Internat. J. Math. & Math. Sci.
VOL. 17 NO. 3 (1994) 587-596

SOLUTIONS TO LYAPUNOV STABILITY PROBLEMS: NONLINEAR SYSTEMS WITH CONTINUOUS MOTIONS

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(Received January 23, 1991 and in revised form May 27, 1992)

Abstract. The necessary and sufficient conditions for accurate construction of a Lyapunov function and the necessary and sufficient conditions for a set to be the asymptotic stability domain are algorithmically solved for a nonlinear dynamical system with continuous motions. The conditions are established by utilizing properties of o-uniquely bounded sets, which are explained in the paper. They allow arbitrary selection of an o-uniquely bounded set to generate a Lyapunov function.

Simple examples illustrate the theory and its applications.

Key Words and Phrases: Stability, Lyapunov Method, Lyapunov Functions, Nonlinear Systems, Dynamical Systems.

1991 Mathematics Subject Classification Codes: 34A34, 34C35, 34D20, 70K15, 70K20, 73H10, 93D05, 93D20.

1. INTRODUCTION

In his fundamental dissertation [1] Lyapunov referred to papers by Poincaré [2], [3] as those inspiring him to establish a method that has become fundamental for qualitative and stability analysis of motions of a very general class of nonlinear systems.

The promising methodological effectiveness of the Lyapunov method has not been fully achieved due to the need to construct a system Lyapunov function. Significant results on a Lyapunov function generation were initiated by Zubov [14]. The literature on the Lyapunov method is too vast [9]-[11],[13],[14] to be referred to herein.

The problem of the necessary and sufficient conditions for constructing a Lyapunov function and the problem of the necessary and sufficient conditions for a set to be the asymptotic stability domain have not yet been solved. Solutions to these problems will be established by using properties of o-uniquely bounded sets. Their features will be explained briefly by referring to [7],[8], where they were discovered and studied.

2. NOTATION

$A, R^n \supseteq A$	- an open connected neighborhood of $x = 0$,
$B_\delta = \{x: \|x\| < \delta\}, R^n \supseteq B_\delta$,	- an open hyperball,
$\overline{B}_\delta = \{x: \|x\| \le \delta\}, R^n \supseteq \overline{B}_\delta$,	- the closure of B_δ,
$\partial B_\delta = \{x: \|x\| = \delta\}, R^n \supset \partial B_\delta$,	- the boundary of both B_δ and $\overline{B}_\delta$,
$C(S)$	- the set of all functions of x continuous on S,

$D_a, D_s, D, R^n \supseteq D_{(\cdot)}$,	- the domain of attraction, of stability, of asymptotic stability, respectively, of $x = 0$,
$D^+v(x) = \limsup\{\langle v[\mathrm{x}(\theta;x)] - v(x)\rangle/\theta: \theta \to 0^+\}$	- the Dini derivative of v along the system motion (Yoshizawa [13]),
$E(S;f)$	- a family of functions determined by Definition 5,
$f: R^n \to R^n$	- a given nonlinear vector function,
$I_0, R_+ \supseteq I_0$,	- the largest subinterval of R_+ over which a motion $\mathrm{x}(t;x_0)$ exists,
$n \in \{1,2,\dots\}$	- the dimension of the system,
$N, R^n \supseteq N$,	- an open connected neighborhood of $x = 0$,
$\mathring{N}$	- the interior of N (in fact $\mathring{N} = N$),
R	- the set of real numbers,
R_+	$= [0,+\infty[= \{\alpha: \alpha \in R, 0 \le \alpha < +\infty\}$,
$S, R^n \supseteq S$,	- an open neighborhood of $x = 0$,
$U, R^n \supset U$,	- an o-uniquely bounded set,
$u: R^n \to R$	- the generating function of the o-uniquely bounded set U,
$U_\zeta = \{x: u(x) < \zeta\}$	- a set generated by the function u and a positive number ζ,
$v: R^n \to R$	- a tentative Lyapunov function of the system,
$\mathrm{x}: R_+ x R^n \to R^n$	- the system motion (solution), $\mathrm{x}(t;x_0) = x(t)$, $\mathrm{x}(0;x_0) = x_0$,
$\|\cdot\|: R^n \to R_+$	- Euclidean norm on R^n,
$\varnothing$	- the empty set.

3. SYSTEM DESCRIPTION

Systems to be analyzed are described by the following equation

$$\frac{dx}{dt} = f(x). \tag{3.1}$$

They are assumed to possess either of the following two features:

Weak Smoothness Property:

(i) There is an open neighborhood S of $x = 0, R^n \supseteq S$, such that for every $x_0 \in S$

(a) the system (1) has the unique solution $\mathrm{x}(t;x_0)$ through x_0 at $t = 0$, and

(b) the motion $\mathrm{x}(t;x_0)$ is defined and continuous in $(t,x_0) \in I_0 \times S$.

