

EXTENSIONS OF HARDY-LITTLEWOOD INEQUALITIES

ZENGJIAN LOU

Mathematics Department
Qufu Normal University
Qufu Shandong 273165
P. R. China

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ABSTRACT. For a function $f \in H^s(B_n)$, with $f(0) = 0$, we prove

① If $0 < p \leq s$, then

$$\int_0^1 r^{-1} \left(\log \frac{1}{r} \right)^{s-p-1} M_p(r, R^\beta f) dr \leq \|f\|_{p,s,\beta}^{s-p} \|f\|_{p,s,\beta}$$

② If $s \leq p < \infty$, then

$$\|f\|_{p,s,\beta} \leq \|f\|_{p,s,\beta}^{s-p} \int_0^1 r^{-1} \left(\log \frac{1}{r} \right)^{s-p-1} M_p(r, R^\beta f) dr$$

where $R^\beta f$ is the fractional derivative of f . These results generalize the known cases $s = 2, \beta = 1$ ([1]).

KEY WORDS AND PHRASES. $H^s(B_n)$ space, fractional derivative.

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1. INTRODUCTION.

Let C^n denote the n -dimensional vector space over C . Let B_n denote the unit ball in C^n with boundary ∂B_n and let σ denote the rotation-invariant positive measure on ∂B_n for which $\sigma(\partial B_n) = 1$.

We assume that f is holomorphic in B_n . Let $R^\beta f(z) = \sum_{\alpha \geq 0} |\alpha|^\beta a_\alpha z^\alpha$ be the fractional derivative of $f(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha$ ($\beta > 0$).

For $0 < p, s, \beta < \infty$, we set

$$M_p(r, f) = \int_{\partial B_n} |f(r\zeta)|^p d\sigma(\zeta)$$

and

$$\|f\|_{p,s,\beta} = \left(\int_0^1 \int_{\partial B_n} |f(r\zeta)|^{p-s} |R^\beta f(r\zeta)|^s \left(\log \frac{1}{r} \right)^{s-p-1} r^{-1} d\sigma(\zeta) dr \right)^{1/s}$$

As usual, for $0 < p < \infty$, $H^s(B_n)$ denotes the space of holomorphic functions on B_n for which the means $M_p(r, f)$ are bounded and the norm of $f \in H^s(B_n)$ is defined by

$$\|f\|_p = \sup_{0 < r < 1} M_p(r, f).$$

Throughout this note, we assume that $f \in H^p(B_n)$, with $f(0) = 0$.

In [1], Hardy-Littlewood proved the following well-known theorem about $H^p(B_1)$.

THEOREM HL. If $0 < p \leq 2$, $f \in H^p(B_1)$, then

$$\int_0^1 (1-r) M_p^2(r, f) dr < \infty \quad (*)$$

If $2 \leq p < \infty$, then $(*)$ implies $f \in H^p(B_1)$.

In this note, we generalize these results to the unit ball B_n , with a new and short proof. That is, we prove the following

THEOREM. ① If $0 < p \leq s$, then

$$\int_0^1 r^{-1} \left(\log \frac{1}{r} \right)^{s-p-1} M_p^2(r, R^\theta f) dr \leq \|f\|_p^{s-p} \|f\|_{s,s,p}$$

② If $s \leq p < \infty$, then

$$\|f\|_{s,s,p} \leq \|f\|_p^{s-p} \int_0^1 r^{-1} \left(\log \frac{1}{r} \right)^{s-p-1} M_p^2(r, R^\theta f) dr$$

Set $s = 2, \beta = 1$ in the Theorem; by the following

LEMMA. For $0 < p < \infty$, then

$$\|f\|_p^2 = p^2 \|f\|_{p,2,1}^2$$

we have the following corollary, which extends Theorem HL (note that for $\zeta \in B_n$, $R^1 f(\lambda \zeta) = \lambda f'(\zeta)$, where $f'(\zeta) = f(\lambda \zeta)$, $\lambda \in B_1$, and $r \log \frac{1}{r} \sim 1-r$)

COROLLARY. ① If $0 < p \leq 2$, then

$$\int_0^1 r^{-1} \left(\log \frac{1}{r} \right) M_p^2(r, R^1 f) dr \leq \frac{1}{p^2} \|f\|_p^2$$

② If $2 \leq p < \infty$, then

$$\|f\|_p^2 \leq p^2 \int_0^1 r^{-1} \left(\log \frac{1}{r} \right) M_p^2(r, R^1 f) dr$$

2. PROOF OF THE MAIN RESULTS.

PROOF of the Theorem. Let $0 < p \leq s$. Assume without loss of generality that $\|f\|_p \neq 0$.

