Internat. J. Math. & Math. Sci.
VOL. 17 NO. 1 (1994) 27-30

COMPLETELY POSITIVE LINEAR OPERATORS FOR BANACH SPACES

MINGZE YANG

Department of Mathematics
University of Saskatchewan
Saskatoon, Sask. Canada S7N 0W0

(Received January 9, 1992 and in revised form February 25, 1992)

ABSTRACT. Using ideas of Pisier, the concept of complete positivity is generalized in a different direction in this paper, where the Hilbert space $\mathcal{H}$ is replaced with a Banach space and its conjugate linear dual. The extreme point results of Arveson are reformulated in this more general setting.

KEY WORDS AND PHRASES: Banach spaces, completely positive operators, extreme points, pure elements.

1980 AMS SUBJECT CLASSIFICATION CODES: 46L05, 47A67

1. INTRODUCTION.

In [6], Pisier studied completely bounded maps from a C^*-algebra to $B(X,Y)$, the space of bounded operators between two arbitary Banach spaces X and Y. Of course , there is a generalization of ordinary completely bounded maps. In this paper, we first define complete positivity for a map from C^*-algebra to $B(X,\overline{X^*})$, where $\overline{X^*}$ denotes the antilinear dual space of X (the set of all conjugate linear functionals on X). Then we give a representation theorem, and give complete solutions to three extremal problems.

In this paper, the C^*-algebra A always has an identity.

2. COMPLETELY POSITIVE OPERATORS.

DEFINITION 2.1. Let X be a Banach space, and $T \in B(X,\overline{X^*})$. We call T positive if, for all positive integers n and $x_1,\ldots,x_n \in X$, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} T(x_i)(x_j) \geq 0.$$

REMARK 2.2. We have $\overline{\ell_2^n(X)^*} = \ell_2^n(\overline{X^*})$, and so $M_n(B(X,\overline{X^*})) = B(\ell_2^n(X),\ell_2^n(\overline{X^*})) = B(\ell_2^n(X),\overline{\ell_2^n(X)^*})$. Thus we may define positivity for $M_n(B(X,\overline{X^*}))$.

DEFINITION 2.3. Let A be a C^*-algebra, ϕ a linear map from A to $B(X,\overline{X^*})$ and let $\phi_n(a_{ij}) = (\phi(a_{ij}))$ for $(a_{ij}) \in M_n(A)$. If ϕ_n is positive for all n, then we say ϕ is completely positive.

THEOREM 2.4. Let $\phi : A \longrightarrow B(X,\overline{X^*})$ be a completely positive map. There is a Hilbert space $\mathcal{H}$, a representation π of A on $\mathcal{H}$ and a bounded operator $V \in B(X,\mathcal{H})$ such that, for all $a \in A$,

$$\phi(a) = \overline{V^*}\pi(a)V,$$

and $\mathcal{H} = [\pi(A)VX]$, where $\overline{V^*}(h)(x) =< h, V(x) >$, for all $h \in \mathcal{H}, x \in X$.

PROOF: Consider the vector space tensor product $A \otimes X$ and define a bilinear form as follows: If $u = x_1 \otimes \xi_1 + \ldots + x_m \otimes \xi_m$, $v = y_1 \otimes \eta_1 + \ldots + y_n \otimes \eta_n$,

$$< u, v >= \sum_{i,j} (\phi(y_i^* x_j)(\xi_j))(\eta_i).$$

Because ϕ is completely positive, we have the fact that $<,>$ is positive semi-definite. For each $a \in A$, define a linear transformation $\pi_0(a)$ on $A \otimes X$ by

$$\pi_0(a)(\sum_{j=1}^{n} x_j \otimes \xi_j) = \sum (ax_j) \otimes \xi_j.$$

π_0 is an algebra homomorphism for which

$$< u, \pi_0(a)v >=< \pi_0(a^*)u, v >$$

for all $u, v \in A \otimes X$.

For fixed u, $\rho(a) =< \pi_0(a)u, u >$ defines a positive linear functional on A; i.e, $\rho(a^*a) \geq 0$. Hence, $< \pi_0(a)u, \pi_0(a)u >=< \pi_0(a^*a)u, u >= \rho(a^*a) \leq \|a^*a\|\rho(1) = \|a\|^2 < u, u >$, where 1 is the identity of A.

Now let $R = \{u \in A \otimes X :< u, u >= 0\}$. R is a linear subspace $A \otimes X$, invariant under $\pi_0(a)$, for all $a \in A$. So $<,>$ determines a positive definite inner product on the quotient $(A \otimes X)/R$ in the usual way.