(ii) For every $x_0 \in (R^n - S)$ every motion $\mathrm{x}(t;x_0)$ of the system (1) is continuous in $t \in I_0$.

Strong Smoothness Property:

(i) The system (1) has Weak Smoothness Property.

(ii) If the boundary ∂S of S is non-empty then every motion of the system (1) passing through $x_0 \in \partial S$ at $t = 0$ obeys $\inf[\|\mathrm{x}(t;x_0)\|: t \in I_0] > 0$ for every $x_0 \in \partial S$.

4. DEFINITIONS

4.1 ON THE DEFINITIONS OF STABILITY DOMAINS

For the definitions of the attraction domain D_a see [4]-[6],[9],[11],[14]. The stability domain D_s and the asymptotic stability domain D of $x = 0$ are defined in [5],[6]. We shall refer to those definitions in the sequel.

For the system (1) with Weak Smoothness Property, the stability domains are mutually related as follows:

LEMMA 1. If the state $x = 0$ of the system (1) possessing Weak Smoothness Property has both the domain of attraction D_a, $S \supseteq D_a$, and the domain of stability D_s, then they and the asymptotic stability domain D are interrelated by

$$D_s \supseteq D_a, \quad D = D_a.$$

PROOF. Let $x = 0$ have D_a, $S \supseteq D_a$, and D_s. Then it has also D because $D = D_a \cap D_s$ and both D_a and D_s are neighborhoods of $x = 0$ [5],[6]. Let $x_0 \in D_a$. Then $\mathbf{x}(t;x_0) \to 0$ as $t \to +\infty$. This and continuity of $\mathbf{x}(t;x_0)$ in $t \in I_0$ (Weak Smoothness Property) imply $\max[\|\mathbf{x}(t;x_0)\| : t \in R_+] = \alpha < +\infty$. Let $\varepsilon = 2\alpha$. Hence, $\|\mathbf{x}(t;x_0)\| < \varepsilon$, $\forall t \in R_+$, which yields [5],[6] $x_0 \in D_s$ so that $D_s \supseteq D_a$ and $D = D_a = D_a \cap D_s$ [5],[6].

4.2 ON THE DEFINITION OF A POSITIVE DEFINITE FUNCTION

The notion of a positive definite function is used in a broader Lyapunov sense [1].

DEFINITION 1. A function $v: R^n \to R$ is a positive definite if and only if there is an open connected neighborhood A of $x = 0$, $R^n \supseteq A$, such that

1) $v(x)$ is uniquely determined by $x \in A$ and v is continuous on A: $v(x) \in C(A)$,
2) $v(0) = 0$, and
3) $v(x) > 0$ for every $(x \neq 0) \in A$.

4.3 DEFINITIONS AND PROPERTIES OF O-UNIQUELY BOUNDED SETS

O-uniquely bounded sets were introduced, defined and studied in [7],[8].

DEFINITION 2. A set U, $R^n \supset U$, is o-uniquely bounded if and only if it is bounded and for every $(x \neq 0) \in R^n$ there is exactly one positive number λ, $\lambda = \lambda(x;U)$, such that $(\lambda x) \in \partial U$.

DEFINITION 3. A function $u: R^n \to R$ is radially increasing on an open neighborhood N of $x = 0$ if and only if for every $(x \neq 0) \in N$ and any μ_i, $i = 1,2$, obeying both $0 \le \mu_1 < \mu_2$ and $\mu_i x \in N$ it satisfies $u(\mu_1 x) < u(\mu_2 x)$.

PROPERTY U. Let N be an open neighborhood of $x = 0$ and U, $N \supset \overline{U}$, be a given bounded set. There is a function $u: R^n \to R$ that obeys the following:

(a) u is continuous on N: $u(x) \in C(N)$,

(b) if $N = R^n$ then $u(x) \to +\infty$ as $\|x\| \to +\infty$,

(c) $u(0) = 0$,

(d) $u(x) > 0$ for all $(x \neq 0) \in N$,

(e) there is positive number ξ, $\xi = \xi(U)$, such that both 1. and 2. hold:

1. $u(x) \le \xi$ for $x \in N$ if and only if $x \in \overline{U}$,
2. $u(x) = \xi$ for $x \in N$ if and only if $x \in \partial U$,

(f) $u(\lambda_i x) = \xi$, $i = 1,2$, holds for any $(x \neq 0) \in N$ if and only if $\lambda_1 = \lambda_2 = \lambda(x;U) \in]0,+\infty]$,

(g) u is radially increasing on N.

Definition 2 implies the next result due to Definition 2, Corollary 1 and Proposition 4 in [8].

LEMMA 2. For a bounded subset U of an open neighborhood N of $x = 0$ to be o-uniquely bounded it is both necessary and sufficient that it possesses Property U.

