(y_{2n+2}, y_{2n+3})$, hence for arbitrary n ,

$$\phi(d(y_n, y_{n+1})) \geq d(y_{n+1}, y_{n+2}).$$

Now, if $y_{2n} = y_{2n+1}$ for any n , one has that y_{2n} is a fixed point of S from the definition $\{y_n\}$. It then follows that, also, $y_{2n+1} = y_{2n+2}$, which implies that $\{y_{2n}\}$ is also a fixed point of T .

For an arbitrary n , we have

$$d(y_n, y_{n+1}) \leq \phi(d(y_{n-1}, y_n)) \leq \cdots \leq \phi^n(d(y_0, y_1)).$$

By Lemma, $\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0$.

Now, using the technique of Kang [4], one would prove that $\{y_n\}$ a Cauchy sequence and it converges to some point y in $S(X)$. Consequently, the subsequences $\{y_{2n}\}$, $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ converge to y . Let $y = S^2u$ and $y = TSv$ for some u and v in X , respectively. From inequalities (2.1)~(2.4), it follows that at least one of the following inequalities must be true for an infinite number of values of n :

$$\phi(d(y_{2n}, y)) \geq d(Sx_{2n+1}, Sv)$$

$$\phi(d(y_{2n}, y)) \geq \frac{1}{2}[d(Sx_{2n+1}, Sv) + d(TSv, Sv)]$$

$$\phi(d(y_{2n}, y)) \geq \frac{1}{2}[d(Sx_{2n+1}, S^2x_{2n+1}) + d(Sx_{2n+1}, Sv)]$$

$$\phi(d(y_{2n}, y)) \geq \frac{1}{3}[d(Sx_{2n+1}, S^2x_{2n+1}) + d(TSv, Sv) + d(Sx_{2n+1}, Sv)]$$

Taking the limit as $n \rightarrow \infty$ in each case yields $y = Sv$. A similar argument applies to proving that $y = Su$. Therefore, y is a common fixed point of S and T . This completes the proof.

THEOREM 2.2. *Let S and T be continuous mappings from a metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists $\phi \in \mathcal{F}$ such that*

$$\phi(d(S^2x, TSy)) \geq \min\{d(Sx, S^2x), d(TSy, Sy), d(Sx, Sy)\} \quad (2.5)$$

for all x, y in X .

Then S or T has a fixed point or S and T have a common fixed point.

PROOF. Define a sequence $\{y_n\}$ as in Theorem 2.1. If $y_n = y_{n+1}$ for any n , then S or T has a fixed point.

It is easy to show that, for each of the inequality (2.5), that we have $\phi(d(y_{2n}, y_{2n+1})) \geq d(y_{2n+1}, y_{2n+2})$. Then one can show that $\phi(d(y_{2n+1}, y_{2n+2})) \geq d(y_{2n+2}, y_{2n+3})$, hence for arbitrary n ,

$$\phi(d(y_n, y_{n+1})) \geq d(y_{n+1}, y_{n+2}).$$

For any $m < n$,

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, y_{m+1}) + d(y_{m+1}, y_n) \\ &\leq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \cdots + d(y_{n-1}, y_n) \\ &\leq \phi^m(d(y_0, y_1)) + \cdots + \phi^{n-1}(d(y_0, y_1)). \end{aligned}$$

Hence, it follows that $\{y_n\}$ is a Cauchy sequence and it converges to some point y in $S(X)$. Consequently, $\{y_{2n}\}$, $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ converge to y . By the continuity of S and T ,

$$S^2x_{2n+1} = Sy_{2n+1} = y_{2n} \rightarrow Sy \quad \text{and} \quad TSx_{2n+2} = Ty_{2n+2} = y_{2n+1} \rightarrow Ty \quad \text{as } n \rightarrow \infty.$$

Thus, S and T have a common fixed point.

COROLLARY 2.3. (1) *Let S and T be mappings from a metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists real numbers $h > 1$ such that for each x, y in X , at least one of the following conditions holds:*

$$\begin{aligned} d(S^2x, TSy) &\geq h d(Sx, Sy). \\ d(S^2x, TSy) &\geq \frac{h}{2}[d(Sx, Sy) + d(TSy, Sy)]. \\ d(S^2x, TSy) &\geq \frac{h}{2}[d(Sx, S^2x) + d(Sx, Sy)]. \\ d(S^2x, TSy) &\geq \frac{h}{3}[d(Sx, S^2x) + d(TSy, Sy) + d(Sx, Sy)]. \end{aligned}$$

Then either S or T has a fixed point, or S and T have a common fixed point.

(2) *Let S and T be continuous mappings from a metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists $h > 1$ such that*

$$d(S^2x, TSy) \geq h \min\{d(Sx, S^2x), d(TSy, Sy), d(Sx, Sy)\}$$

for all x, y in X .

Then S or T has a fixed point or S and T have a common fixed point.

PROOF. For $\phi \in \mathcal{F}$, we define $\phi : [0, \infty) \rightarrow [0, \infty)$ by $\phi(t) = \frac{1}{h}(t)$, where $h > 1$. From Theorem 2.1 and 2.2, we obtain (1) and (2), respectively.