Let $\mathcal{H} = \overline{(A \otimes X)/R}$. There is a unique representation π of A on $\mathcal{H}$ such that

$$\pi(a)(u + R) = \pi_0(a)u + R$$

$a \in A$, $u \in A \otimes X$.

We define a linear map V: $X \longrightarrow \mathcal{H}$ by

$$V(\xi) = 1 \otimes \xi + R$$

for all $\xi \in X$.

We may verify that V is bounded, and $\phi(a) = \overline{V^*}\pi(a)V$ for all $a \in A$.

Let $R_1 = [\pi(A)VX] \subseteq \mathcal{H}$, and $\pi_1(a) = \pi(a)|_{R_1}$ for all $a \in A$. Because $\pi(1) = I$, so $V(X) \subseteq R_1$. We have $\overline{V^*}\pi(a)V(x_1) = \overline{V^*}\pi(a)|_{R_1}V(x_1) = \overline{V^*}\pi_1(a)V(x_1) = \phi(a)(x_1)$, for all $x_1 \in X$, $a \in A$. So we may assume that $\mathcal{H} = [\pi(A)VX]$.

Suppose ϕ: $A \longrightarrow B(X, \overline{X^*})$ is a completely positive map. If there exists Hilbert spaces $\mathcal{H}_i$, representations π_i of A on $\mathcal{H}_i$, and bounded operators $V_i : X \longrightarrow \mathcal{H}_i$ then

$$\phi(a) = \overline{V_i^*}\pi_i(a)V_i,$$

for $i = 1, 2$, where $\mathcal{H}_i = [\pi_i(A)V_iX]$. Define U: $\mathcal{H}_1 \longrightarrow \mathcal{H}_2$ by

$$U(\sum_{i=1}^{n} \pi_1(a_i)V_1\xi_i) = \sum_{i=1}^{n} \pi_i(a_i)V_2\xi_i,$$

for all $a_1, \ldots, a_n \in A, \xi_1, \ldots, \xi_n \in X$. Then we need to extend to $\mathcal{H}_1$. We may verify that $UV_1 = V_2$ and $U\pi_1(a) = \pi_2(a)U$ for all $a \in A$.

Next we verify that U is an unitary.

$$\begin{aligned}
<\sum_{i=1}^{n}\pi_2(a_i)V_2\xi_i,\sum_{i=1}^{n}\pi_2(a_i)V_2\xi_i> &= \sum_i\sum_j <\pi_2(a_i)V_2\xi_i,\pi_2(a_j)V_2\xi_j> \\
&= \sum_i\sum_j <\pi_2(a_j^*a_i)V_2\xi_i,V_2\xi_j> \\
&= \sum_i\sum_j \phi(a_j^*a_i)(\xi_i)(\xi_j) \\
&= \sum_i\sum_j <\pi_1(a_j^*a_i)V_1\xi_i,V_1\xi_j> \\
&= <\sum_i\pi_1(a_i)V_1\xi_i,\sum_i\pi_1(a_i)V_1\xi_i>.
\end{aligned}$$

So the representation given in Theorem 2.4 is unique up to unitary equivalence.

3. PREPARATIONS.

NOTATION 3.1. Let $CP(A,X)$ denote all completely positive linear maps from A to $B(X,\overline{X^*})$.

LEMMA 3.2. Let ϕ_1 and ϕ_2 belong to $CP(A,X)$, and suppose that $\phi_1 \leq \phi_2$. Let $\phi_i(a) = \overline{V_i^*}\pi_i(a)V_i$ be the canonical expression of ϕ_i, where π_i is a representation of A on R_i such that $[\pi_i(A)V_iX] = R_i$, $i = 1,2$. Then there exsists a contraction $T \in B(R_2,R_1)$ such that

$$TV_2 = V_1,$$

$$T\pi_2(a) = \pi_1(a)T$$

for all $a \in A$.

PROOF: For every $\xi_1\dots,\xi_n \in X, a_1,\dots,a_n \in A$,

$$\begin{aligned}
\|\sum_{j=1}^{n}\pi_1(a_j)V_1\xi_j\|^2 &= <\sum_{j=1}^{n}\pi_1(a_j)V_1\xi_j,\sum_{j=1}^{n}\pi_1(a_j)V_1\xi_j> \\
&= \sum_i\sum_j \pi_1(a_j^*a_i)V_1(\xi_i)(\xi_j) \\
&= \sum_i\sum_j \phi_1(a_j^*a_i)(\xi_i)(\xi_j) \\
&\leq \sum_i\sum_j \phi_2(a_j^*a_i)(\xi_i)(\xi_j) \\
&= \|\sum_{j=1}^{n}\pi_2(a_j)V_2\xi_j\|^2
\end{aligned}$$

Define T: $R_2 \longrightarrow R_1$ by

$$T(\sum_{j=1}^{n}\pi_2(a_j)V_2\xi_j) = \sum_{j=1}^{n}\pi_1(a_j)V_1\xi_j$$

We can verify that above two statements hold.