DEFINITION 4. (i) A function u is the generating function on N of an o-uniquely bounded set U if and only if they have Property U.

(ii) The function u is the generating function of the uniquely bounded set U if and only if they obey (i) for $N = R^n$.

Lemma 2 and Definition 4 imply the following corollary [8].

COROLLARY 1. If a function u is the generating function on N of an o-uniquely bounded set U then for any $\zeta > 0$ for which $N \supseteq N_\zeta$ the subset U_ζ of N is a connected open neighborhood of $x = 0$ that is also an o-uniquely bounded set with the generating function u on N.

5. SOLUTIONS VIA O-UNIQUELY BOUNDED SETS

We shall make use of the family $E(S;f)$ defined as follows.

DEFINITION 5. A function $u: R^n \to R$ belongs to the family $E(S;f)$ if and only if

1) u is continuous on S; $u(x) \in C(S)$, and
2) the following equations along the motions of the system (3.1),

$$D^+v(x) = -u(x), \qquad (5.1a)$$

$$v(0) = 0, \qquad (5.1b)$$

have a solution v that is well defined in R and continuous for every $x \in \overline{B}_\mu$ for some $\mu \in]0, +\infty[, \mu = \mu(u,f)$.

THEOREM 1. In order for the state $x = 0$ of the system (1) with Strong Smoothness Property to have the domain D of asymptotic stability and for a set N, $R^n \supseteq N$, to be the domain of its asymptotic stability, $N = D$, it is both necessary and sufficient that

1) the set N is an open connected neighborhood of $x = 0$ and $S \supseteq N$,
2) $f(x) = 0$ for $x \in N$ if and only if $x = 0$, and
3) for arbitrarily selected o-uniquely bounded set U, $S \supset \overline{U}$, with the generating function u on S obeying $u \in E(S;f)$, the equations (5.1) have a unique solution function v on N with the following properties:
   (i) v is positive definite on N, and
   (ii) if the boundary ∂N of N is non-empty then $v(x) \to +\infty$ as $x \to \partial N, x \in N$.

PROOF. *Necessity.* Let $x = 0$ of the system (3.1) with Strong Smoothness Property have the asymptotic stability domain D. Definitions of D_a and D [5],[6] show that it has also the attraction domain $D_a, D_a \supseteq D$. It is a neighborhood of $x = 0$ due to Definition of D_a, and S is a neighborhood of $x = 0$ in view of the smoothness property. Hence, $D_a \cap S \neq \emptyset$. Let us prove $S \supseteq D_a$. If $\partial S = \emptyset$, then $S = R^n$ and $S \supseteq D_a$ due to $R^n \supseteq D_a$. If $\partial S \neq \emptyset$, then we shall consider both $x_0 \in \partial S$ and $x_0^* \in (R^n - \overline{S})$. If $x_0 \in \partial S$, then $x_0 \notin D_a$ due to (ii) of Strong Smoothness Property. Therefore, $\partial S \cap D = \emptyset$. If $x_0^* \in (R^n - \overline{S})$, then for $\mathbf{x}(t:x_0^*) \to 0$ as $t \to +\infty$ it is necessary that there is $t^* \in R_+$ such that $\mathbf{x}(t^*;x_0^*) \in \partial S$, because D and S are neighborhoods of $x = 0, x_0^* \notin \overline{S}$ and the motion $\mathbf{x}(t;x_0)$ is continuous in $t \in R_+$ due to (ii) of Weak Smoothness Property ensured by (i) of Strong Smoothness Property. However, $\mathbf{x}(t^*;x_0^*) \in \partial S$ implies that $\mathbf{x}(t;x_0)$ does not converge to $x = 0$ because of (ii) of Strong Smoothness Property. This yields $x_0^* \notin D$ and $(R^n - \overline{S}) \cap D = \emptyset$. By connecting the above results, that is $D_a \cap S \neq \emptyset, D_a \cap \partial S = \emptyset$ and $D_a \cap (R^n - \overline{S}) = \emptyset$, we conclude that $S \supseteq D_a$. Therefore, $D = D_a$ (Lemma 1) and $S \supseteq D$. Let $N = D$ so that $S \supseteq N$. Hence, N is open connected neighborhood of $x = 0$ due to (i-b) of Weak Smoothness Property, $N = D = D_a$, and invariance of D_a with respect to system motions (Theorem 1.5.14 by Bhatia and Szegö [4], Theorem 33.3 by Hahn [9]). This proves necessity of the condition 1). From $N = D = D_a, D_s \supseteq D_a$, and Definitions of D_a and D it results that $x = 0$ is the unique equilibrium state of the system (1) in N, which implies $f(x) = 0$ for $x \in N = D$ if and only if $x = 0$ (Proposition 7 in [6]) and proves necessity of the condition 2).