Set $\mu(\zeta) = \frac{|f(r\zeta)|^p}{\|f\|_p^p}$, then $\int_{B_n} \mu(\zeta) d\sigma(\zeta) \leq 1$; we have, by Jensen's inequality, for each r ,

$$\begin{aligned} & \left(\int_{B_n} |f(r\zeta)|^{p-s} |R^\theta f(r\zeta)|^s d\sigma(\zeta) \right)^{p/s} \\ &= \left(\|f\|_p^p \int_{B_n} \left| \frac{R^\theta f(r\zeta)}{f(r\zeta)} \right|^s \mu(\zeta) d\sigma(\zeta) \right)^{p/s} \\ &\geq \|f\|_p^{p/s} \int_{B_n} \left| \frac{R^\theta f(r\zeta)}{f(r\zeta)} \right|^p \mu(\zeta) d\sigma(\zeta) \\ &= \|f\|_p^{p/s-p} \int_{B_n} |R^\theta f(r\zeta)|^p d\sigma(\zeta) \\ &= \|f\|_p^{p/s-p} M_p^2(r, R^\theta f) \end{aligned}$$

So

$$\int_{B_n} |f(r\zeta)|^{p-s} |R^\theta f(r\zeta)|^s d\sigma(\zeta) \geq \|f\|_p^{s-p} M_p^2(r, R^\theta f)$$

Therefore

$$\begin{aligned}\|f\|_{p,\beta}^{-\beta} \|f\|_{p,s,\beta} &= \|f\|_{p,\beta}^{-\beta} \int_0^1 \int_{\partial B_r} |f(r\zeta)|^{p-s} |R^\beta f(r\zeta)|^s \left(\log \frac{1}{r}\right)^{s\beta-1} r^{-1} d\sigma(\zeta) dr \\ &\geq \int_0^1 r^{-1} \left(\log \frac{1}{r}\right)^{s\beta-1} M_p^s(r, R^\beta f) dr\end{aligned}$$

The case $p \geq s$ is treated in a similar way to obtain, for each r ,

$$\int_{\partial B_r} |f(r\zeta)|^{p-s} |R^\beta f(r\zeta)|^s d\sigma(\zeta) \leq \|f\|_{p,\beta}^{-\beta} M_p^s(r, R^\beta f)$$

So

$$\begin{aligned}\|f\|_{p,s,\beta} &= \int_0^1 \int_{\partial B_r} |f(r\zeta)|^{p-s} |R^\beta f(r\zeta)|^s \left(\log \frac{1}{r}\right)^{s\beta-1} r^{-1} d\sigma(\zeta) dr \\ &\leq \|f\|_{p,\beta}^{-\beta} \int_0^1 r^{-1} \left(\log \frac{1}{r}\right)^{s\beta-1} M_p^s(r, R^\beta f) dr\end{aligned}$$

This completes the proof of the Theorem.

Now, we use the technique of [2] to give the proof of the lemma.

For $\zeta \in B_r$, $R^1 f(\lambda\zeta) = \lambda f'_\zeta(\lambda)$, where $f'_\zeta(\lambda) = f(\lambda\zeta)$, $\lambda \in B_1$.

By the Hardy-Stein identity for one complex variable ([3]) we have

$$\begin{aligned}M_p^s(r, f_\zeta) &= \frac{p^2}{2\pi} \int_0^r \int_0^{2\pi} |f'_\zeta(\rho e^{i\theta})|^{p-2} |f''_\zeta(\rho e^{i\theta})|^2 \left(\log \frac{r}{\rho}\right) \rho d\theta d\rho \\ &= \frac{p^2}{2\pi} \int_0^r \int_0^{2\pi} |f(\rho\zeta e^{i\theta})|^{p-2} |R^1 f(\rho\zeta e^{i\theta})|^2 \left(\log \frac{r}{\rho}\right) \rho^{-1} d\theta d\rho\end{aligned}$$

Integrating with respect to $d\sigma(\zeta)$, using the Fubini theorem and the formula

$$\int_{\partial B_r} g(\zeta) d\sigma(\zeta) = \frac{1}{2\pi} \int_{\partial B_r} d\sigma(\zeta) \int_0^{2\pi} g(e^{i\theta}\zeta) d\theta, \quad g \in L^1(\sigma)$$

(see [4, P. 15]), we have

$$M_p^s(r, f) = p^2 \int_0^r \int_{\partial B_r} |f(\rho\zeta)|^{p-2} |R^1 f(\rho\zeta)|^2 \left(\log \frac{r}{\rho}\right) \rho^{-1} d\sigma(\zeta) d\rho$$

Letting $r \rightarrow 1$, we obtain the Lemma.

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