NOTATION 3.3. For $\phi \in CP(A,X)$, let $[0,\phi] = \{\psi \in CP(A,X);\psi \leq \phi\}$. Let $\phi(a) = \overline{V^*}\pi(a)V$ for all $a \in A$. For each operator $T \in \pi(A)'$, define a map $\phi_T(a) = \overline{V^*}T\pi(a)V$. Then $T \longrightarrow \phi_T$ is linear. If $\phi_T = 0$, we have

$$<T\pi(a)V\xi,\pi(b)V\eta>=<T\pi(b^*a)V\xi,V\eta>=\phi_T(b^*a)(\xi)(\eta) = 0$$

$$<T(\sum_{i=1}^{n}\pi_i(a_i)V\xi_i),\sum_{i=1}^{n}\pi(b_j)V\xi_j>= 0.$$

So $T = 0$. That is, $T \longrightarrow \phi_T$ is injective.

THEOREM 3.4. $T \longrightarrow \phi_T$ is an affine order isomorphism of the partially ordered convex set of $\{T \in \pi(A)' : 0 \leq T \leq I\}$ onto $[0, \phi]$.

The proof of this theorem is exactly the same way as the proof of theorem in Arveson's paper [1].

4. THE THREE EXTREMAL PROBLEMS.

Now we come to discuss three extremal problems.

DEFINITION 4.1. A completely positive map $\phi \in CP(A, X)$ is pure if, for every $\psi \in CP(A, X)$, $\psi \leq \phi$ implies that ψ is a scalar multiple of ϕ.

REMARK 4.2. According to [3], the extreme rays of $CP(A, X)$ can be characterized as the half lines $\{t\phi : t \geq 0\}$, where ϕ is a pure element of $CP(A, X)$.

We state the following theorems without proofs, for the proofs are almost the same as those in Arveson's paper [1].

THEOREM 4.3. All nonzero pure elements of $CP(A, X)$ are precisely those of the form $\phi(a) = \overline{V^*}\pi(a)V$, where π is an irreducible representation of A on some Hilbert space R and $V \in B(X, R)$, such that $R = [\pi(A)VX]$.

THEOREM 4.4. Let $\phi \in CP(A, X)$ and let $\phi(a) = \overline{V^*}\pi(a)V$ be its canonical representation. The extreme points of $[0, \phi]$ are those maps of the form $\overline{V^*}P\pi(a)V$, where P is a projection in $\pi(A)'$.

We consider the extreme points of the set $CP(A, X; K) = \{\phi \in CP(A, X); \phi(1) = K\}$, where K is a fixed positive operator in $B(X, \overline{X^*})$.

THEOREM 4.5. Let $\phi \in CP(A, X; K)$ and let $\pi(a) = \overline{V^*}\pi(a)V$ be its canonical representation with $\overline{V^*}V = K$. Then ϕ is an extreme point of $CP(A, X; K)$ if and only if $[VX]$ is a faithful subspace for the commutant $\pi(A)'$ of $\pi(A)$.

ACKNOWLEDGEMENT. The author thanks professor K. F. Taylor for helpful suggestions.

References

1. W.B.Arveson, Subalgebras of C^*-algebras, Acta Math., 123 (1969), 142-224

2. W.B.Arveson, An Invitation to C^*-algebra, Springer-Verlag, New York, Heidelberg, Berlin, 1976.

3. G.Choquet, Lectures on Analysis , W.A. Benjamin Inc., 1969.

4. E.Christensen and A.M. Sinclair, A Survey of Completely Bounded Operators, Bull. London Math 21 (1989) 417-448.

5. V.I.Paulsen, Completely Bounded Maps and Dilations, Pitman Research Notes in Math. Series, 1986.

6. G. Pisier, Completely Bounded Maps Between Sets of Banach Space Operators, Indiana Univ. Mat 39 (1990), No. 1, 249-277.

7. C.E.Rickart, Banach Algebras, Von Nostrand, Princeton, 1960.

8. S.Sakai, C^*-algebras and W^*-algebras, Berlin-Heidelberg-New York, Springer 1971.

9. W.F.Stinespring, Positive Functions on C^*-algebras, Proc. Amer. Math. Soc. 6 (1955), 211-216.

10. M.Takesaki, Theory of Operator Algebra I, Springer-Verlag, Berlin, 1979.

Journal of Applied Mathematics and Decision Sciences

Special Issue on Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/jamds/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	June 1, 2009
First Round of Reviews	September 1, 2009
Publication Date	December 1, 2009

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be