From $N = D$ it follows that the interval I_0 of existence of $\mathbf{x}(t;x_0)$ equals R_+, $I_0 = R_+$, for every $x_0 \in N$, due to Definitions of D_a, D_s and D [5],[6]. Let U be arbitrarily selected open o-uniquely bounded set such that $N \supset \overline{U}$ and its generating function u on S obeys $u \in E(S;f)$. Such a set U exists because S is open neighborhood of $x = 0$ (Lemma 2). Definition 3, Property U, and Lemma 2 show that the function u is also positive definite on S. Since $S \supseteq N = D$ then the function u is the positive definite generating function on N, too. The property of $u \in E(S;f)$ ensures existence of $\mu > 0$ such that there exists a solution function v to the equations (5.1), which is well defined in R and continuous for every $x \in \overline{B}_\mu$, that is that

$$|v(x)| < +\infty \quad \text{for every} \quad x \in \overline{B}_\mu \quad \text{and} \quad v(x) \in C(\overline{B}_\mu). \tag{5.2}$$

Let $\zeta \in]0,+\infty[$ be such that

$$\overline{B}_\mu \cap U \supseteq \overline{U}_\zeta. \tag{5.3}$$

The existence of such ζ is assured by Corollary 1. Let $\tau \in [0,+\infty[$, $\tau = \tau(x_0;f;u;\zeta)$, be such that for any $x_0 \in N$ the following condition holds,

$$\mathbf{x}(t;x_0) \in U_\zeta \quad \text{for every} \quad t \in [\tau,+\infty[. \tag{5.4}$$

Such τ exists in view of Definitions of D_a and D, $D_a = D$, $N = D$ and $x_0 \in N$. Notice that $x_0 \in N$ implies also

$$\mathbf{x}(+\infty;x_0) = 0. \tag{5.5}$$

After integrating (5.1a) from $t \in R_+$ to $+\infty$ we derive

$$v[\mathbf{x}(+\infty;x_0)] - v[\mathbf{x}(t;x_0)] = -\int_t^{+\infty} u[\mathbf{x}(\sigma;x_0)]d\sigma \quad \text{for every} \quad (t,x_0) \in R_+ x N. \tag{5.6}$$

Since $u \in E(S;f)$ then the following holds,

$$v(0) = 0. \tag{5.7}$$

Now, (5.5)-(5.7) yield

$$v[\mathbf{x}(t;x_0)] = \int_t^{+\infty} u[\mathbf{x}(\sigma;x_0)]d\sigma \quad \text{for every} \quad (t,x_0) \in R_+ x N. \tag{5.8}$$

This can be written in the following form,

$$v[\mathbf{x}(t;x_0)] = \int_t^{+\infty} u[\mathbf{x}(\sigma;x_0)]d\sigma \quad \text{for every} \quad (t,x_0) \in R_+ x N. \tag{5.9}$$

Positive invariance of D with respect to system motions, $N = D$, continuity of the motions $\mathbf{x}$ due to the smoothness property, continuity of u on N, the definition of τ (5.4) and (5.2), and compactness of $[\tau, t]$ for any $t \in R_+$ prove

$$\left| \int_t^{\tau} u[\mathbf{x}(\sigma;x_0)]d\sigma \right| < +\infty \quad \text{for every} \quad (t,x_0) \in R_+ x N. \tag{5.10}$$

From (5.2)-(5.4) we obtain

$$\left| \int_\tau^{+\infty} u[\mathbf{x}(\sigma;x_0)]d\sigma \right| < +\infty \quad \text{for every} \quad x_0 \in N. \tag{5.11}$$

(5.9)-(5.11) together prove boundedness of $v[\mathbf{x}(t;x_0)]$ expressed as

$$|v[\mathbf{x}(t;x_0)]| < +\infty \quad \text{for every} \quad (t,x_0) \in R_+ x N. \tag{5.12}$$

Hence, by setting $t = 0$ and $x_0 = x$ in (5.12) we derive

$$|v(x)| < +\infty \quad \text{for every} \quad x \in N. \tag{5.13}$$

Continuity of the motion $\mathbf{x}$ in $x_0 \in N$, continuity of u in $x \in N$, and of v in $x \in \overline{B}_\mu$, $\overline{B}_\mu \supseteq \overline{U}_\xi$, positive invariance of $N = D$ with respect to system motions, (5.4), (5.9) and (5.12) prove continuity of v in $x \in N$

$$v(x) \in C(N). \tag{5.14}$$

Positive invariance of N with respect to system motions, positive definiteness of u on N and (5.8) imply

$$v(x) > 0 \quad \text{for all} \quad (x \neq 0) \in N. \tag{5.15}$$

Now, (5.7), (5.14) and (5.15) prove necessity of the positive definiteness of v on N.