THEOREM 2.4. *Let S and T be mappings from a metric space (X, d) into itself such that $S(X) \subseteq S^2(X)$, $S(X) \subseteq TS(X)$ and $S(X)$ is complete. Suppose that there exists non-negative real numbers $\alpha < 1$, $\beta < 1$ and γ ($\alpha + \beta + \gamma > 1$) such that*

$$d(S^2x, TSy) \geq \alpha d(Sx, S^2x) + \beta d(TSy, Sy) + \gamma d(Sx, Sy)$$

for all x, y in X .

Then S and T have a common fixed point.

PROOF. Define a sequence $\{y_n\}$ as in Theorem 2.1. Suppose that $y_{2n} = y_{2n+1}$ for some n . Then

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(S^2x_{2n+1}, TSx_{2n+2}) \\ &\geq \alpha d(Sx_{2n+1}, S^2x_{2n+1}) + \beta d(TSx_{2n+2}, Sx_{2n+2}) \\ &\quad + \gamma d(Sx_{2n+1}, Sx_{2n+2}) \\ &= \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n+1}, y_{2n+2}) + \gamma d(y_{2n+1}, y_{2n+2}), \end{aligned}$$

that is, $d(y_{2n}, y_{2n+1}) \geq \left(\frac{\beta+\gamma}{1-\alpha}\right)d(y_{2n+1}, y_{2n+2})$, which says that $y_{2n+1} = y_{2n+2}$ since $\beta + \gamma \neq 0$. Thus, y_{2n} is a common fixed point of S and T . Similarly, $y_{2n+1} = y_{2n+2}$ gives that y_{2n+1} is a common fixed point of S and T .

Now, suppose that $y_n \neq y_{n+1}$ for each n . Then

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(S^2x_{2n+1}, TSx_{2n+2}) \\ &\geq \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n+1}, y_{2n+2}) + \gamma d(y_{2n+1}, y_{2n+2}). \end{aligned}$$

Thus, we have

$$d(y_{2n+1}, y_{2n+2}) \leq p_1 d(y_{2n}, y_{2n+1}), \quad \text{where } p_1 = \frac{1-\alpha}{\beta+\gamma} < 1.$$

Similarly, we have

$$d(y_{2n+2}, y_{2n+3}) \leq p_2 d(y_{2n+1}, y_{2n+2}), \quad \text{where } p_2 = \frac{1-\beta}{\alpha+\gamma} < 1.$$

Putting $p = \max\{p_1, p_2\}$, we have

$$d(y_n, y_{n+1}) \leq p d(y_{n-1}, y_n).$$

Since $p < 1$, by Lemma of Jungck [2], $\{y_n\}$ is a Cauchy sequence and it converges to some point y in $S(X)$. Consequently, the subsequences $\{y_{2n}\}$, $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ converge to y . Let $y = S^2u$ and $y = TSv$ for some u and v in X , respectively. Then

$$d(y_{2n}, y) = d(S^2x_{2n+1}, TSv).$$

Taking the limit as $n \rightarrow \infty$, we have $0 \geq (\beta + \gamma)d(y, Sv)$, so that $y = Sv$. Similarly, $y = Su$. Therefore, S and T have a common fixed point.

REMARK. Our results improve several results of Kang [3], Khan et al. [5], Rhoades [9] and Taniguchi [10]. Furthermore, we have used non-surjective mappings.

REFERENCES

1. BOYD, D. W. and WONG, J. S. W. On nonlinear contractions, *Proc. Amer. Math. Soc.* **20** (1969), 458–464.
2. JUNGCK, G. Commuting maps and fixed points, *Amer. Math. Monthly* **83** (1976), 261–263.
3. KANG, S. M. Fixed points for expansion mappings, *Math. Japonica* **38** (1993), 713–717.
4. KANG, S. M. and RHOADES, B. E. Fixed points for four mappings, *Math. Japonica* **37** (1992), 1053–1059.
5. KHAN, M. A., KHAN, M. S. and SESSA, S. Some theorems on expansion mappings and their fixed points, *Demonstratio Math.* **19** (1986), 673–683.
6. MATKOWSKI, J. Fixed point theorems for mappings with contractive iterate at a point, *Proc. Math. Soc.* **62** (1977), 344–348.
7. PARK, S. and RHOADES, B. E. Some fixed point theorems for expansion mappings, *Math. Japonica* **33** (1988), 129–132.
8. RHOADES, B. E. A comparison of various definitions of contractive mappings, *Trans. Amer. Math. Soc.* **226** (1977), 257–290.
9. RHOADES, B. E. Some fixed point theorems for pairs of mappings, *Jñānābha* **15** (1985), 151–156.
10. TANIGUCHI, T. Common fixed point theorems on expansion type mappings on complete metric spaces, *Math. Japonica* **34** (1989), 139–142.
11. WANG, S. Z., LI, B. Y., GAO, Z. M. and ISÉKI, K. Some fixed point theorems on expansion mappings, *Math. Japonica* **29** (1984), 631–636.

Special Issue on Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/jamds/guidelines.html>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

Manuscript Due	June 1, 2009
First Round of Reviews	September 1, 2009
Publication Date	December 1, 2009

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be