To prove uniqueness of the solution v to (5.1.*ab*) we shall suppose that there are two solutions v_1 and v_2 to (5.1). Hence,

$$v_1(x_0) - v_2(x_0) = \int_0^{+\infty} \{u[\mathbf{x}_1(\sigma;x_0)] - u[\mathbf{x}_2(\sigma;x_0)]\} d\sigma \quad \text{for every} \quad x_0 \in N. \tag{5.16}$$

Since $u(x)$ is uniquely determined by $x \in N$, due to (a) of Property U and Definition 4, and the motion of the system is unique through x_0, $\mathbf{x}_1(\sigma;x_0) = \mathbf{x}_2(\sigma;x_0)$ and $u[\mathbf{x}_1(\sigma;x_0)] = u[\mathbf{x}_2(\sigma;x_0)]$ so that $v_1(x_0) - v_2(x_0) = 0$ for every $x_0 \in N$. This proves uniqueness of the solution v to (5.1) and completes the proof of 3(i).

Let ∂N be non-empty, $x_1, x_2, \ldots, x_k, \ldots$ be a sequence converging to x', $x_k \to x'$ as $k \to +\infty$, where $x_k \in N$, for all $k = 1,2,\ldots$, and $x' \in \partial N$. Let $\xi \in]0,+\infty[$ be arbitrarily chosen so that $U_\xi = \{x : u(x) < \xi\}$, $U \supseteq U_\xi$. Such ξ exists because the set U is o-uniquely bounded and the function u is its generating function on N (Definitions 2 and 3, Property U, Lemma 2 and Definition 4). The set U_ξ is a connected open neighborhood of $x = 0$ (Corollary 1). Let T_k, $T_k = T(x_k, \xi) \in [0,+\infty[$, be the first instant obeying the following

$$\mathbf{x}(t;x_k) \in \overline{U}_\xi \quad \text{for all} \quad t \in [T_k, +\infty[. \tag{5.17}$$

The existence of such T_k is guaranteed by $x_k \in N$ and $N = D$ (Definitions of D_a and D [5], [6]). Continuity of the motions $\mathbf{x}$ in $(t,x_0) \in R_+ x N$ due to Strong Smoothness Property and $\mathring{N} = N = D$ (Theorem 33.1 by Hahn [9]) and $S \supseteq D$ imply $T_k \to +\infty$ as $k \to +\infty$ (Theorem 33.2 by Hahn [9]). Let m be a natural number such that $x_k \in (N - \overline{U}_\xi)$ for all $k = m, m+1, \ldots$. Such m exists because N is open, $N \supset \overline{U}_\xi$ and $x_k \to \partial N$ as $k \to +\infty$. Let α' be defined by (18),

$$\alpha' = \min[u(x) : x \in (N - U_\xi)]. \tag{5.18}$$

The o-unique boundedness of the set U_ξ, the fact that the function u is its generating function on N (Corollary 1), $N \supset \overline{U}$, and $U \supseteq U_\xi$ guarantee (Property U and Lemma 2) that α' defined by (5.18) satisfies

$$\alpha' = \xi \in]0,+\infty[. \tag{5.19}$$

From (5.9) we get, after replacing τ by T_k,

$$v[\mathbf{x}(t;x_k)] = \int_t^{Tk} u[\mathbf{x}(\sigma;x_k)] d\sigma + \int_{Tk}^{+\infty} u[\mathbf{x}(\sigma;x_k)] d\sigma \quad \text{for every} \quad (t,x_k) \in R_+ x N,$$
$$\text{and for} \quad k = m, m+1, \ldots. \tag{5.20}$$

Setting $t = 0$ in (5.20) and using (5.18) and (5.19) we derive

$$v(x_k) \geq \int_0^{Tk} \xi d\sigma + \int_{Tk}^{+\infty} u[\mathbf{x}(\sigma;x_k)] d\sigma \quad \text{for} \quad x_k \in N \quad \text{and all} \quad k = m, m+1, \ldots. \tag{5.21}$$

Positive invariance of $N = D$ with respect to system motions, positive definiteness of u on N, and (5.21) imply

$$v(x_k) \geq \xi T_k \quad \text{for} \quad x_k \in N \quad \text{and all} \quad k = m, m+1, \ldots. \tag{5.22}$$

Since $T_k \to +\infty$ as $k \to +\infty$, the last inequality, the definitions of T_k, $T_k = T(x_k, \xi)$, and of x_k, and $\alpha > 0$ imply

$$v(x_k) \to +\infty \quad \text{as} \quad x_k \to \partial N \quad \text{due to} \quad k \to +\infty, \quad x_k \in N,$$

which proves necessity of the condition (3-ii).

Sufficiency. Let all the conditions of Theorem 1 hold. Then, $S \supseteq N$. Two possible cases will be considered separately: a) N is a bounded set, b) N is an unbounded set.

a) Let N be a bounded set. Then, under the conditions of the theorem to be proved all the conditions of Theorem 1 by Vanelli and Vidyasagar [12] are satisfied, which proves $N = D_a$. Since $D_a = D$ (in view of Weak Smoothness Property implied by Strong Smoothness Property and Lemma 1), $N = D$.

b) Let N be an unbounded set. Under the conditions of the theorem to be proved the zero state $x = 0$ of the system (1) is asymptotically stable (cf. Yoshizawa [13]). Hence, it has the domain of asymptotic stability D. Since both N and D are open connected neighborhoods of $x = 0$,

$$N \cap D \neq \varnothing. \tag{5.23}$$

Since $S \supseteq N$, S is also unbounded. If ∂S is empty, then $S = R^n$, which implies $S \supseteq D$. If ∂S is non-empty, then $\partial S \cap D = \varnothing$ due to (ii) of Strong Smoothness Property and Definitions of D_a, D_s and D [5],[6]. This result implies $S \supseteq D$ because both D and S are neighborhoods of $x = 0$ and D is also connected. Altogether, in both cases $S \supseteq D$. We shall treat separately the cases of non-empty ∂D and of empty ∂D. The definition of the function v, $S \supseteq D$, and the proof of the necessity part prove continuity of v on D and $v(x) \to +\infty$ as $x \to \partial D$, which together with continuity of v also on N, $S \supseteq N$ and $v(x) \to +\infty$ as $x \to \partial N$ [the condition 3(ii)] imply both

$$\partial D \cap N = \varnothing \quad \text{and} \quad D \cap \partial N = \varnothing.$$

These equations and (5.23) prove both $\partial D = \partial N$ and $D = N$ due to the fact that both D and N are open connected neighborhoods of $x = 0$. Let now ∂D be empty. Then $D = R^n$. Hence, v is positive definite on R^n (see the proof of the necessity part). Thus, it is continuous on R^n, which implies $v(x) < +\infty$ for every $x \in R^n$. Therefore, $\partial N \cap R^n = \varnothing$ due to the conditions 3(ii), which yields $N = R^n$ so that $N = D$. Finally, $N = D$ in all the cases, which completes the proof. ▮

The conditions slightly change if the system (3.1) possesses Weak Smoothness Property rather than Strong Smoothness Property.

THEOREM 2. For the state $x = 0$ of the system (1) possessing Weak Smoothness Property to have the domain D of asymptotic stability and that a subset N of S, $S \supseteq N$, equals $D : N = D$, it is both necessary and sufficient that

1) the set N is an open connected neighborhood of $x = 0$,
2) $f(x) = 0$ for $x \in N$ if and only if $x = 0$, and
3) for arbitrarily selected o-uniquely bounded set U, $S \supset \overline{U}$, with the generating function u on R^n obeying $u \in E(S;f)$, the equations (5.1) have a unique solution function v on N with the following properties:
   (i) v is positive definite on N, and
   (ii) if the boundary ∂N of N is non-empty then $v(x) \to +\infty$ as $x \to \partial N$, $x \in N$.

PROOF. *Necessity.* Let the system (3.1) possess Weak Smoothness Property. Let $x = 0$ have the asymptotic stability domain D, $S \supseteq D$, and let N, $S \supseteq N$, be equal to D. Let an o-uniquely bounded set U, $S \supset U$, with the generating function u obeying $u \in E(S;f)$, be arbitrarily selected. From this point on we have to repeat the proof of the necessity part of Theorem 1 to show that the conditions 1)-3) of Theorem 2 hold. In that way we complete the proof of the necessity part.

Sufficiency. Let the system (3.1) possess Weak Smoothness Property and the conditions 1)-3) be valid. Then $x = 0$ of the system (3.1) is asymptotically stable [1]. Therefore, $x = 0$ has the domain of asymptotic stability (Definitions of D_a, D_s and D [5],[6]). Let $x_0 \in (R^n - \overline{N})$. Since $\mathrm{x}(t;x_0)$ is continuous in $t \in I_0$, then it can enter N iff it passes through ∂N. But $v(x) \to +\infty$ as $x \to \partial N$, $x \in N$ [the condition 3(ii)]. This and $D^+v(x) < 0$ for $x \in (R^n - N)$ in view of positive definiteness of u on R^n and (5.1a), show that $\mathrm{x}(t;x_0)$ cannot reach ∂N. Hence, $\mathrm{x}(t;x_0) \in (R^n - \overline{N})$ for all $t \in I_0$. Therefore, $\overline{N} \supset D$. Furthermore, (5.1a) and positive definiteness of u on R^n imply (see the proof of the necessity part of Theorem 1) $v(x) \to +\infty$ as $x \to \partial D$, $x \in D$, which together with the condition 3(i) proves $\partial D \cap N = \varnothing$. This result, $\overline{N} \supset D$, and the fact that D and N are non-empty open connected neighborhoods of $x = 0$ imply $D = N$ and complete the proof. ▮

The properties of the generating function u of an o-uniquely bounded set U are essential for the accurate one-shot determination of the asymptotic stability domain. However, such properties are not needed for asymptotic stability of $x = 0$ only. This is clarified by the next result.

THEOREM 3. For the state $x = 0$ of the system (3.1) possessing Weak Smoothness Property to be asymptotically stable it is both necessary and sufficient that for any positive definite function $p \in E(S;f)$ there exists a unique solution function v to (5.24) with (5.24*a*) determined along system motions,

$$D^+v(x) = -p(x), \tag{5.24a}$$

$$v(0) = 0, \tag{5.24b}$$

which is also positive definite.

PROOF. *Necessity.* Let the system (3.1) possess the Weak Smoothness Property. Let $x = 0$ be asymptotically stable. Then it has D_a, D_s and D, and $D_a \cap S \neq \varnothing$, $D_s \cap S \neq \varnothing$ and $D \cap S \neq \varnothing$, because D_a, D_s, D and S are neighborhoods of $x = 0$. Let $p \in E(S;f)$ be an arbitrarily selected positive definite function (Definition 1). Such properties of p and its membership to $E(S;f)$ guarantee existence of a solution v to the equations (5.24), which is well defined in R and continuous (see the proof of the necessity part of Theorem 1) on the set A determined in Definition 1. The set $L = A \cap D, D \supseteq L$, is also an open connected neighborhood of $x = 0$ (see the proof of Theorem 1 for such a property of D). Let ε satisfying $L \supseteq B_\varepsilon$ be arbitrarily selected. Then $D \supseteq B_\varepsilon$. Let $\rho \in]0,\varepsilon[$ obeying $D_s(\varepsilon) \supseteq B_\rho$ be also arbitrarily selected, where $D_s(\varepsilon)$ is defined [5],[6] as the neighborhood of $x = 0$ such that $\|\mathbf{x}(t;x_0)\| < \varepsilon$ for all $t \in R_+$ holds iff $x_0 \in D_s(\varepsilon)$. By following the proofs of (5.13) and (5.14), we prove that v, defined by (5.24), has the following properties since $A \supseteq L \supseteq B_\varepsilon \supseteq D_s(\varepsilon) \supseteq B_\rho$,

$$|v(x)| < +\infty \quad \text{for every} \quad x \in B_\rho, \tag{5.25a}$$

$$v(x) \in C(B_\rho). \tag{5.25b}$$

Notice that $D_s(\varepsilon) \supseteq B_\rho$ and the definitions of $D_s(\varepsilon)$ and D guarantee [5],[6] $\mathbf{x}(t;x_0) \in B_\varepsilon$ for every $(t,x_0) \in R_+ x B_\rho$. This result, $A \cap D \supseteq B_\varepsilon$, positive definiteness of the function p on A, $\mathbf{x}(+\infty;x_0) = 0$ for every $x_0 \in B_\rho$ (because $D \supseteq B_\rho$) and (5.24*a*), integrated from $t = 0$ to $t = +\infty$, together with (5.24*b*) prove (5.26),

$$v(x_0) > 0 \quad \text{for every} \quad (x_0 \neq (0)) \in B_\rho. \tag{5.26}$$

Now, (5.24*b*) through (5.26) prove positive definitness of the solution v to (5.24) on B_ρ. Its uniqueness is proved in the same way as in the proof of Theorem 1, which completes the proof of the necessity part.

Sufficiency. Sufficiency of the conditions of Theorem 3 for asymptotic stability of $x = 0$ of the system (3.1) with Weak Smoothness Property is well known [13]. This completes the proof of Theorem 3. ▮

6. EXAMPLES

Example 1. Let $n = 1$,

$$\frac{dx}{dt} = -x + h(x), \quad h(x) = \begin{cases} x\,|x| & \text{for} \quad |x| \in [0,1], \\ x(|x|)^{1/2} & \text{for} \quad |x| \in [1,+\infty[\end{cases} \tag{6.1}$$

The system possesses Strong Smoothness Property because $f(x) = -x + h(x)$ is Lipschitzian on R^1. The equilibrium states are $x_{e1} = -1$, $x_{e2} = 0$ and $x_{e3} = +1$. They suggest $S =]-1,+1[$ and $U = \{x : x \in R^1, |x| < \alpha\} =]-\alpha,+\alpha[$, for $\alpha \in]0,1[$. The generating function u on N, $u(x) = |x|$, of the o-uniquely bounded set U and (5.1*ab*) yield

$$D^+v(x) = -|x|, \quad x \in S.$$

The solution v to this equation is

$$v(x) = -\ln(1-|x|), \quad x \in S. \tag{6.2}$$

The function $v(27)$ and the set $N = S =]-1,+1[$ satisfy all the requirements of Theorem 1, that is that,

1) $N =]-1,+1[$ is an open connected neighborhood of $x = 0$ and $N = S$,
2) $f(x) = -x + h(x) = 0$ for $x \in N$ iff $x = 0$,
3) (i) $v(x) = 0$ for $x \in N$ iff $x = 0$, $v(x) \in C(N)$, and $v(x) > 0$ for every $(x \neq 0) \in N$, which prove positive definiteness of v on N,
   (ii) $v(x) \to +\infty$ as $x \to \partial N = \{-1,+1\}, x \in N$.

Hence $N =]-1,+1]$ is the domain D of asymptotic stability of $x = 0$,

$$D =]-1,+1[.$$

Notice that $|f(x)| = |x|\,|1-|x||, x \in N$, is not a generating function on N of any o-uniquely bounded set because it is not radially increasing on N.

Example 2. Let the function h be defined as in Example 1 and

$$\frac{dx}{dt} = x - h(x). \tag{6.3}$$

It is clear that the system possesses Strong Smoothness Property on R^1 and has the equilibrium states $x_{e1} = -1$, $x_{e2} = 0$ and $x_{e3} = +1$ (see Example 1). Let, again, $U = \{x : x \in R^1, |x| < \alpha\} =]-\alpha,+\alpha[$ so that $u(x) = |x|$. From (5.1a) we get

$$D^+v(x) = -|x|, \quad x \in N.$$

Integrating this equation along motions of the system (6.3) we derive

$$v(x) = \ln(1-|x|), \quad x \in N,$$

which is negative definite on N and, thus, does not satisfy the necessary and sufficient conditions for asymptotic stability of $x = 0$ of the system (6.3). Hence, $x = 0$ of the system (6.3) is not asymptotically stable and does not have the asymptotic stability domain.

Example 3. Let $n = 2$ and

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-x_1(1+|x_1|x_2^2)(1-|x_1|)\\ -x_2(1-x_1^2|x_2|)(1-|x_2|)\end{bmatrix} = f(x). \tag{6.4}$$

The function f is globally Lipschitz continuous. The system has Strong Smoothness Property on R^2. The set S_e of its equilibrium states is determined by

$$S_e = \{x : x \in R^2, (x = 0) \text{ or } (|x_1| = 1, |x_2| = 1)\}.$$

This suggests $S = \{x : x \in R^2, |x_1| < 1, |x_2| < 1\}$. The system (6.4) has Weak Smoothness Property on S. Let $U = \{x : x \in R^2, |x_1| + |x_2| < \alpha\}$, $\alpha \in]0,1[$, so that U is o-uniquely bounded set with the generating function u on R^2 defined by $u(x) = |x_1| + |x_2|$, which together with (5.1) and (6.4) yields

$$v(x) = -\ln[(1-|x_1|)(1-|x_2|)].$$

The function v and the set $N = S$ obey all the conditions of Theorem 2. Therefore, $x = 0$ of the system (6.4) is asymptotically stable with the domain D of its asymptotic stability obtained as $D = N = S$, that is that

$$D = \{x : x \in R^2, |x_1| < 1, |x_2| < 1\}.$$

7. CONCLUSION

The necessary and sufficient conditions for asymptotic stability of the zero equilibrium state and for a set to be the domain of its asymptotic stability are proved in an algorithmic form that enables accurate construction of a system Lyapunov function. If a function v obtained from $D^+v = -u$ for an arbitrarily chosen u, which is a generating function of an o-uniquely bounded set, is not positive definite then the zero state is not asymptotically stable. There is no sense to try with another function u. However, if so derived function v is positive definite then the zero state is asymptotically stable. In this way the problem of an algorithm to construct accurately and directly a system Lyapunov function has been solved. However, it imposes other very complex mathematical problems: the problem of finding conditions on u guaranteeing existence of well defined and continuous v satisfying (5.1) on anyhow small neighborhood $\overline{B}_\mu$ of $x = 0$, and the problem of solving (5.1). These problems have not been solved.

Theorems of the paper open and initiate new directions in the Lyapunov stability analysis.

ACKNOWLEDGEMENT. The author is grateful to all the members of the Department of Electrical and Computer Engineering, Louisiana State University, Baton Rouge, LA, U.S.A., for their wonderful hospitality that was helpful to him for his research.

The author also expresses his gratitude to Professor Lokenath Debnath, Editor, and reviewers for their comments and thanks to Ecole Central de Lille, 59651 Villeneuve d'Ascq-Cedex, France, for the opportunity to be with the School and to revise the paper effectively.

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